NEUTRON TRANSPORT THEORY

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OXFORD
AT THE CLARENDON PRESS

Oxford University Press, Amen House, London E.C.4

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
BOMBAY CALCUTTA MADRAS KARACHI KUALA LUMPUR
CAPE TOWN IBADAN NAIROBI ACCRA

FIRST EDITION 1957
REPRINTED 1958

PRINTED IN GREAT BRITAIN

PREFACE

THE purpose of this book is to give a comprehensive and up-to-date account of all the principal mathematical methods used in neutron transport theory, together with a brief discussion of their relative suitability in various situations.

Some of these methods have already been described in previous books on related subjects such as nuclear reactor theory and the theory of radiative transfer in stellar atmospheres. Many other equally important methods, however, have been available only in technical reports, or in isolated articles in periodicals. This is particularly true of the more refined forms of the spherical harmonics method, which is in fact the most powerful method in existence, of the Serber–Wilson method, which is very useful in certain problems, despite its limited range of applicability, and of the chief methods, apart from age theory, for dealing with energy-dependent problems, namely the method of polynomial approximations, Feynman's method, etc. The author's purpose has been to attempt to fill this gap.

The subject is developed in a self-contained manner; that is, no previous knowledge of it is necessary. It was felt, however, that the majority of readers of a book giving a comprehensive survey of the more advanced methods will, in fact, have some elementary knowledge of the subject. The order and method of presentation therefore give priority to the needs of such readers, rather than to those who use the book as their first introduction to the subject. In particular, instead of starting with a simplified physical picture, deriving the more elementary approximations, and then proceeding to refine these, we have preferred to begin with a precise formulation of the problem and to present any simplified treatment that may be available as an approximate method of solving the exact equations. It was felt that this would allow a more unified presentation of the subject, and an immediate assessment of the range of applicability of each approximate method. Admittedly this means that a reader who wishes for a first introduction to the subject will need to go through a greater amount of formal work before he can understand the problems involved. To start with the precise formulation of the problem does not, of course, preclude the use of tentative and semi-intuitive arguments in seeking more profitable approximations, and considerable use has been made of such arguments for this purpose.

The knowledge of mathematics which the reader is assumed to possess

is similar to that required in other branches of mathematical physics. Besides the material contained in Whittaker and Watson's *Modern Analysis*, he is presumed to know the definition, inversion formulae, and elementary properties of Fourier, Laplace, and Mellin transforms. No previous knowledge of their applications is assumed. No use is made of quantum mechanics in neutron transport theory, provided that the laws governing the results of individual collisions are assumed; these laws, however, are to some extent of quantum-mechanical origin.

In conclusion, the author would like to stress that he would have had great difficulty in writing the book, had it not been for the generous assistance of a number of people. He is particularly indebted to Dr. J. B. Sykes, whose collaboration has almost amounted to co-authorship. Dr. J. H. Tait and Mr. K. T. Spinney have written Chapter XXVI on Holte's method, while Mr. K. W. Morton has made appreciable contributions to Chapter XVI on the Monte Carlo method. It is a pleasant duty to thank Dr. B. H. Flowers, Dr. J. H. Tait, and other members of the Theoretical Physics Division, A.E.R.E. Harwell, for many helpful suggestions and constructive criticisms, and Dr. Flowers, Professor R. E. Peierls, and Dr. G. Placzek† for their encouragement. Professor H. A. Bethe, Dr. J. P. Elliott, Professor R. P. Feynman, Dr. J. C. Mark, and Professor R. Serber have kindly given permission to include hitherto unpublished contributions to neutron transport theory. Finally, the author is greatly indebted to Professor W. H. Watson and the Computation Centre of the Physics Department, University of Toronto, for facilities which enabled him to complete the book.

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June 1955

† Since deceased.

Note added in second impression. The author's thanks are due to Dr. A. Hassitt for a number of comments which have been incorporated in the second impression (pp. 359–60, 416).

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THE SUBJECT OF NEUTRON TRANSPORT THEORY

NEUTRON transport theory is concerned with the migration of neutrons through bulk media. This migration involves a large number of random events, namely collisions, and to study it we must first ascertain the laws governing the results of individual collisions, and then solve the statistical problem of determining the result of a large number of random collisions governed by these laws. The former problem belongs to the field of nuclear physics, and of classical and quantum mechanics, and neutron transport theory begins with given laws governing the results of individual collisions and considers the statistical problem of determining the distribution of the neutrons in space, angle, and energy.

The chief direct applications of neutron transport theory are in the study and design of nuclear reactors and in the interpretation of experiments in nuclear physics. In these applications, the neutron energies lie between a few million electron-volts and about one-fortieth of an electron-volt. The upper limit corresponds to neutrons emitted in fission or by natural-radioactivity sources, while the lower limit corresponds to neutrons in thermal equilibrium with the surrounding medium; these cannot on the average lose any further energy in subsequent collisions. Accordingly, neutron transport theory is concerned with neutrons whose energies are either in this range, or such that the laws governing the results of individual collisions (which we may call the *collision laws*) are the same as for energies in this range.

The fact that one of the main applications of neutron transport theory is to the study and design of nuclear reactors allies it to nuclear reactor theory. The latter, however, is a branch of engineering, which relies on neutron transport theory for information concerning the distribution of the neutrons in space, angle, and energy, much as neutron transport theory in turn relies on nuclear physics and mechanics for information concerning the collision laws. Accounts of nuclear reactor theory contain, of course, some discussion of the migration of neutrons, but this is usually based on a simplified physical picture, omitting the detailed consideration of neutron migration as a series of individual collisions and passing directly to its effects on the spatial distribution, just as the equations of heat conduction or of chemical diffusion are usually derived by disregarding the individual events concerned and passing directly

to the net result of these. We shall see that such a simplified treatment leads, under suitable conditions, to a valid approximation to the spatial neutron distribution at large distances from all sources and boundaries, provided that certain quantities are suitably interpreted. By 'large distances' we here mean those large compared with the *mean free path*, that is, the mean distance travelled by a neutron between successive collisions. This is the basis of the *diffusion approximation*, which is one of the approximate methods available in dealing with large bodies. If, however, we are concerned with bodies of only moderate size, or with boundary effects, more accurate methods, based on a more exact consideration of neutron migration, have to be employed.

Neutron transport theory is sometimes called *neutron diffusion theory*. We prefer the former name, however, in order to emphasize that our treatment is based on the more exact consideration of neutron migration, and not on the simplified model referred to above. The treatment developed will therefore be applicable to bodies of any size, and not only to large ones.

Much of the early work on neutron transport theory was concerned with the distribution of neutrons in space and angle rather than in energy, and an approximation was developed in which the energy dependence was removed by integration. This is the *constant cross-section approximation*, often called *one-velocity-group theory*. However, this procedure is only approximate, and is not very accurate except in a few special cases, although these are of some practical importance. The constant cross-section approximation retains its importance chiefly as a means of familiarizing oneself with the mathematical techniques used in neutron transport theory, and as a convenient method of solving a number of auxiliary problems. We shall therefore begin our treatment with a fairly detailed discussion of the constant cross-section approximation, but afterwards we shall give equal prominence to energy-dependent problems.

We shall now indicate the relation between neutron transport theory and other branches of physics. It has already been stated that neutron transport theory is concerned with a statistical problem. Several other such problems are of interest to the physicist: the transfer of radiation in stellar atmospheres and radiative equilibrium in them; the penetration of X-rays and γ-rays through scattering media; cosmic ray showers, and so on. These processes differ only in that the collision laws are different, and even these are very similar for some of the processes mentioned. This is particularly true of neutron transport theory and

radiative transfer in stellar atmospheres. In fact, the equations of the constant cross-section approximation are identical with those for the corresponding case (a grey atmosphere with unpolarized light) in radiative transfer theory. Thus several fundamental problems of neutron transport theory have been considered, and partly solved, by astrophysicists such as Schwarzschild, Milne, Hopf, and others, before the neutron was even discovered, besides the more recent work of Chandrasekhar. Conversely, many of the methods recently developed in neutron transport theory could advantageously be used in problems of radiative transfer.

However, even in this approximation there is an appreciable difference between the two branches of physics, in that neutron transport theory presents a much greater variety of problems. In the radiative transfer case, we are usually concerned with a star, which is a spherical body whose radius is large compared with the mean free path, and can usually be approximated by a half-space. In neutron transport theory, the migration of neutrons through bodies of many shapes and sizes has to be considered. Secondly, radiative transfer is always concerned with a given star, whereas the ultimate object in neutron transport theory is often to design a nuclear reactor, and this involves finding the critical size, i.e. solving an eigenvalue problem. Both these features call for a greater variety and flexibility of methods for use in neutron transport theory than are needed for the theory of radiative transfer.

Thus the scope of neutron transport theory is in some ways rather wider than might have been expected. In other respects, however, it is found to be more limited than the name might suggest. In particular, while we shall consider systems containing more than one medium, each medium will be regarded as homogeneous. In practice we often encounter heterogeneous media of certain types, such as the lattice formed by placing rods of one material in another material, with the radius of the rod and the spacing of the lattice lying within definite limits. The determination of the neutron distribution in such a medium, using the constants given by nuclear physics, might be regarded as a problem of neutron transport theory. However, the development of methods of calculation suited to such a system, and the selection of the best experiments to supplement our knowledge of the fundamental constants and to avoid lengthy calculations of quantities more easily determined experimentally, belong rather to the theory of nuclear reactors, and will not be discussed in this book.

A second limitation of the scope of neutron transport theory, in the

sense now current, is connected with the energy range considered. We have said above that, in the chief applications of neutron transport theory, the energy range concerned is from a few MeV to about $\frac{1}{40}$ eV. The collision laws applied are therefore those valid for neutrons in this energy range. Outside the above limits, the collision laws are different. At energies of the order of 30 MeV, new processes which cannot occur at lower energies are found, such as nuclear disintegrations giving the *stars* observed in photographic emulsions. The collision laws for events of this kind are as yet insufficiently well known to form the basis of a statistical treatment. Moreover, the information required concerning neutrons of such high energies is different from that desired regarding neutrons in the range mentioned above. If we consider neutrons of energy below thermal, the problem is even more different. Here the main problem is not that of the result of random collisions of neutrons with atomic nuclei, but that of the interaction of a beam of neutrons with crystal lattices. This problem, however, belongs to a different branch of physics, *neutron diffraction theory*, and is completely outside the scope of transport theory.

PART I

GENERAL ANALYSIS

I

THE PHYSICAL PROPERTIES OF NEUTRONS

1.1. The properties underlying the concept of diffusion

A NEUTRON is a heavy uncharged elementary particle. Of the forces which act upon it, the nuclear forces are by far the most important, and they are the only ones that need be taken into account under the conditions in which one is interested in the diffusion of neutrons. Since these nuclear forces have an extremely short range, it follows that:

(A) the motion of a neutron can be described in terms of its collisions with atomic nuclei and with other freely moving neutrons;
(B) these collisions are well-defined events;
(C) between such collisions a neutron moves with a constant velocity —that is, in a straight line with a constant speed.

Three further points are to be noted.

Firstly, even with the strongest neutron sources and the most powerful nuclear reactors, the number of neutrons present at any time in any volume is negligibly small compared with the number of atomic nuclei present in that volume. Consequently—

(D) the mutual collisions of freely moving neutrons may safely be neglected, and only the collisions of neutrons with the atomic nuclei of the surrounding medium need be taken into account.

Secondly, nuclear radii are very small compared with atomic radii, the ratio being of the order of 10^{-4}. Thus, at a point outside the atom, the solid angle subtended by the nucleus is of the order of 10^{-8} compared with that subtended by the whole atom. Hence, on classical reasoning, the neutron will pass through some 10^8 atomic systems between successive collisions with nuclei. This number is so large that we may safely conclude that

(E) for a neutron travelling at a given speed through a given medium, the probability of collision per unit path length is a constant.

In actual fact, classical reasoning is inexact; we should take into account such quantum-mechanical effects as uncertainty of localization of the neutron, resonances, etc. These depend on the speed of the neutron, and so the number of atoms traversed between collisions will also depend thereon. However, the above order-of-magnitude estimate will be sufficiently accurate unless there are very strong resonances near the thermal energies, and these occur in only a few elements. Even in these cases the strongest resonances are usually capture resonances; thus the neutrons are almost certain to be captured, rather than scattered, if they undergo a collision. In these conditions it clearly matters little whether (E) is strictly correct or not.

Thirdly, it is fairly natural to assume that

(F) the neutron or neutrons emerging from a collision do so at the point in space where the collision took place.

This is strictly true except in the case of fission, and there the error involved in the assumption is negligible.

1.2. The results of individual collisions

1.2.1. *The relevant energy range. Thermal and non-thermal neutrons*

As has been pointed out in the Introduction, neutron transport theory is primarily concerned with neutrons that are originally emitted by fission or by some natural-radioactivity source of neutrons. The energy of such neutrons is of the order of a few MeV. This energy is many orders of magnitude greater than the energy ($\frac{1}{40}$ eV at room temperature) at which a neutron would be in thermal equilibrium with the medium in which it diffuses. Thus, when a fission (etc.) neutron undergoes scattering, it must lose energy, and will continue to do so until it either is absorbed by some nucleus or has reached thermal equilibrium with the surrounding medium. The neutron energies in which we shall be interested, therefore, range from a few MeV down to about $\frac{1}{40}$ eV.

When a neutron in this energy range undergoes a collision, it may be *captured*; it may cause *fission*, with the subsequent release of several other neutrons; it may be *scattered—elastically* if the scattering nucleus is left in its initial state, or *inelastically* if that nucleus is left in an excited state. In the last case, the energy of excitation is later released in one of a number of forms, as will be seen presently (§ 1.2.3).

The rest mass of a neutron, expressed in energy units, is about 10^3 MeV, i.e. well above our energy range, so that relativistic effects may be neglected in examining the results of collisions. On the other hand, the molecular binding energies are of the order of a few eV and the energies

of thermal motion are even less, so that these effects can also be neglected except at the lower end of the range, and we may assume that

(i) the atomic nuclei are initially at rest;
(ii) they recoil freely after a collision.

In the case of fission, these assumptions can clearly always be applied, regardless of the energy of the neutron which causes the fission. Inelastic scattering, as we shall see later, can be undergone only by neutrons sufficiently energetic for assumptions (i) and (ii) to be satisfied in any event. With elastic scattering, however, cases may arise where those assumptions are not strictly correct.

Neutrons whose energy is high enough to justify the use of assumptions (i) and (ii) in considering their scattering are called *non-thermal* or *fast* neutrons. (The term *fast* is, however, sometimes reserved for neutrons in the MeV energy region.) Neutrons whose energy is not high enough to justify the use of (i) and (ii) are called *thermal* neutrons.†

1.2.2. *Elastic scattering*

We consider first the most frequent result of a collision, namely elastic scattering, and restrict ourselves for the moment to non-thermal neutrons. In an elastic collision, the atomic nucleus is left after the collision in its initial state, that is, the ground state. The energy of the neutron in the centre-of-gravity system‡ is therefore unaltered, and we have only to examine the angular distribution of the scattered neutrons. Thus, if ω, say, is the final direction of the neutron in the C system, $g(\omega, E)\,d\omega$ is the probability that for a neutron of energy E this direction will lie within $d\omega$ about ω, and Θ is the scattering angle (i.e. the angle between the initial and final directions of the neutron in the C system), we have only to examine the function§

$$g(\omega, E) = (1/4\pi)\{1+3g_1(E)P_1(\cos\Theta)+5g_2(E)P_2(\cos\Theta)+\ldots\}, \quad (1.1)$$

where P_1, P_2, etc., are Legendre polynomials.

A simple qualitative estimate of the magnitudes of the various $g_n(E)$

† It may be more consistent to restrict the term *thermal* to those neutrons which are in or near thermal equilibrium with the surrounding medium, and to introduce an intermediate energy region called *epithermal*, where assumptions (i) and (ii) are no longer strictly applicable, but thermal equilibrium has not yet been reached. However, this region is small compared with the non-thermal region and contains many fewer neutrons than the thermal region. It can therefore usually be safely disregarded.

‡ We shall refer to the centre-of-gravity system as the C system and to the laboratory system as the L system.

§ To take $g(\omega, E)$ as depending only on Θ, as far as direction is concerned, presupposes that we are dealing with unpolarized neutrons and that the atomic nuclei are randomly oriented. This will be the case in all applications of neutron transport theory.

in (1.1) can be obtained as follows. According to quantum mechanics, it is impossible to say exactly where the neutron is at a given moment; the measure of the uncertainty in its position for each energy is given by its de Broglie wavelength. If this is greater than the nuclear radius, i.e. if

$$E < 10/M^{\frac{2}{3}} \tag{1.2}$$

(where E is the neutron energy in MeV and M is the mass number of the nucleus), it is impossible to give an account of how the collision happened. (In classical language, it is impossible to say at what angle the neutron hit the nucleus.) In this case, all directions for the scattered neutron will be equally likely in the C system, i.e. all $g_n(E)$ for $n = 1, 2,$ etc., will be negligible.

Similarly, if the de Broglie wavelength of the neutron is less than the nuclear radius but more than half the nuclear radius, i.e. if

$$10/M^{\frac{2}{3}} < E < 40/M^{\frac{2}{3}},$$

the best description of the geometrical conditions of the collision can involve only one parameter. It is readily seen that this implies that all but the first two terms in (1.1) will be negligible. That is, $g_1(E)$ cannot be neglected, but $g_n(E)$ can be neglected for $n \geqslant 2$. In general, $g_n(E)$ can be expected to be negligible if

$$E < n^2 . 10/M^{\frac{2}{3}}. \tag{1.2'}$$

If the theory of elastic scattering is rigorously worked out, it is found that there are in fact two kinds of elastic scattering, *potential* scattering and *resonance* scattering. The estimate (1.2') is valid for potential scattering but not for resonance scattering. However, elastic resonance scattering is usually unimportant, so that (1.2') may safely be used.

On applying these estimates to the heaviest nuclei, we see that in this case elastic scattering should be approximately isotropic in the C system up to about 250 KeV. Beyond this energy, the elastic scattering can no longer be regarded as isotropic, but up to about 1 MeV it is safe to regard it as linear in the cosine of the scattering angle, and so on. For lighter nuclei, the limits in question will be correspondingly higher.

Thus, for the elastic scattering of non-thermal neutrons of energies not greater than a few MeV, the scattering law in the C system is always given by a polynomial of comparatively low order in the cosine of the scattering angle. The situation regarding the elastic scattering of *thermal* neutrons is much more involved, and we shall not examine it for the time being; see §§ 4.4 and 20.4.3.

1.2.3. *Inelastic scattering*

We next consider inelastic scattering. In this case the nucleus is left after the collision in an excited state. The excitation energy is supplied by the kinetic energy of the colliding neutron, and is eventually released —usually in the form of a γ-ray, but sometimes by means of some other process, such as the emission of a proton, of an α-particle, or of a neutron. The last case is referred to as an $(n, 2n)$ reaction, since in the end we have two freely migrating neutrons instead of one.

For highly symmetrical and fairly light nuclei (such as carbon or oxygen), the energy difference between the ground state and the lowest excited state of the nucleus is of the order of a few MeV; for heavier or less symmetrical nuclei it is less, but even for the heaviest nuclei it is, *with a few exceptions, of the order of 40 KeV. Further, if a neutron has just enough energy to make an inelastic scattering energetically possible,* the probability of this event is still zero, and it can become appreciable only for energies well above the threshold value. Finally, for any one excited state, the probability that the nucleus will be left in that state is usually small. Hence, for the majority of nuclei, inelastic scattering becomes really important only when the neutron is so energetic that the nucleus can be left in any of a large number of alternative excited states.

These remarks show that, for the lighter nuclei, inelastic scattering can be neglected altogether in the energy range with which we are concerned; in the middle of the periodic table (mass numbers say 50 to 200), inelastic scattering may have to be taken into account for a few exceptional nuclei towards the top of our energy range; for the heaviest nuclei inelastic scattering becomes a general occurrence towards the top of the energy range.

The angular distribution of inelastically scattered neutrons is usually assumed to be isotropic. This assumption is not strictly correct; if we consider inelastic scattering which leaves the nucleus in a particular excited state, the angular distribution will depend upon the nucleus and upon the state in which it is left. However, if the neutron is energetic enough and the nucleus is heavy enough, so that the latter may be left in any of a large number of excited states, we shall really be interested in the mean angular distribution, averaged over all excited states in a given range of excitation energies. While the angular distributions corresponding to individual excited states may be anisotropic, the deviations from isotropy will usually almost cancel on taking the mean, so that the mean angular distribution will usually be nearly isotropic.

On the other hand, if the original energy of the neutron is just sufficient

to make appreciable the probability of inelastic scattering by the given nucleus, then after the collision the neutron will be so slow that in the subsequent elastic collisions it will be scattered isotropically (see § 1.2.2). In this case, the scattering law in the inelastic collision is unlikely to affect noticeably the general neutron distribution. Thus the only case where the assumption of isotropic inelastic scattering is likely to be inadequate for our purposes is that where only a few excited levels are available, yet the neutron emerges from the collision with sufficient energy to be anisotropically scattered in subsequent elastic collisions. Such situations, if they are possible at all, are too rare to be of interest.

One further point should be noticed with respect to inelastic scattering. Since, in our energy range, this process can occur only with fairly heavy nuclei, and since the energy expended on nuclear excitation is always large compared with that expended on nuclear recoil, the difference between the C and L systems may always be disregarded in dealing with inelastic scattering. In particular, the scattering may be treated as isotropic in the L system. This may be shown as follows: if the nucleus is heavy, isotropy of scattering in the C system is nearly equivalent to isotropy in the L system, unless the velocity of the scattered neutron is comparable with, or less than, that of the centre of gravity of the system neutron–nucleus. In this case, however, the neutron energy in the C system after the collision is of the order of $1/M^2$ compared with its energy before the collision, where M is the mass number of the nucleus. That is to say, the relative excess of available energy over the energy needed for inelastic scattering is of the order of $1/M^2$, which (e.g.) is about 4×10^{-3} for oxygen. For such low excess energies, the probability of inelastic scattering is negligible. If more than one excited level is available, the same argument can be applied to each resulting energy level of the nucleus separately.

The same argument can also be applied to the case of the $(n, 2n)$ reaction, which is of importance only in beryllium at high energies. The angular distribution may again be assumed isotropic.

1.2.4. *Fission*

We finally consider fission. When fission is brought about by a neutron, the fissile nucleus normally first absorbs the neutron, and this brings the nucleus into a highly excited state, so that it enters a state of violent and, at first, very irregular motion. The energy of this motion is gradually concentrated into one particular mode of vibration, namely periodic elongation and contraction of the nucleus, and this finally

results in fission. Thus fission is a comparatively slow process on the nuclear time-scale, though it is practically instantaneous on most time-scales. It is obvious that, during the period in which the nuclear matter is in irregular motion, before the energy is concentrated in a particular mode of vibration, all trace is lost of the direction in which the neutron originally entered the nucleus; that is, all directions are equally likely in the C system for the secondary neutrons emitted after fission. Further,

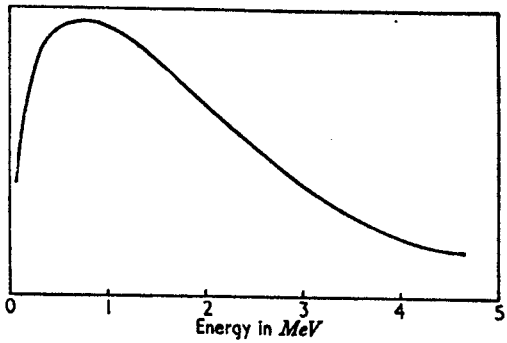


FIG. 1.

since fission occurs only in the heaviest nuclei, whose mass is over 200 times that of the neutron, and since (if the original neutron energy was in the range of interest to us) the secondary neutrons will have energies at least comparable with that of the original neutron, the mass of the fissile nucleus may be taken as infinite to a close approximation. That is, the C and L systems may be identified. Thus we can say that:

All directions of the secondary neutrons emitted in fission are equally likely in the laboratory system of coordinates.

There are certain cases of resonance fission where the above description of the fission process is not strictly applicable, and so the angular distribution is not necessarily isotropic. However, it is usually treated as isotropic; this assumption may be justified in the same general manner as was done for inelastic scattering.

The energy spectrum of neutrons emitted in fission has been measured by experiment. It is shown schematically in Fig. 1.

It is seen from this graph that the mean energy of fission neutrons is of the order of a few MeV, and that the energies of the fastest and slowest neutrons emitted in an appreciable amount differ by a factor of ~ 10. On the other hand, the neutron energy varies over the range in which

we are interested by a factor of $\sim 10^8$. Hence, though the spread of the fission spectrum is appreciable on the *linear* energy scale, it may frequently be neglected on the *logarithmic* energy scale.

1.2.5. *Prompt and delayed neutrons*

In order to be able to deal with time-dependent, as well as stationary, problems it is necessary to examine possible delays in the emission of secondary neutrons. It has been mentioned that fission, elastic scattering, and inelastic scattering are practically instantaneous on any but the nuclear time-scale. While this is strictly true for both kinds of scattering, for fission the situation is more involved. The fission itself may be regarded as instantaneous, but at least a certain proportion of the fission neutrons emerge from the fission fragments when the fission is completed. In the great majority of cases, this process can still be regarded as instantaneous. There are, however, a small number of cases where the last of the fission neutrons cannot be released before a β-ray has been emitted. The time-scale for β-emission is much longer than that for neutron emission, and so the time delay before the neutron is emitted may well be significant. Since the fission fragment is charged, it can travel only a negligible distance in space during this time compared with the distances travelled by neutrons. The delayed neutrons may therefore be regarded as emitted from the same point in space as the prompt neutrons, and this justifies the assumption (F) of § 1.1. The energy spectrum of the delayed neutrons may also be expected to differ from that of prompt neutrons; it will involve rather lower energies.

1.3. The mean free path, the cross-section, the mean number of secondaries, etc.

1.3.1. *Definitions and notations*

We now introduce certain definitions which are in constant use in dealing with phenomena of neutron transport. According to assumption (E) of § 1.1, for a neutron travelling with a given speed through a given medium, the probability of a collision per unit path length is a constant. The reciprocal of this probability is known as the *total mean free path*, or briefly as the *mean free path*, and is usually denoted by l_{tot}, or l simply. To indicate that it may depend on the speed v, i.e. on the energy E of the neutron, we shall sometimes write it as $l_{\text{tot}}(v)$, $l(v)$, $l_{\text{tot}}(E)$, or $l(E)$. The probability itself, referred to in assumption (E), is accordingly known as the *inverse total mean free path*, or the *inverse mean free path* simply, and is sometimes denoted by α_{tot}, α, $\alpha_{\text{tot}}(v)$, etc.

It is obvious from the argument which led us to assumption (E) that, as the density of the medium through which the neutrons travel increases, the inverse mean free path will increase proportionately. Further, the contribution of the various kinds of atom to the inverse mean free path will be additive. Thus we may write

$$1/l_{\text{tot}} = \alpha_{\text{tot}} = \sum_i N_i \sigma_{\text{tot},i}, \tag{1.3}$$

where N_i is the number of atoms of the ith kind per unit volume, and the $\sigma_{\text{tot},i}$ are certain coefficients appertaining to the various kinds of atom, and are known as the *total cross-sections*. The argument which led to (1.3) suggests that each of these total cross-sections depends on the energy of the neutron and on the properties of the nucleus to which it refers, but not on the number of that kind of nucleus per unit volume or on the presence or absence of other kinds of nucleus. This is in fact the case, provided that the neutron energy is large enough compared with the energies of molecular binding, so that the recoil of the nucleus is independent of how the atom was bound before the collision. If, however, the neutron energy is comparable with, or smaller than, the energy of molecular binding, the result of the collision may depend on how the atom, whose nucleus is hit, was bound before the collision. In this case, by the laws of quantum mechanics, the probability of the collision, that is the cross-section, will also depend on the nature of the molecular binding, and so may be affected by the presence or absence of other material.

When a neutron undergoes a collision, the result may be its capture, or scattering, or a fission accompanied by the release of several new neutrons. Thus the number of neutrons is not conserved, and it will be convenient to introduce the *mean number of secondaries per collision*, denoted by c. If the medium through which the neutron travels is 'pure', i.e. contains only one kind of nucleus, and if for this kind of nucleus we denote the probabilities of scattering, capture, and fission by

$$\sigma_s/\sigma_{\text{tot}}, \quad \sigma_c/\sigma_{\text{tot}}, \quad \text{and} \quad \sigma_f/\sigma_{\text{tot}} \tag{1.4}$$

respectively, then the mean number of secondaries per collision will clearly be

$$c = (\sigma_s + \nu\sigma_f)/\sigma_{\text{tot}},$$

where ν is the *mean number of secondaries per fission*. In a medium containing several kinds of nucleus, the mean number of secondaries per collision will be

$$c = \sum_i N_i[\sigma_{s,i} + \nu_i \sigma_{f,i}] \Big/ \sum_i N_i \sigma_{\text{tot},i}, \tag{1.5}$$

where ν_i, $\sigma_{s,i}/\sigma_{\text{tot},i}$, etc., are the values of ν, $\sigma_s/\sigma_{\text{tot}}$, etc., for the ith kind of

nucleus, and N_i and $\sigma_{\text{tot},i}$ have been defined in connexion with (1.3). The formula (1.5) follows from the facts that the probability for a collision to be with a nucleus of the jth kind is $N_j \sigma_{\text{tot},j} \Big/ \sum_i N_i \sigma_{\text{tot},i}$, while the mean number of secondaries in such a collision is $(\sigma_{s,j}+\nu_j \sigma_{f,j})/\sigma_{\text{tot},j}$. The numerator in (1.5) is called the *mean number of secondaries per unit path*, and is denoted by β.

All the quantities which appear in these formulae may depend on the energy of the neutron. However, it is generally considered that the mean number of secondaries per fission, ν, is practically independent of energy up to a few MeV, that is, in the energy range with which we shall be concerned.

The use of the notation (1.4) for the probabilities of various results of the collision is equivalent to representing the total cross-section σ_{tot} as the sum of three terms:

$$\sigma_{\text{tot}} = \sigma_s+\sigma_c+\sigma_f, \tag{1.6}$$

which are called the *scattering cross-section*, the *capture cross-section*, and the *fission cross-section* respectively. The scattering cross-section is sometimes divided into the *elastic cross-section* and the *inelastic scattering cross-section* (the shorter term *inelastic cross-section* is avoided, since it is often understood to include capture and fission as well as inelastic scattering):

$$\sigma_s = \sigma_{\text{el}}+\sigma_{\text{in}}. \tag{1.6'}$$

The concepts of the *scattering mean free path*, the *capture mean free path*, etc., are also often used; they are defined, analogously to (1.3), by

$$1/l_s = \alpha_s = \sum_i N_i \sigma_{s,i}; \qquad 1/l_c = \alpha_c = \sum_i N_i \sigma_{c,i}, \quad \text{etc.} \tag{1.3'}$$

Formulae (1.6) and (1.6′) then imply that

$$1/l_{\text{tot}} = 1/l_s+1/l_c+1/l_f; \qquad 1/l_s = 1/l_{\text{el}}+1/l_{\text{in}}. \tag{1.7}$$

1.3.2. *General features of the energy dependence of cross-sections*

The dependence of the various cross-sections defined above on energy has been determined experimentally, and curves showing this dependence may be found, for instance, in (**26**). We shall not reproduce these data, but we shall now discuss some typical features of the curves, derived partly from experiment and partly from the theory of nuclear physics.

We first consider the capture cross-sections. A nucleus of mass number M can capture a neutron if such capture results in a possible excited state of the isotope of mass number $M+1$. Each of these excited states is associated with a particular energy, though these energy levels are

somewhat broadened owing to the finite lifetime of the excited states. Breit and Wigner (6) have investigated the effect of this broadening on the cross-section, and have shown that, if only one excited state of the isotope of mass number $M+1$ is available, then the capture cross-section for the nucleus of mass number M is given by

$$\sigma_c(E) = \frac{1}{\sqrt{E}} \frac{A}{(E-E_r)^2+\frac{1}{4}\Gamma^2}, \tag{1.8}$$

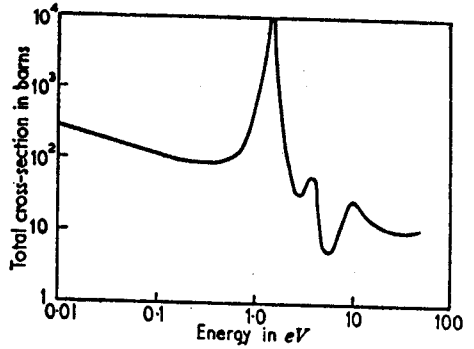
where E is the energy of the neutron and A, E_r, and Γ depend on M but not on E. The curve showing the variation of this function has a peak at $E = E_r$, known as the *capture resonance*; E_r is called the *resonance energy* and Γ the *width* of the resonance. If more than one excited state of the isotope of mass number $M+1$ is available but these states are widely spaced, then the capture cross-section can be approximately represented by the superposition of terms of the form (1.8), one for each state; the quantity A, however, in this case is a slowly varying function of E. If the excited states available are not widely spaced, the expression becomes more complicated. For heavy nuclei and very fast neutrons, the excited states of the isotope of mass number $M+1$ become quite dense, and the resonance character of the capture cross-section curve is altogether lost.

The formula (1.8) and its extensions just described refer to capture by an isolated nucleus, and the energy is referred to the C system. This means that it is applicable only to neutrons whose energy is greater than the molecular binding energy. However, taking (1.8) as it stands, we see that, when E is sufficiently small (depending on the value of Γ), $\sigma_c(E)$ behaves as $1/\sqrt{E}$, that is, as $1/v$. For some nuclei this behaviour is masked by the thermal and molecular-binding effects; for others it is discernible in the non-thermal region, while for boron it continues up to remarkably high energies.

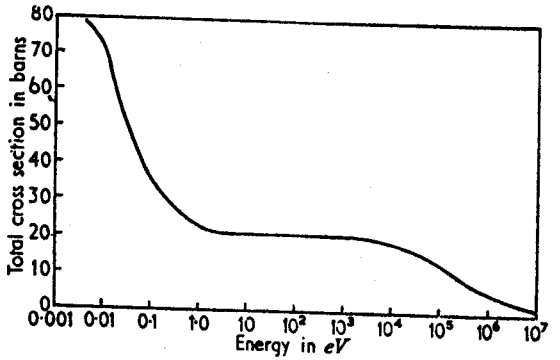
We now consider elastic scattering. As was mentioned in § 1.2.2, this consists of two parts, the potential scattering, for which the cross-section is largely independent of energy, and the resonance scattering, for which the cross-section is given by a formula which differs from the type (1.8) only by the absence of the factor $1/\sqrt{E}$. In the resonances, the resonance scattering is usually just comparable with the potential scattering, while outside the resonances it is negligible in comparison. Thus, if the resonances concerned are fairly narrow, it is often possible to neglect resonance scattering altogether.

For other processes, the cross-section curves are largely similar to

those described above. For threshold processes, however, i.e. those which are energetically impossible for $E <$ some E_t, the cross-section is zero for $E \leqslant E_t$, and rises slowly from zero for greater energies, finally taking on the character of the curves discussed above. The curves for the total cross-section are obtained by superposing those for the individual processes, and are therefore particularly liable to be irregular. Figs. 2*a* and 2*b* show the variation of the total cross-section with energy for indium and for hydrogen in paraffin over certain ranges. The cross-sections in these graphs are given in barns (1 barn $= 10^{-24}$ cm^2).



(*a*)



(*b*)

FIG. 2.

The peak of the indium curve at $E \cong 1{\cdot}44$ eV is a capture resonance of the type represented by (1.8). The hydrogen curve has no such peak, i.e. within the energy range shown the proton has no sharp resonances for neutron processes. The rise of this curve below 1 eV is connected with the effect of the chemical binding, and this part of the curve depends to

some extent on the chemical compound considered; for instance, in the case of hydrogen in water, the rising part of the curve is displaced a little to the right. On the other hand, the part of the curve for energies greater than 1 eV is independent of the chemical compound considered.

The absence of resonances in the hydrogen cross-section curve has to do with the fact that the proton is the simplest existing nucleus. Most other elements exhibit resonances, although for some there are large energy ranges devoid of resonances; in carbon, for instance, the total cross-section is remarkably constant from about 2 eV to about 500 KeV.

1.4. Preview of the main problems of neutron transport theory

We here conclude our summary of the physical properties of the neutron. Before beginning the mathematical treatment of neutron transport and migration, however, we shall give a short qualitative preview of the problems for whose solution neutron transport theory is required.

Considering the variation in time of the total number of neutrons, n, in a system, we see that

$$\text{Production by independent sources} + \text{production by fission and } (n, 2n) \text{ reaction} = \text{capture} + \text{escape from the system} + \partial n/\partial t, \quad (1.9)$$

where $\partial n/\partial t$, if positive, is the increase per unit time in the neutron population of the system; if negative, it is the decrease per unit time, taken with a negative sign. 'Production by fission' is understood to mean the excess of the number of neutrons released in fission over the number absorbed in causing fission.

Such independent sources as cosmic-ray neutrons are always present in a system, but usually they are negligible compared with the artificial sources present or the production by fission. Thus, if artificial sources are absent, the left side of (1.9) reduces to production by fission.

Let us now imagine the linear dimensions of the system increased in some ratio a, the chemical composition and density remaining unchanged. The production by fission will be approximately proportional to the volume occupied by fissile material, and so will change by a factor a^3. Capture will also vary as a^3, and since it is necessarily smaller than the production it will represent a constant fraction of the latter. The escape of neutrons from the system, however, will be approximately proportional to the surface area, and so will increase by a factor a^2. Thus (1.9) gives

$$\partial n/\partial t \cong n[Aa^3 - Ba^2], \quad (1.10)$$

where the factor n is introduced because production by fission, capture,

and escape are all proportional to the existing neutron population. A and B are approximately constant. The formula (1.10) shows at once that there should be a value of a, a_0 say, such that for $a < a_0$, $\partial n/\partial t$ is negative, and in the absence of independent sources the neutron density will decay exponentially (in time-dependent problems the decay may be only approximately exponential, since A and B may themselves be functions of the time, for instance if the fractions of the neutron population in different parts of the system vary with time). For $a > a_0$, however, $\partial n/\partial t$ is positive, and the neutron population will increase exponentially (or approximately so). If $a = a_0$, the system is said to be *critical*, and if a is a characteristic linear dimension a_0 is called the *critical value* of that dimension. The name arises since for $a > a_0$ the neutron population increases indefinitely, and unless the system varies with time the results will eventually be catastrophic.

The determination of the critical size is the most important problem of neutron transport theory. Occasionally one has also to deal with time-dependent problems and those where the independent neutron sources have to be taken into account.

In what follows, we shall always keep in mind the critical-size problem and the problem of the neutron distribution due to independent sources. As regards time-dependent problems, however, we shall derive the fundamental equations in a form applicable to both stationary and non-stationary problems, and we shall show (in Chapter III) how any time-dependent problem may be reduced to a stationary one; thereafter, we shall confine ourselves to stationary problems alone. A more detailed treatment of time-dependent problems is outside the scope of the work.

II

THE MATHEMATICAL FORMULATION OF THE LAWS OF NEUTRON MIGRATION

2.1. The form of the transport equation

We now turn to the mathematical formulation of the laws of neutron migration. The six assumptions (A) to (F) of § 1.1 allow us to construct the basic equation governing the diffusion and migration of neutrons. We define the following symbols:

t = time;

$\mathbf{r}$ = the position vector of a neutron;

$\mathbf{v} = v\boldsymbol{\Omega}$, its velocity vector, v being the speed, that is, the magnitude of the velocity, and $\boldsymbol{\Omega}$ a unit vector in the direction of motion;

$N\,dV\,dv\,d\Omega = N(\mathbf{r}, v\boldsymbol{\Omega}, t)\,dV\,dv\,d\Omega$, the probable number of neutrons at time t in the volume element dV around the point $\mathbf{r}$, 'belonging to $dvd\Omega$' (i.e. travelling with speed between v and $v+dv$ in a direction lying within a solid angle $d\Omega$ around the direction $\boldsymbol{\Omega}$);

$S\,dV\,dv\,d\Omega\,dt = S(\mathbf{r}, v\boldsymbol{\Omega}, t)\,dV\,dv\,d\Omega\,dt$, the probable number of neutrons belonging to $dvd\Omega$ emitted during the time t to $t+dt$ in the volume element dV by the independent neutron sources.

(By independent sources we here mean sources whose existence and strength do not depend on the neutron population of the system. Such sources include spontaneous fission, natural radioactivity, and particle accelerators used to generate neutrons. These independent sources will sometimes be referred to as *sources* simply.)

$c_t f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega};\, t')\,dv\,d\Omega\,dt'$, the probable number of neutrons belonging to $dvd\Omega$ which result from a collision undergone by a neutron with velocity $v'\boldsymbol{\Omega}'$, and which emerge during a time t' to $t'+dt'$ after the collision.

We now consider a packet of neutrons specified by the values of $v\boldsymbol{\Omega}$ and $[\mathbf{r}+tv\boldsymbol{\Omega}]$, at times close to $t = 0$. As t increases by dt, the probable number of neutrons in the packet will decrease on account of capture, and on account of scattering out of the packet; it will increase on account of contributions from the independent sources and on account of scattering into the packet. The net increase, by definition, is

$$(dN/dt)\,dV\,dv\,d\Omega\,dt. \tag{2.1}$$

The decrease due to capture and scattering out of the packet will be given directly by the probable number of neutrons undergoing collision during the time dt. This is seen as follows: a scattering collision alters either the speed or the direction of motion of the neutron, otherwise no collision can be said to have taken place; in a fission collision, on the other hand, the probability that any resulting neutron will have the same direction and speed as the original neutron is of the second order of small quantities. The decrease during the time dt is given by (E) of § 1.1 (see also § 1.3.1) as

$$\{1/l_{\text{tot}}(v)\}N\,dV\,dv\,d\Omega.v\,dt, \tag{2.2}$$

where $v\,dt$ is the distance travelled by the neutron in the time dt.

The increase due to scattering into the packet arises from collisions of other neutrons in dV. It is found as follows: the expression corresponding to (2.2) for all other speeds and directions at the time $(t-t')$ is multiplied by the probable number of neutrons, resulting from each collision, which emerge with a delay between t' and $t'+dt'$ and belong to $dvd\Omega$; this product is then integrated over all speeds and all directions of the initial neutron and over all values of the delay time. The result is

$$dV\,dv\,d\Omega\,dt\iiiint\{v'/l_{\text{tot}}(v')\}N(\mathbf{r},v'\Omega',t-t')c_{\nu}f(v'\Omega'\to v\Omega;t')\,dt'dv'd\Omega'. \tag{2.3}$$

The contribution from the independent sources is given, by definition, by

$$S\,dV\,dv\,d\Omega\,dt.$$

We also notice that

$$dN/dt=\partial N/\partial t+v\Omega.\mathbf{grad}\,N,$$

where the dot denotes the scalar product and the gradient is taken with respect to the position coordinates. Collecting these results, cancelling the differentials, and transferring the decrease term to the left side, we have finally

$$\left\{\begin{aligned}&\frac{\partial N}{\partial t}+v\Omega.\mathbf{grad}\,N+\frac{vN}{l_{\text{tot}}(v)}\\&=\int\frac{v'dv'}{l_{\text{tot}}(v')}\iiint N(\mathbf{r},v'\Omega',t-t')c_{\nu}f(v'\Omega'\to v\Omega;t')\,dt'd\Omega'+S.\end{aligned}\right. \tag{2.4}$$

This is the so-called *transport equation* or *Boltzmann equation*, which forms the basis of the entire theory of neutron transport.

In stationary problems, the quantity $c_{\nu}f(v'\Omega'\to v\Omega;t')$ will enter this equation only in the form

$$\int_0^{\infty}c_{\nu}f(v'\Omega'\to v\Omega;t')\,dt', \tag{2.5}$$

and it is convenient to have a simple notation for this integral. If (2.5) is integrated over all v and Ω we obtain the mean number of secondaries per collision, which we have previously called $c(v')$ (§ 1.3.1). We shall therefore write the integral (2.5) as $c(v')f(v'\Omega' \to v\Omega)$, where $f(v'\Omega' \to v\Omega)$ is normalized so that

$$\iiint f(v'\Omega' \to v\Omega)\, dvd\Omega = 1, \tag{2.6}$$

and $f(v'\Omega' \to v\Omega)\, dvd\Omega$ may be described as the probability that a neutron, entering a collision with velocity $v'\Omega'$, will belong to $dvd\Omega$ after the collision.

Using this notation, the Boltzmann equation for a stationary problem is written

$$v\Omega.\mathbf{grad}\, N + vN/l_{\mathrm{tot}}(v) = \int \frac{v'c(v')\, dv'}{l_{\mathrm{tot}}(v')} \iint N(\mathbf{r}, v'\Omega') f(v'\Omega' \to v\Omega)\, d\Omega' + S. \tag{2.4'}$$

2.2. Discussion of the form of the function $c_f f(v'\Omega' \to v\Omega; t')$

Before the equation (2.4) can be used, the form of the function $c_f f(v'\Omega' \to v\Omega; t')$ must be specified. This may be done by expressing mathematically the conclusions of § 1.2. Let $[c_f f(v'\Omega' \to v\Omega; t')]_f$ be the value which the function $c_f f(v'\Omega' \to v\Omega; t')$ would have if every collision resulted in fission, and let a similar notation be adopted for elastic and inelastic scattering. Since scattering is always instantaneous and the number of neutrons in scattering is conserved, we have

$$[c_f f(v'\Omega' \to v\Omega; t')]_{\mathrm{is}} = [f(v'\Omega' \to v\Omega)]_{\mathrm{is}}\, \delta(t'),$$

and a similar expression for elastic scattering; $\delta(t')$ is Dirac's delta function and $[f(v'\Omega' \to v\Omega)]_{\mathrm{is}}$ is defined similarly to $[c_f f(v'\Omega' \to v\Omega; t')]_{\mathrm{is}}$. Thus

$$\frac{1}{l_{\mathrm{tot}}(v')} c_f f(v'\Omega' \to v\Omega; t') = \frac{1}{l_f(v')}[c_f f(v'\Omega' \to v\Omega; t')]_f + \\ + \frac{\delta(t')}{l_{\mathrm{is}}(v')}[f(v'\Omega' \to v\Omega)]_{\mathrm{is}} + \sum_i \frac{\delta(t')}{l_{\mathrm{el},i}(v')}[f(v'\Omega' \to v\Omega)]_{\mathrm{el},M_i}, \tag{2.7}$$

where, according to the notation of § 1.3, $l_{\mathrm{tot}}(v')/l_f(v')$ is the probability that a collision of a neutron of speed v' will result in fission, and similarly for elastic and inelastic scattering. The contribution from elastic scattering is represented as the sum of terms arising from various kinds of nucleus; M_i is the mass number of the ith kind of nucleus and $l_{\mathrm{el},i}$ is the corresponding mean free path for elastic scattering.

Let $\nu(t')$ be the mean number of secondaries emitted in fission with a delay time not greater than t' (so that $\nu(0)$ is the number of prompt neutrons and $\nu(\infty)$ equals ν of § 1.3), and let $F_f(v, t')$ be the number of neutrons emitted with speed v and delay t', so normalized that

$$\int_0^\infty F_f(v, t')\, dv = 1.$$

Then, since fission is regarded as isotropic in the L system (see § 1.2.4),

$$[c_f f(v'\Omega' \to v\Omega; t')]_f = \frac{1}{4\pi}\left[\frac{d\nu(t')}{dt'} F_f(v, t') + \nu(0)\delta(t')F_f(v, 0)\right], \quad (2.8)$$

where the normalization factor $1/4\pi$ is introduced so that (2.8) gives $d\nu(t')/dt'$ on integration over all v and Ω. For stationary problems, where (2.4′) holds, (2.8) can be replaced by the simpler form

$$[c(v')f(v'\Omega' \to v\Omega)]_f = (\nu/4\pi)F_f(v), \quad (2.8')$$

obtained by integrating (2.8) over all t'.

Inelastic scattering is also regarded as isotropic in the L system (see § 1.2.3), so that analogously to (2.8′) we have for stationary problems

$$[f(v'\Omega' \to v\Omega)]_{is} = \frac{1}{4\pi} F_{is}(v' \to v), \quad (2.9)$$

where $F_{is}(v' \to v)$ is the number of neutrons with initial speed v' and final speed v.

In the case of elastic scattering, the L and C systems cannot always be identified, and the difference between them should be allowed for in calculating the elastic scattering contribution to (2.7). In doing this, we shall limit ourselves to non-thermal neutrons, since the elastic scattering of thermal neutrons has not yet been discussed; for simplicity we shall deal only with the case of isotropic scattering in the C system, that is when (1.2) is satisfied, since for the more general case where more than one term of (1.1) must be retained the results are too complex to quote; and for brevity we shall omit the suffix i which denotes the kind of nucleus involved.

Under these assumptions, the molecular binding can be neglected, and the problem reduces to one of two bodies, the neutron and the nucleus, of which the latter can be taken as initially at rest. Momentum is conserved, and so is kinetic energy, because in an elastic collision the nucleus is left in the ground state. These conservation conditions can be written

$$\left.\begin{aligned} v'^2 &= v^2 + Mu^2 \\ \mathbf{v}' &= \mathbf{v} + M\mathbf{u} \end{aligned}\right\}, \quad (2.10)$$

where $\mathbf{v}' = v'\Omega'$ is the velocity of the neutron before the collision, $\mathbf{v} = v\Omega$ is its velocity after the collision, $\mathbf{u}$ is the velocity of the recoiling nucleus, and M is the ratio of the nuclear mass to the neutron mass. Eliminating $\mathbf{u}$ we have

$$v^2 = v'^2 - |\mathbf{v}' - \mathbf{v}|^2/M, \tag{2.11}$$

that is,

$$\Omega.\Omega' = \frac{(M+1)v^2 - (M-1)v'^2}{2vv'}. \tag{2.12}$$

Thus, if all collisions resulted in elastic scattering by nuclei of mass M, we should have

$$[f(v'\Omega' \to v\Omega)]_{\text{el},M} = \delta\left(\Omega.\Omega' - \frac{(M+1)v^2 - (M-1)v'^2}{2vv'}\right) f(v, v'), \tag{2.13}$$

where $f(v, v')$ is a function of v and v' only.

The equation (2.11) may alternatively be written

$$v^2 = v'^2 - |\mathbf{v}'_C - \mathbf{v}_C|^2/M = v'^2 - [v'^2_C - 2v_C v'_C \cos\Theta + v^2_C]/M, \tag{2.14}$$

where $\mathbf{v}'_C$ and $\mathbf{v}_C$ are the initial and final velocities of the neutron in the C system, $v'_C = |\mathbf{v}'_C|$, $v_C = |\mathbf{v}_C|$, and Θ is the angle between $\mathbf{v}'_C$ and $\mathbf{v}_C$, sometimes called the *scattering angle*. Rewriting the conservation equations (2.10) in the C system, we have

$$v_C = v'_C = Mv'/(M+1),$$

and hence (2.14) becomes

$$v^2 = v'^2\left[1 - \frac{2M}{(M+1)^2}(1 - \cos\Theta)\right]. \tag{2.15}$$

That is, for elastic scattering of fast neutrons, the final energy in the L system is a linear function of the cosine of the scattering angle.

Hitherto the analysis has been general. We now restrict ourselves to the case where (1.2) is satisfied, i.e. elastic scattering is isotropic in the C system. If ω specifies the direction in the C system, then $d\omega = |d\phi\, d\cos\Theta|$, ϕ being the azimuthal angle, so that, if all values of ω are equally probable, it follows that all values of $\cos\Theta$ are equally probable. Then, from (2.15), we see that the probability of any permissible value of v, which is $f(v, v')$ in (2.13), is proportional to dv^2/dv, that is, to v. From (2.15) we also see that the permissible values of v lie in the range

$$\{(M-1)/(M+1)\}v' \leqslant v \leqslant v',$$

so that outside this range $f(v, v')$ is zero. The proportionality factor in

$\bar{f}(v, v')$ is found from (2.6), and on substituting the resulting $\bar{f}(v, v')$ into (2.13) we find

$$[f(v'\Omega' \to v\Omega)]_{\mathrm{el},M}$$

$$= \begin{cases} \dfrac{(M+1)^2 v}{4\pi M v'^2}\,\delta\left(\Omega.\Omega' - \dfrac{(M+1)v^2-(M-1)v'^2}{2vv'}\right) & \text{if } \dfrac{M-1}{M+1}v' \leqslant v \leqslant v', \\ 0 \quad \text{otherwise.} \end{cases} \tag{2.16}$$

Two features of this expression are to be noted. Firstly, for any given M, it takes the form

$$[f(v'\Omega' \to v\Omega)]_{\mathrm{el}\,M} = (1/v') \times \text{a function of } v/v' \text{ and } \Omega.\Omega' \text{ only.} \tag{2.16'}$$

Secondly, on integrating (2.16) over all v, and recalling that

$$\delta(x-x_0) = [dy/dx]_{x=x_0}\,\delta[y(x)-y(x_0)],$$

we have

$$\int_0^\infty [f(v'\Omega' \to v\Omega)]_{\mathrm{el},M}\,dv = \frac{1}{4\pi M}\,\frac{[\{M^2-1+(\Omega.\Omega')^2\}^{\frac{1}{2}}+\Omega.\Omega']^2}{\{M^2-1+(\Omega.\Omega')^2\}^{\frac{1}{2}}}. \tag{2.17}$$

Using (2.17), it is seen that, as $M \to \infty$, the expression (2.16) becomes

$$[f(v'\Omega' \to v\Omega)]_{\mathrm{el},\infty} = \frac{1}{4\pi}\,\delta(v-v'), \tag{2.18}$$

which corresponds to scattering that is isotropic and without energy loss in the L system.

If the anisotropy of scattering in the C system is not negligible, i.e. further terms in (1.1) have to be taken into account, the corresponding $\bar{f}(v, v')$ in (2.13) can be worked out similarly.

Finally, substituting (2.8), (2.9), and (2.16) or (2.18) (or their appropriate modifications) into (2.7) gives the function $c_{t'} f(v'\Omega' \to v\Omega; t')$ appearing in (2.4).

2.3. The boundary conditions

The equation (2.4) is an integro-differential equation, and to make the problem of its solution determinate it is necessary to specify boundary conditions. These, in fact, follow at once from the physical interpretation of $N(\mathbf{r}, v\Omega, t)$, and we shall limit ourselves at present to showing how they are formulated in some illustrative cases.

2.3.1. *The interface between two media*

If two media are in direct contact, that is without any other matter interposed between them, then any packet of neutrons characterized by the vectors $v\Omega$ and $\mathbf{r}+tv\Omega$ will contain exactly the same number of neutrons when it enters one medium as when it left the other. This can be expressed mathematically thus:

$N(\mathbf{r}+R\Omega, v\Omega, t+R/v)$ is a continuous function of R for $\mathbf{r}+R\Omega$ at the interface. (2.19)

If, however, there is a third medium interposed, then the effects of crossing it may have to be taken into account, and then the condition (2.19) must be modified.

It is to be noticed that the condition (2.19) places a condition only on what happens as the interface is crossed following the direction of the neutron packet. That is,

$$N(\mathbf{r}+R\Omega, v\Omega', t+R/v)$$

is stipulated to be continuous in R only if $\Omega = \Omega'$. Though continuity will also usually be present even if $\Omega \neq \Omega'$, this is not stipulated, and to do so might in some cases result in the problem being over-determined.

2.3.2. *The free surface of a medium*

The term *free surface* is used in neutron transport theory to denote a surface or part of one on which no neutrons fall from outside. For instance, if an isolated system has a convex (or, more generally, non-re-entrant) surface, this will be a free surface. (By a non-re-entrant surface is meant one such that a straight line emerging from it cannot meet it again.) If, however, the system has a re-entrant surface, those parts of the surface that receive neutrons emerging from other parts of the surface are not considered as a free surface. This definition implies that

$N(\mathbf{r}, v\Omega, t) = 0$ for all Ω entering the system and $\mathbf{r}$ on the surface. (2.20)

2.3.3. *The condition at infinity*

No physical system can extend to infinity, and the concept of such a system is a mathematical idealization. The condition at infinity will therefore always depend on the actual situation which is to be approximated. It should be noticed, however, that a remote neutron source can justifiably be regarded as 'at infinity' only if the number of neutrons coming from this source directly (i.e. without collision) to the region

under consideration is negligibly small. Accordingly, the condition at infinity should always imply that *the number of neutrons coming directly from infinity is zero.*

2.3.4. *The initial conditions*

In dealing with a time-dependent problem it is necessary to specify not only boundary conditions in space, but also initial conditions. If delayed neutrons are neglected, i.e. if (2.8) reduces to

$$[c_r f(v'\Omega' \to v\Omega; t')]_f = (\nu/4\pi)F_f(v)\delta(t'),$$

then it is sufficient to specify $N(\mathbf{r}, v\Omega, t)$ at the initial time $t = t_0$, say. If, however, delayed neutrons are to be taken into account, then the presence of the integral

$$\int_0^\infty N(\mathbf{r}, v'\Omega', t-t')F_f(v, t')\frac{d\nu(t')}{dt'}\,dt' \tag{2.21}$$

makes it necessary to give N for all $t \leqslant t_0$. This necessity can be avoided, but only by introducing a more general treatment, which lies outside the scope of this book. This would consist in the use of functions R_j, say, which characterize the distribution of fission fragments capable of emitting delayed neutrons. The equation (2.4) is supplemented by equations governing the R_j, and the integral (2.21) is expressed in terms of the R_j.

2.4. The integral equation

The integro-differential equation (2.4), with the boundary conditions (2.19) and (2.20) and appropriate initial conditions, can easily be transformed into an integral equation. Inasmuch as a time-dependent problem can always be reduced to a stationary one (see below), we shall show this transformation for stationary problems, i.e. starting from equation (2.4').

We shall first carry out the derivation of the integral equation for the simple case when (i) the system consists of a single non-re-entrant homogeneous body; (ii) the sources, if any, are isotropic, i.e.

$$S(\mathbf{r}, v\Omega) = S(\mathbf{r}, v); \tag{2.22}$$

(iii) $f(v'\Omega' \to v\Omega)$ is assumed to be independent of Ω' and Ω, i.e. to have the form

$$f(v'\Omega' \to v\Omega) = f(v' \to v)/4\pi, \tag{2.23}$$

where the factor $1/4\pi$ is inserted so that the normalization of $f(v' \to v)$ by

$$\int_0^\infty f(v' \to v)\,dv = 1 \tag{2.24}$$

is consistent with (2.6). The last assumption implies that all $g_n(E)$ $(n \geqslant 1)$ in (1.1) are neglected, and that for heavy nuclei (2.16) is replaced by (2.18) or by

$$[f(v'\Omega' \to v\Omega)]_{\mathrm{el},M} \cong \begin{cases} (M+1)^2 v/8\pi M v'^2 & \text{if } \dfrac{M-1}{M+1}v' \leqslant v \leqslant v', \\ 0 \quad \text{otherwise,} \end{cases} \tag{2.25}$$

say, while light nuclei are assumed to be altogether absent.

In dealing with this case, it will be convenient to introduce the notation

$$n(\mathbf{r}, v) = \iint N(\mathbf{r}, v\Omega)\, d\Omega. \tag{2.26}$$

In order to understand the significance of the factors in the kernel of the integral equation, we shall first derive it from first principles and then establish its equivalence to the Boltzmann equation.

From (2.26) and the definition of N it follows that $n(\mathbf{r}, v)\,dV dv$ represents the probable number of neutrons, regardless of direction, present at a given time in the volume dV about $\mathbf{r}$ and having speeds between v and $v+dv$. This number should be equal to the integral over the entire volume of the system of the contributions to the density arising from other volume elements dV'. The contribution from the volume element dV' about $\mathbf{r}'$ is, however, given by the product of the following four factors:

(1) the probable number of neutrons with speed between v and $v+dv$ produced in dV' per unit time (where 'production' includes those resulting from scattering);
(2) the probability that a neutron produced in dV' will have a direction that passes through dV;
(3) the probability that it will escape collision between dV' and dV;
(4) the length of time during which the neutron can have been produced in dV' so as to be in dV at the time when $n(\mathbf{r}, v)\,dV dv$ is estimated.

The first factor, from the same arguments as were used to derive (2.4), is

$$dV' dv \int n(\mathbf{r}', v')\{v' c(v')/l_{\mathrm{tot}}(v')\} f(v' \to v)\, dv' + 4\pi S(\mathbf{r}', v)\, dV' dv, \tag{2.27}$$

where the factor 4π in the second term arises because $S(\mathbf{r}', v)\,dV' dv$ is the source strength per unit solid angle; cf. (2.22) and the definition of S in § 2.1.

The second factor is found by expressing dV in the form $ds.dA$, where ds is measured in the direction from dV' to dV, and dA lies in a plane perpendicular to this direction. Then, since by (2.22) and (2.23) all directions are equally probable for a neutron produced in dV', the probability that it will pass through dV is given by the solid angle subtended at dV' by dA, that is

$$dA/4\pi|\mathbf{r}-\mathbf{r}'|^2. \tag{2.28}$$

The third factor follows from assumption (E) of § 1.1, provided that the path from dV' to dV lies entirely within the medium (which is necessarily true for non-re-entrant media) and the medium is homogeneous. This factor is

$$\exp[-|\mathbf{r}-\mathbf{r}'|/l_{\text{tot}}(v)]. \tag{2.29}$$

The fourth factor is clearly equal to the time spent by the neutron in dV, and this is ds/v, where $dV = ds.dA$. Combining these results and cancelling the differentials, we have

$$n(\mathbf{r},v) = \frac{1}{4\pi v}\iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|}{l_{\text{tot}}(v)}\right]\times$$
$$\times\left\{\int n(\mathbf{r}',v')\frac{v'c(v')}{l_{\text{tot}}(v')}f(v'\to v)\,dv'+4\pi S(\mathbf{r}',v)\right\}. \tag{2.30}$$

It is sometimes convenient to rewrite (2.30) as follows: v' is replaced by v'' and v by v', and the equation is multiplied by

$$v'c(v')f(v'\to v)/4\pi l_{\text{tot}}(v')$$

and integrated over all v', putting

$$Q(\mathbf{r},v) = \frac{1}{4\pi}\int dv'\frac{v'c(v')}{l_{\text{tot}}(v')}n(\mathbf{r},v')f(v'\to v)+S(\mathbf{r},v). \tag{2.31}$$

Equation (2.30) then becomes

$$Q(\mathbf{r},v) = \frac{1}{4\pi}\iiint\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\int dv'\frac{c(v')}{l_{\text{tot}}(v')}\times$$
$$\times Q(\mathbf{r}',v')f(v'\to v)\exp\left[-\frac{|\mathbf{r}-\mathbf{r}'|}{l_{\text{tot}}(v')}\right]+S(\mathbf{r},v). \tag{2.32}$$

The quantity $Q(\mathbf{r},v)$ is sometimes called the *emission density* of neutrons; it is the number of neutrons either scattered into or generated in $dvdV$ per unit time.

2.5. The equivalence of the Boltzmann and integral equations

We shall now show the equivalence of the Boltzmann and integral equations by deriving the latter, not from first principles, but from the Boltzmann equation (2.4′) with the boundary condition (2.20).

We first notice that, in the case considered in § 2.4, i.e. when (2.22) and (2.23) hold, the right side of (2.4′) is identical with (2.31), so that in this case the Boltzmann equation can be rewritten

$$v\Omega.\mathbf{grad}\,N(\mathbf{r},v\Omega)+vN(\mathbf{r},v\Omega)/l_{\text{tot}}(v)=Q(\mathbf{r},v). \tag{2.33}$$

The first step is to express $N(\mathbf{r},v\Omega)$ in terms of $Q(\mathbf{r},v)$. Since $\Omega.\mathbf{grad}$ is simply the derivative taken along the direction of Ω, the equation (2.33), $\mathbf{r}$ being replaced by $\mathbf{r}-R\Omega$, can be rewritten

$$-\frac{d}{dR}N(\mathbf{r}-R\Omega,v\Omega)+\frac{1}{l_{\text{tot}}(v)}N(\mathbf{r}-R\Omega,v\Omega)=\frac{1}{v}Q(\mathbf{r}-R\Omega,v). \tag{2.34}$$

If $Q(\mathbf{r}-R\Omega,v)$ is supposed known, (2.34) can easily be solved for $N(\mathbf{r}-R\Omega,v\Omega)$:

$$N(\mathbf{r}-R\Omega,v\Omega)=N(\mathbf{r}-R_0\Omega,v\Omega)e^{(R-R_0)/l_{\text{tot}}(v)}+$$
$$+\frac{1}{v}\int_R^{R_0}Q(\mathbf{r}-R'\Omega,v)e^{(R-R')/l_{\text{tot}}(v)}\,dR'. \tag{2.35}$$

We now put $R=0$ and notice that, if the line $\mathbf{r}-R_0\Omega$ (where $\mathbf{r}$ and Ω are fixed and R_0 is variable) cuts the surface of the system for some $R_0>0$, then, for this value of R_0, $N(\mathbf{r}-R_0\Omega,v\Omega)$ vanishes by (2.20). In this case (2.35) becomes

$$N(\mathbf{r},v\Omega)=\frac{1}{v}\int_0^{R_0}Q(\mathbf{r}-R'\Omega,v)e^{-R'/l_{\text{tot}}(v)}\,dR', \tag{2.36}$$

where R_0 has the value specified above. Since $Q(\mathbf{r}',v)$ is clearly zero in the vacuum outside the system (where the mean free path is infinite), the upper limit of integration in (2.36) can be replaced by infinity. Changing the dummy variable we have

$$N(\mathbf{r},v\Omega)=\frac{1}{v}\int_0^{\infty}Q(\mathbf{r}-R\Omega,v)e^{-R/l_{\text{tot}}(v)}\,dR. \tag{2.37}$$

If, on the other hand, the medium extends to infinity and $\mathbf{r}-R_0\Omega$ does not reach the surface for any positive R_0, then instead of (2.20) the condition at infinity must be used (see § 2.3.3). This condition implies that

$$\lim_{R_0\to\infty}N(\mathbf{r}-R_0\Omega,v\Omega)e^{-R_0/l_{\text{tot}}(v)}=0,$$

since otherwise the number of neutrons coming directly from infinity would not be zero. Taking R_0 infinite, the formula (2.35) again reduces to (2.37).

If we now integrate (2.37) over all Ω, put $\mathbf{r}-R\Omega=\mathbf{r}'$ and notice that $dRd\Omega = dV'/R^2 = dV'/|\mathbf{r}-\mathbf{r}'|^2$, we find

$$n(\mathbf{r},v)=\frac{1}{v}\iiint\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}Q(\mathbf{r}',v)e^{-|\mathbf{r}-\mathbf{r}'|/l_{tot}(v)}. \qquad (2.38)$$

Combining this with (2.31) we have (2.30), as is to be expected, on eliminating $Q(\mathbf{r},v)$, or (2.32) on eliminating $n(\mathbf{r},v)$.

2.6. The case of re-entrant bodies and inhomogeneous media

In the case considered in § 2.4 it was simpler to derive the integral equation from first principles; in more general cases, however, it is easier to derive it from the Boltzmann equation, following the procedure of § 2.5. In the present section we shall still retain the assumptions (2.22) and (2.23), i.e. the independent sources and the scattering law are regarded as isotropic in the L system. The scattering medium, however, need no longer be a single homogeneous non-re-entrant body, and when this restriction has been removed the mean free path and the number of secondaries may depend on the position as well as on the velocity. This is the case even when the system is single and homogeneous but re-entrant, since the Boltzmann equation must then be applied not only to the points inside the body, but to those outside the body lying on any straight line joining two points inside the body. At such points the mean free path is infinite, i.e. different from that in the medium itself.

If now the mean free path and number of secondaries per collision depend on position, then (2.31) and (2.34) are replaced by

$$Q(\mathbf{r},v)=\frac{1}{4\pi}\int dv'\frac{v'c(\mathbf{r},v')}{l_{tot}(\mathbf{r},v')}n(\mathbf{r},v')f_r(v'\to v)+S(\mathbf{r},v) \qquad (2.31')$$

and

$$-\frac{d}{dR}N(\mathbf{r}-R\Omega,v\Omega)+\frac{N(\mathbf{r}-R\Omega,v\Omega)}{l_{tot}(\mathbf{r}-R\Omega,v)}=\frac{1}{v}Q(\mathbf{r}-R\Omega,v), \qquad (2.34')$$

and the solution of (2.34') is

$$N(\mathbf{r}-R\Omega,v\Omega)=N(\mathbf{r}-R_0\Omega,v\Omega)\exp\left[\int_{R_0}^{R}\frac{dR''}{l_{tot}(\mathbf{r}-R''\Omega,v)}\right]+$$

$$+\frac{1}{v}\int_{R}^{R_0}Q(\mathbf{r}-R'\Omega,v)\exp\left[\int_{R'}^{R}\frac{dR''}{l_{tot}(\mathbf{r}-R''\Omega,v)}\right]dR'. \qquad (2.35')$$

The last expression can be written in a simpler form using the notation

$$\tau_v(\mathbf{r}-R\Omega,\mathbf{r}-R'\Omega)=\left|\int_{R}^{R'}\frac{dR''}{l_{tot}(\mathbf{r}-R''\Omega,v)}\right|. \qquad (2.39)$$

This quantity is called† the *optical distance* between $\mathbf{r}-R\Omega$ and $\mathbf{r}-R'\Omega$. It may be described as the geometrical distance between the points, multiplied by the inverse mean free path averaged over the line segment between them.

Expressing the integrals in the exponents in (2.35′) in terms of optical distance, and proceeding then as in § 2.5, we find instead of (2.38):

$$n(\mathbf{r},v)=\frac{1}{v}\iiint\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\,Q(\mathbf{r}',v)e^{-\tau_v(\mathbf{r},\mathbf{r}')}. \tag{2.38′}$$

Combining this with (2.31′) and eliminating either $Q(\mathbf{r},v)$ or $n(\mathbf{r},v)$, we obtain the integral equation for $n(\mathbf{r},v)$ or $Q(\mathbf{r},v)$ respectively:

$$n(\mathbf{r},v)=\frac{1}{4\pi v}\iiint\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}e^{-\tau_v(\mathbf{r},\mathbf{r}')}\times$$
$$\times\left\{\int n(\mathbf{r}',v')\frac{v'c(\mathbf{r}',v')}{l_{\mathrm{tot}}(\mathbf{r}',v')}f_{\mathbf{r}'}(v'\to v)\,dv'+4\pi S(\mathbf{r}',v)\right\}, \tag{2.30′}$$

$$Q(\mathbf{r},v)=\frac{1}{4\pi}\iiint\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\int dv'\,\frac{c(\mathbf{r},v')}{l_{\mathrm{tot}}(\mathbf{r},v')}Q(\mathbf{r}',v')f_{\mathbf{r}}(v'\to v)e^{-\tau_v(\mathbf{r},\mathbf{r}')}+$$
$$+S(\mathbf{r},v). \tag{2.32′}$$

2.7. The integral equation in the general case

If the assumption (2.23) is not satisfied, for instance if the system contains light nuclei such as hydrogen, the Boltzmann equation can still be reduced to the integral equation as before, but with the difference that the right side of (2.4′) (i.e. what we have called the neutron emission density) will depend on Ω. In the most general case, therefore, the natural unknown in the integral equation is not the quantity $n(\mathbf{r},v)$, but rather $N(\mathbf{r},v\Omega)$ itself. The equation will then take the form

$$N(\mathbf{r},v\Omega)=\iiint dV'\iiint dv'd\Omega'N(\mathbf{r}',v'\Omega')K(\mathbf{r}',v'\Omega'\to\mathbf{r},v\Omega)+$$
$$+\text{independent source terms}; \tag{2.40}$$

but the actual form of the kernel $K(\mathbf{r}',v'\Omega'\to\mathbf{r},v\Omega)$ will not concern us for the time being. We shall also leave for the moment the cases (where $f(v'\Omega'\to v\Omega)$ is a low-order polynomial in $\Omega'.\Omega$) for which (2.40) can be reduced to an equation for $n(\mathbf{r},v)$. We can also introduce, similarly to (2.31), the *neutron emission angular distribution* $Q(\mathbf{r},v\Omega)$, and derive for it an equation similar to (2.40).

† From the use of a similar quantity in optics.

III

STATIONARY AND TIME-DEPENDENT PROBLEMS. THE ADJOINT EQUATION

3.1. Stationary problems and critical-size problems

We shall be concerned chiefly with stationary problems. These will be divided into two major classes. In one, the geometrical dimensions and chemical composition of the system are given and it is required to determine the distribution of neutrons due to given sources. In the other, it is required to find the geometrical dimension and/or chemical composition for which a stationary distribution of neutrons is possible in the absence of independent sources. The latter case leads to an eigenvalue problem with a homogeneous equation.

The chemical composition of a system is characterized by the values of the various N_i, the number of atoms of the ith kind per unit volume, in different parts of the system. The various inverse mean free paths are linear functions of the N_i (see § 1.3.1), and the coefficients and the kernel in equation (2.4) are linear functions of the inverse mean free paths. Thus we have a homogeneous equation linear with respect to the parameter of chemical composition, and the general properties of such equations may be utilized; we shall mention some of these later.

In some cases, certain changes in the chemical composition will result in replacing the homogeneous equation corresponding to (2.4′) by

$$v\Omega.\mathbf{grad}\,N+vN/l_{\mathrm{tot}}(v)=\gamma\times[\text{right side of (2.4′) without } S], \tag{3.1}$$

where γ expresses the degree of change in the chemical composition. In general this will not be so, but (3.1) may still be considered as a mathematical device, and we shall refer to its eigenvalues γ_n and eigenfunctions $N_{c,n}$ as eigenvalues and eigenfunctions *under varying chemical composition*.

It is a well-known property of homogeneous equations that only one of the possible eigenfunctions can be everywhere positive. This will be proved in § 3.3. Since on physical grounds $N(\mathbf{r}, v\Omega)$ cannot be negative,† it follows that only one eigenvalue of (3.1) will be of physical interest.

† It should be remarked that, although $N(\mathbf{r}, v\Omega)$ itself cannot be negative, it may appear so in two cases: (i) if the analytical expression is extrapolated beyond the region where it is valid into one where it has no physical significance, (ii) if N is expressed as the sum of two or more terms, in which case the separate terms may become negative, though their sum is always positive. In such cases we may speak of 'a negative number of neutrons.'

This is called the *critical value*, as explained in § 1.4. If, on the other hand, we vary the size of the system instead of its chemical composition, the smallest size for which (3.1) with $\gamma = 1$ has a solution is called the *critical size*.

3.2. Time-dependent problems and their reduction to stationary problems

Time-dependent problems can always be at least formally reduced to stationary ones, assuming of course that the properties of the medium do not change with time. The most general method is to use a Laplace transform. To determine the solution of (2.4) for $t > 0$, we take $N(\mathbf{r}, v\boldsymbol{\Omega}, t)$ as zero for $t < 0$ (delayed neutrons from fissions occurring before $t = 0$ are included in the source term S), and perform a Laplace transformation with respect to t. Using (2.7) and (2.8) we find:

$$v\boldsymbol{\Omega}.\mathbf{grad}\,\bar{N} + \left[\lambda + \frac{v}{l_{\mathrm{tot}}(v)}\right]\bar{N}$$

$$= \iiint v'dv'd\Omega'\bar{N}(\mathbf{r}, v'\boldsymbol{\Omega}', \lambda)\left[\frac{\bar{F}_f(v,\lambda)\bar{\nu}(\lambda)}{4\pi l_f(v')} + \frac{1}{l_s(v')}\{f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})\}_s\right] + \bar{S}, \tag{3.2}$$

where

$$\bar{N} = \bar{N}(\mathbf{r}, v\boldsymbol{\Omega}, \lambda) = \int_0^\infty N(\mathbf{r}, v\boldsymbol{\Omega}, t)e^{-\lambda t}\,dt, \tag{3.3}$$

$$\bar{S} = N(\mathbf{r}, v\boldsymbol{\Omega}, 0) + \int_0^\infty S(\mathbf{r}, v\boldsymbol{\Omega}, t)e^{-\lambda t}\,dt,$$

$$\bar{\nu}(\lambda) = \lambda\int_0^\infty \nu(t')e^{-\lambda t'}\,dt', \tag{3.4}$$

and $$\bar{F}_f(v,\lambda)\bar{\nu}(\lambda) = \nu(0)F_f(v,0) + \int_0^\infty F_f(v,t')\frac{d\nu(t')}{dt'}e^{-\lambda t'}\,dt'$$

(so that $\bar{F}_f(v,\lambda)$ satisfies the normalization condition

$$\int_0^\infty \bar{F}_f(v,\lambda)\,dv = 1\Big).$$

For simplicity, the terms due to elastic and inelastic scattering have been combined.

Equation (3.2) is of exactly the same form as (2.4'), and can be regarded as the equation governing the stationary neutron distribution in a certain fictitious medium. In this fictitious medium, the inverse total mean free path is increased by λ/v, compared with its value in the actual

medium; the mean number of secondaries per fission is changed from $\nu(\infty)$ to $\bar{\nu}(\lambda)$, with the appropriate change in the fission spectrum, while all other constants are left unaltered. Since the scattering and fission mean free paths remain unchanged, the increase in the inverse total mean free path means an increase in the inverse capture mean free path of λ/v, and this term is therefore sometimes referred to as *fictitious capture*. Further, since $\nu(t')$ increases monotonically with t', it follows from (3.4) that $\bar{\nu}(\lambda)$ for $\lambda > 0$ decreases monotonically with increasing λ, and $\bar{\nu}(\infty) = \nu(0)$, $\bar{\nu}(0) = \nu(\infty)$. That is, the number of secondaries per fission in the fictitious medium is rather less than in the actual medium, and decreases with increasing λ.

If $\bar{S}$ vanishes, the equation again becomes homogeneous, i.e. an eigenvalue problem, whose parameter is now λ. If also the system does not extend to infinity, then, by a well-known property of eigenvalue problems for finite domains, the eigenvalues λ_n will form a discrete set. It follows from our discussion of (3.2) that the eigenvalues λ_n cannot be arbitrarily large, real, and positive. It can also be shown (though the proof is difficult) that the real parts of complex eigenvalues (if any exist) must be smaller than the largest real eigenvalue. Arranging the eigenvalues in order of decreasing real part and calling them λ_0, λ_1, λ_2, etc., we thus see that the most general solution of the homogeneous equation corresponding to (3.2) is

$$\bar{N}(\mathbf{r}, v\boldsymbol{\Omega}, \lambda) = \sum_{n=0}^{\infty} a_n N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega})\delta(\lambda - \lambda_n), \tag{3.5}$$

where the a_n are some constants, and hence the solution of equation (2.4) in this case is

$$N(\mathbf{r}, v\boldsymbol{\Omega}, t) = \sum_{n=0}^{\infty} a_n N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega})e^{\lambda_n t}. \tag{3.6}$$

If any of the eigenvalues is degenerate, i.e. corresponds to more than one eigenfunction, then we include more than one term in (3.5) correspondingly. It may be shown that λ_0 cannot be degenerate, except in very unusual circumstances.†

Since, however, λ_0 is algebraically greater than any other λ_n, it follows that for sufficiently large t all terms in (3.6) after the first will be negligibly small compared with the first term, and the solution (3.6) reduces to

$$N(\mathbf{r}, v\boldsymbol{\Omega}, t) \cong a_0 N_{t,0}(\mathbf{r}, v\boldsymbol{\Omega})e^{\lambda_0 t}. \tag{3.7}$$

Accordingly, our aim is usually to determine only λ_0 and $N_{t,0}$ rather than the complete set of eigenvalues and eigenfunctions. Since $N(\mathbf{r}, v\boldsymbol{\Omega}, t)$

† Such circumstances would arise, for example, in the case of a system divided by a black slab, where the parts of the system on each side of the slab are isolated from each other and no neutrons can pass between them.

must be non-negative everywhere, the same is true of $N_{t,0}(\mathbf{r}, v\boldsymbol{\Omega})$, from (3.7). This can also be proved without appeal to physical considerations, by showing that, if $N_{t,0}(\mathbf{r}, v\boldsymbol{\Omega})$ changes sign, there must be an eigenfunction corresponding to an eigenvalue larger than λ_0. This contradicts the supposition that λ_0 is the largest eigenvalue.

The value of λ_0 is called the *time constant* of the system. If it is negative, then (3.7) shows that the number of the neutrons in the system will decrease exponentially with time; if it is positive, this number will increase exponentially. Since $\lambda_0 = 0$ is the largest value for which this increase does not take place, it is called the *critical* value. Systems for which λ_0 is negative are called *subcritical*, and those for which it is positive are called *supercritical*. The effects of the sources, i.e. the solution of the inhomogeneous equation, will normally be of interest only for subcritical systems.

This also shows (cf. § 3.1) that only the eigenvalue λ_0 is of interest in stationary problems. For, if the dimensions or the chemical composition have been chosen so that a stationary solution is possible but this solution changes sign as $\mathbf{r}$ and $v\boldsymbol{\Omega}$ vary, then in the time-dependent case it is one of the eigenvalues other than λ_0 which vanishes, and λ_0 is positive. Thus, though a stationary distribution is possible, so also is an exponentially increasing one, and it is the latter which would be encountered if such a system were constructed.

3.3. The adjoint equation and an orthogonality relation

We now turn to the inhomogeneous equation, i.e. to the case where $\bar{S}$ in (3.2) does not vanish. It is often found advantageous to expand the solution of an inhomogeneous equation in terms of the eigenfunctions of the corresponding homogeneous equation, and we shall employ this approach in the present case. We shall, however, neglect the delayed neutrons, i.e. $\bar{F}_f(v, \lambda)\bar{\nu}(\lambda)$ is assumed independent of λ, and the equation (3.2) simplifies to

$$v\boldsymbol{\Omega}.\mathbf{grad}\,\bar{N}+[\lambda+\{v/l_{\mathrm{tot}}(v)\}]\bar{N} = \iiint \{v'c(v')/l_{\mathrm{tot}}(v')\}f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})\bar{N}(\mathbf{r}, v'\boldsymbol{\Omega}', \lambda)\,dv'd\Omega'+\bar{S}, \quad (3.8)$$

where $f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})$ is given by the same expression as in (2.4′).

The equation (3.8) is now linear in the parameter λ; the corresponding homogeneous equation is

$$v\boldsymbol{\Omega}.\mathbf{grad}\,\bar{N}+[\lambda+\{v/l_{\mathrm{tot}}(v)\}]\bar{N} = \iiint \{v'c(v')/l_{\mathrm{tot}}(v')\}f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega})\bar{N}(\mathbf{r}, v'\boldsymbol{\Omega}', \lambda)\,dv'd\Omega'. \quad (3.8')$$

Let $\lambda_0, \lambda_1, \lambda_2$, etc., be as before the eigenvalues of (3.8′) and $N_{i,0}, N_{i,1}$, etc., the corresponding eigenfunctions. We consider simultaneously with (3.8) and (3.8′) the equation

$$-v\Omega.\operatorname{grad}\bar{N}^\dagger+\left[\lambda+\frac{v}{l_{\text{tot}}(v)}\right]\bar{N}^\dagger = \{vc(v)/l_{\text{tot}}(v)\}\iiint f(v\Omega\to v'\Omega')\bar{N}^\dagger(\mathbf{r},v'\Omega',\lambda)\,dv'd\Omega', \quad (3.9)$$

with the free surface boundary condition

$$\bar{N}^\dagger(\mathbf{r},v\Omega,\lambda)=0 \quad \text{for all } \Omega \text{ leaving the system and } \mathbf{r} \text{ on the surface} \quad (3.10)$$

(that is, in the directions opposite to those appearing in (2.20); the boundary conditions at an interface, however, are still given by (2.19)).

The equation (3.9) is called the *adjoint equation* to (3.8′). We shall give later the physical interpretation of its solution in certain cases, though we shall not discuss the difficult problem of the general significance of that solution.

Let $\lambda_0^\dagger, \lambda_1^\dagger$, etc., be the eigenvalues of (3.9), and let $N_{i,0}^\dagger, N_{i,1}^\dagger$, etc., be the corresponding eigenfunctions. The following orthogonality relation can then easily be proved:

$$(\lambda_i-\lambda_j^\dagger)\iiint dV\iiint dvd\Omega\, N_{i,i}(\mathbf{r},v\Omega)N_{i,j}^\dagger(\mathbf{r},v\Omega)=0, \quad (3.11)$$

where the volume integral is extended over some arbitrary convex body large enough to include the entire system.‡

‡ From (3.11) follows the result of § 3.1 that at most one eigenfunction can be everywhere positive. The proof of this, however, is very long, and we shall do no more than sketch its outlines.

The first step is to show that, if $N(\mathbf{r}, v\Omega, 0)$ is non-negative and the delayed neutrons are neglected, then $N(\mathbf{r}, v\Omega, t)$ is non-negative for all $t > 0$. This is proved by writing out the time-dependent form (2.40′) of (2.40), with $N(\mathbf{r}, v\Omega, 0)$ in the source term, and solving the resulting Volterra equation in a unique Neumann series, whose sum is non-negative because the kernel and free term of the equation are non-negative. From this result we deduce (i) that, since the third iterated kernel of (2.40′) is bounded, $N(\mathbf{r}, v\Omega, t)$ cannot increase faster than $e^{\lambda' t}$ with λ' depending on the kernel only, and therefore there is a largest λ_0, which we call λ_0 (this was proved physically in § 3.2); (ii) that the eigenfunction corresponding to λ_0 is non-negative, since (3.6) for large t becomes (3.7), whose left side is non-negative, as shown above.

The second step in the proof is to show that there is a largest eigenvalue of the adjoint equation, and that the corresponding eigenfunction is non-negative. This is proved by the same means as the first step and its corollaries, considering (instead of $N(\mathbf{r}, v\Omega, t)$) the solution $N^*(\mathbf{r}, v\Omega, t)$ of the Boltzmann equation with $f(v'\Omega'\to v\Omega)$ replaced by $f(-v\Omega\to -v'\Omega')$ and the same boundary conditions. Since $N_{i,0}^*(\mathbf{r}, v\Omega) = N_{i,0}(\mathbf{r}, -v\Omega)$ and $\lambda_0^* = \lambda_0^\dagger$, the second step is proved. Using this result, we deduce from (3.11) and the first step above that $\lambda_0^\dagger = \lambda_0$ and that $N_{i,i}$ cannot be non-negative unless $\lambda_i = \lambda_0$, and similarly that $N_{i,j}^\dagger$ cannot be non-negative unless $\lambda^\dagger = \lambda_0$.

The relation (3.11) can be proved in the usual manner, that is, by multiplying the equation defining $N_{i,i}(\mathbf{r}, v\boldsymbol{\Omega})$ by $N^{\dagger}_{i,j}(\mathbf{r}, v\boldsymbol{\Omega})$, subtracting the same thing with i and j interchanged, and integrating over all $\mathbf{r}$, v, and $\boldsymbol{\Omega}$. On the right side we obtain the difference of two expressions which differ only in the order of integration and in the naming of the variables. Since, however, the integration is over a finite region, the order of integration may be interchanged, and so the right side is zero. On the left side we obtain two terms, the left side of (3.11) and the integral

$$\iiint v\, dv d\Omega \iiint dV\{N^{\dagger}_{i,j}\boldsymbol{\Omega}.\mathbf{grad}\, N_{i,i}+N_{i,i}\boldsymbol{\Omega}.\mathbf{grad}\, N^{\dagger}_{i,j}\}. \quad (3.12)$$

The inner integral in (3.12) clearly reduces to

$$\iint \theta(\boldsymbol{\Omega}, \mathbf{r}) N_{i,i}(\mathbf{r}, v\boldsymbol{\Omega}) N^{\dagger}_{i,j}(\mathbf{r}, v\boldsymbol{\Omega})\, dS, \quad (3.13)$$

where S is the surface of the region of integration and $\theta(\boldsymbol{\Omega}, \mathbf{r})$ is some function of $\boldsymbol{\Omega}$ and $\mathbf{r}$. If at the point $\mathbf{r}$ on S the vector $\boldsymbol{\Omega}$ is pointing inwards, then $N_{i,i}(\mathbf{r}, v\boldsymbol{\Omega})$ vanishes by (2.20); if outwards, then $N^{\dagger}_{i,j}(\mathbf{r}, v\boldsymbol{\Omega})$ vanishes by (3.10). In either case the product $N.N^{\dagger}$ vanishes, so that (3.13) and consequently (3.12) also vanish. This proves (3.11).

It follows from this that, if the $N_{i,i}(\mathbf{r}, v\boldsymbol{\Omega})$ form a *complete* set, then all the eigenvalues $\lambda^{\dagger}_{j}$ of (3.9) are among those of (3.8′), the λ_i; conversely, if the $N^{\dagger}_{i,j}(\mathbf{r}, v\boldsymbol{\Omega})$ form a complete set, then all the λ_i are among the $\lambda^{\dagger}_{j}$. For, if the $N_{i,i}$ form a complete set, and there is a $\lambda^{\dagger}_{j}$, say, which is not among the λ_i, then the corresponding eigenfunction $N^{\dagger}_{i,j}$ would be, by (3.11), orthogonal to every function of a complete set, and therefore would be identically zero, which is contrary to definition.

It has not yet been proved that the $N_{i,i}$ or the $N^{\dagger}_{i,j}$ form a complete set. It is, however, natural to expect that this is so, by analogy with other problems of mathematical physics, and we shall assume this result henceforward. It then follows that the eigenvalues of (3.9) are identical with those of (3.8′). This will be confirmed in cases where the $N^{\dagger}_{i,j}$ have a physical interpretation. See also Appendix A.

3.4. The eigenfunction expansion of the solution of the inhomogeneous equation

We now proceed to the solution of the inhomogeneous time-dependent equation (2.4), which, since delayed neutrons are neglected, will have the same form as (2.4′), but with the additional term $\partial N/\partial t$. This form of the equation we shall call (2.4″). Assuming that the set $N_{i,n}(\mathbf{r}, v\boldsymbol{\Omega})$ is complete, the solution can be represented in the form

$$N(\mathbf{r}, v\boldsymbol{\Omega}, t) = \sum_{n} A_n(t) N_{i,n}(\mathbf{r}, v\boldsymbol{\Omega}), \quad (3.14)$$

where $N_{t,n}$ are the eigenfunctions of (3.8′), and $A_n(t)$ are some functions of the time. Using the orthogonality relation (3.11), and assuming that $N_{t,i}$ and $N^\dagger_{t,j}$ are so normalized that for $i = j$ the integral in (3.11) is unity, we find from (3.14)

$$A_n(t) = \iiint dV \iiint dv d\Omega\, N^\dagger_{t,n}(\mathbf{r}, v\Omega) N(\mathbf{r}, v\Omega, t). \tag{3.15}$$

The equation (2.4″) to be solved is now multiplied by $N^\dagger_{t,n}(\mathbf{r}, v\Omega)$, and from the product is subtracted the equation (3.9) defining $N^\dagger_{t,n}(\mathbf{r}, v\Omega)$, multiplied by $N(\mathbf{r}, v\Omega, t)$. The difference is then integrated over all v, Ω, and $\mathbf{r}$. Using (3.15) and defining

$$s_n(t) = \iiint dV \iiint dv d\Omega\, N^\dagger_{t,n}(\mathbf{r}, v\Omega) S(\mathbf{r}, v\Omega, t), \tag{3.16}$$

we easily find

$$dA_n(t)/dt - \lambda_n A_n(t) = s_n(t), \tag{3.17}$$

all other terms vanishing as in the proof of (3.11); the equation (3.17) can be solved immediately.

This procedure can, of course, be applied also to stationary problems. We give an example, whose result will be of use later. Let there be a subcritical system ($\lambda_0 < 0$) and a *monochromatic collimated* point source of unit strength, i.e. a source emitting neutrons of only one speed v_0 (say) in only one direction Ω_0 (say), and situated at $\mathbf{r}_0$ (say). The mathematical expression of this source is

$$S(\mathbf{r}, v\Omega) = \delta(\mathbf{r} - \mathbf{r}_0)\delta(v - v_0)\delta(\Omega - \Omega_0), \tag{3.18}$$

where $\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$ is the three-dimensional delta function, and $\delta(\Omega - \Omega_0)$ is the angular delta function; the coefficient unity in (3.18) represents the unit strength of the source. Let the resulting neutron distribution be $N(\mathbf{r}_0, v_0\Omega_0 \to \mathbf{r}, v\Omega)$. Substituting (3.18) into (3.16), we have

$$s_n = N^\dagger_{t,n}(\mathbf{r}_0, v_0\Omega_0),$$

and then from (3.17) and (3.14) it follows that

$$N(\mathbf{r}_0, v_0\Omega_0 \to \mathbf{r}, v\Omega) = \sum_n (-1/\lambda_n) N^\dagger_{t,n}(\mathbf{r}_0, v_0\Omega_0) N_{t,n}(\mathbf{r}, v\Omega). \tag{3.19}$$

3.5. Another orthogonality relation

The analysis given above relating to equation (3.8′) is also applicable to (3.1). The adjoint equation is constructed as in § 3.3, and differs from (3.9) only in that λ is absent and the right side is multiplied by γ. It is easily proved, as for (3.11), that the eigenfunctions $N_{c,n}(\mathbf{r}, v\Omega)$ of (3.1)

and $N^{\dagger}_{c,n}(\mathbf{r}, v\Omega)$ of the adjoint equation satisfy the orthogonality relation

$$(\gamma_i - \gamma_j) \iiint dV \iiint N_{c,i}(\mathbf{r}, v\Omega) \times$$
$$\times \{vc(v)/l_{\text{tot}}(v)\}\, dvd\Omega \iiint f(v\Omega \to v'\Omega') N^{\dagger}_{c,j}(\mathbf{r}, v'\Omega')\, dv'd\Omega' = 0. \tag{3.20}$$

Of course, since the variable parameter is now a different quantity, not linearly depending on λ, the eigenfunctions $N_{c,n}(\mathbf{r}, v\Omega)$ of (3.1) are not the same as the $N_{i,n}(\mathbf{r}, v\Omega)$, so that (3.11) and (3.20) are not two sets of orthogonality relations connecting the same set of functions, but relate to two different sets of functions. We shall call (3.11) *orthogonality under variable time-constant* and (3.20) *orthogonality under variable composition*, though the latter term is not always justified (cf. § 3.1).

In stationary problems, the solution of an inhomogeneous equation can be expanded in terms of $N_{c,n}(\mathbf{r}, v\Omega)$ in the same way as was done above using $N_{i,n}(\mathbf{r}, v\Omega)$. In particular, if $N_{c,n}(\mathbf{r}, v\Omega)$ and $N^{\dagger}_{c,n}(\mathbf{r}, v\Omega)$ are normalized so that the integral in (3.20) is unity for $i = j$, we easily find, proceeding as in § 3.4,

$$N(\mathbf{r}_0, v_0\Omega_0 \to \mathbf{r}, v\Omega) = \sum_n (1/\gamma_n) N^{\dagger}_{c,n}(\mathbf{r}_0, v_0\Omega_0) N_{c,n}(\mathbf{r}, v\Omega). \tag{3.21}$$

It should be noted, however, that for purposes of practical computation there is usually no advantage in the use of eigenfunction expansions in stationary problems, and the formulae (3.19) and (3.21) are quoted only for their general interest.

The orthogonality relations (3.20) can easily be generalized. In forming the homogeneous equation corresponding to (2.4′), instead of replacing $c(v)$ by $\gamma c(v)$, we could have first represented $c(v)$ in the form

$$c(v) = c_0(v) + c_1(v) \tag{3.22}$$

and then replaced $c_1(v)$ by $\gamma c_1(v)$, leaving $c_0(v)$ unaltered. The form of the corresponding adjoint equation is obvious, and the corresponding orthogonality relation is

$$(\gamma_i - \gamma_j) \iiint dV \iiint N_{c,i}(\mathbf{r}, v\Omega) \times$$
$$\times \{vc_1(v)/l_{\text{tot}}(v)\}\, dvd\Omega \iiint f(v\Omega \to v'\Omega') N^{\dagger}_{c,j}(\mathbf{r}, v'\Omega')\, dv'd\Omega' = 0, \tag{3.20′}$$

to which we refer as *orthogonality under variable composition*. The actual form and values of the γ_n, $N_{c,n}$, and $N^{\dagger}_{c,n}$ will, of course, depend on how we divide $c(v)$. If the chemical composition varies in only one medium, we should put for this medium $c_1(v) = c(v)$, $c_0(v) = 0$, and for all other media $c_1(v) = 0$, $c_0(v) = c(v)$. The space integration in (3.20′) is then

extended only over the volume occupied by the first medium. In what follows, we shall usually refer to orthogonality under variable composition in the original form (3.20), but it must be remembered that all conclusions are equally applicable to the more general relation (3.20′).

3.6. The adjoint integral equation

The relations (3.20) and (3.20′) contain other factors in the integrand besides $N_{c,i}$ and $N^{\dagger}_{c,j}$. This means that $N^{\dagger}_{c,j}$ is not identical with (although closely related to) the solution of the integral equation adjoint to the homogeneous form of (2.40). This adjoint equation is not, however, generally of interest, because of the unwieldiness of (2.40).

If the scattering is isotropic in the L system, i.e. $f(v'\Omega' \to v\Omega)$ has the form (2.23), the integral equation (2.40) is replaced by a rather simpler one obtained by combining (2.38′) and (2.31′), and in this case the adjoint equation and the connexion between its solution and the $N^{\dagger}_{c}(\mathbf{r}, v\Omega)$ previously defined is also of interest.

The homogeneous integral equation under variable composition (without dividing $c(v)$) which corresponds to the elimination of Q between (2.38′) and (2.31′) is

$$n_c(\mathbf{r}, v) = \frac{\gamma}{4\pi v} \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} e^{-\tau_v(\mathbf{r},\mathbf{r}')} \int dv' \frac{v' c(\mathbf{r}', v')}{l(\mathbf{r}', v')} f_{\mathbf{r}'}(v' \to v) n_c(\mathbf{r}', v'), \tag{3.23}$$

and its adjoint equation is therefore

$$n^{\dagger}_{c}(\mathbf{r}, v) = \frac{\gamma}{4\pi} \frac{v c(\mathbf{r}, v)}{l(\mathbf{r}, v)} \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} \int \frac{dv'}{v'} e^{-\tau_v(\mathbf{r},\mathbf{r}')} f_{\mathbf{r}}(v \to v') n^{\dagger}_{c}(\mathbf{r}', v'). \tag{3.24}$$

To establish the relation between this $n^{\dagger}_{c}(\mathbf{r}, v)$ and $N^{\dagger}_{c}(\mathbf{r}, v\Omega)$ defined earlier, we consider the quantity

$$N^*(\mathbf{r}, v\Omega) = \frac{\gamma}{4\pi} \int_0^{\infty} n^{\dagger}_{c}(\mathbf{r}+R\Omega, v) e^{-\tau_v(\mathbf{r},\mathbf{r}+R\Omega)}\, dR. \tag{3.25}$$

It can easily be shown, as in deriving (2.35′), that this $N^*(\mathbf{r}, v\Omega)$ satisfies the equation

$$-v\Omega.\mathbf{grad}\, N^*(\mathbf{r}, v\Omega) + \frac{v}{l(\mathbf{r}, v)} N^*(\mathbf{r}, v\Omega) = \frac{\gamma v}{4\pi} n^{\dagger}_{c}(\mathbf{r}, v). \tag{3.26}$$

Multiplying (3.25) by

$$\frac{v_0}{v} \frac{c(\mathbf{r}, v_0)}{l(\mathbf{r}, v_0)} f_{\mathbf{r}}(v_0 \to v),$$

integrating over all v and Ω, transforming the right side as in the derivation of (2.38′), and using (3.24), we obtain

$$\frac{v_0 c(\mathbf{r}, v_0)}{l(\mathbf{r}, v_0)} \iiint \frac{dv}{v} d\Omega f_r(v_0 \to v) N^*(\mathbf{r}, v\Omega) = n_c^\dagger(\mathbf{r}, v_0). \tag{3.27}$$

Eliminating $n_c^\dagger$ between (3.27) and (3.26), we see that $N^*(\mathbf{r}, v\Omega)$ satisfies the same equation as $N_c^\dagger(\mathbf{r}, v\Omega)$. It is also clear from the definition (3.25) that $N^*(\mathbf{r}, v\Omega)$ satisfies the same boundary conditions as $N_c^\dagger(\mathbf{r}, v\Omega)$. With suitable normalization, we therefore have

$$N^*(\mathbf{r}, v\Omega) = N_c^\dagger(\mathbf{r}, v\Omega), \tag{3.28}$$

and this establishes the connexion between the solution of the adjoint integral equation and that of the adjoint integro-differential equation, for the case of scattering isotropic in the L system.

In the present case, we see by the use of (3.27) that (3.20) reduces to

$$(\gamma_i - \gamma_j) \iiint dV \int dv\, n_{c,i}(\mathbf{r}, v) n_{c,j}^\dagger(\mathbf{r}, v) = 0. \tag{3.29}$$

This could, of course, have been derived directly from (3.23) and (3.24).

3.7. Some further remarks

The existence, for $f(v'\Omega' \to v\Omega)$ independent of Ω and Ω', of an orthogonality relation between $n_c(\mathbf{r}, v)$ and $n_c^\dagger(\mathbf{r}, v)$ only [(3.29)] could have been foreseen as follows. The kernel of equation (2.30) or (2.30′) is linear in $c(v)$, whilst the parameter γ was introduced into (3.1) as a constant multiplier of $c(v)$. Thus the integral equation for $n_c(\mathbf{r}, v)$ is linear in the parameter γ, and it is a general property of integral equations linear in the parameter that there is an orthogonality relation between their eigenfunctions and those of the adjoint equation.

For eigenfunctions under variable time-constant, the situation is different, since the equation (2.30) or (2.30′) is not linear in the inverse total mean free path, and the parameter λ enters (3.8′) as a constant added to $v/l_{\text{tot}}(v)$. There is no reason, therefore, for an orthogonality relation to be expected, and in fact none can exist, as is shown by the following simple argument (due to Wilson (55)).

Consider a time-dependent problem with no delayed neutrons and no independent sources for $t > 0$, and let the value of $n(\mathbf{r}, v, t)$ $(= \iint N(\mathbf{r}, v\Omega, t)\, d\Omega)$ be given at $t = 0$. Then, by (3.6), the expression for $n(\mathbf{r}, v, t)$ when $t > 0$ will be of the form

$$n(\mathbf{r}, v, t) = \sum_m a_m n_{\lambda_m}(\mathbf{r}, v) e^{\lambda_m t}. \tag{3.30}$$

If there were an orthogonality relation between $n_{i,t}(\mathbf{r}, v)$ and $n^{\dagger}_{i,j}(\mathbf{r}, v)$ only, then the coefficients a_m in (3.30) would be uniquely determined by knowing $n(\mathbf{r}, v, t)$ at $t = 0$. Thus the neutron density at any subsequent time would be known. This is impossible; for, let there be two identical systems, in one of which the neutron distribution at time $t = 0$ is

$$N_1(\mathbf{r}, v\boldsymbol{\Omega}, 0) = n(\mathbf{r}, v, 0)\delta(\boldsymbol{\Omega} - \mathbf{r}/r),$$

and in the other is

$$N_2(\mathbf{r}, v\boldsymbol{\Omega}, 0) = n(\mathbf{r}, v, 0)\delta(\boldsymbol{\Omega} + \mathbf{r}/r),$$

where $r = |\mathbf{r}|$, and $\mathbf{r} = 0$ is the centre of each system. Then at $t = 0$, $n(\mathbf{r}, v, t)$ is the same in each system, but in the first system all the neutrons start by moving outwards, while in the second they all start by moving inwards. Then the initial escape of neutrons must be greater in the first system, and so the values of $n(\mathbf{r}, v, t)$ for $t > 0$ cannot be identical in the two systems. This completes the proof that no orthogonality relation can exist between $n_{i,t}(\mathbf{r}, v)$ and $n^{\dagger}_{i,j}(\mathbf{r}, v)$.

In the discussion (§§ 3.3 and 3.4) of the time-dependent problem in the presence of sources, delayed neutrons have been neglected. If this is no longer permissible, then (cf. § 3.2) the Laplace transform of the corresponding homogeneous integro-differential equation is no longer linear in the parameter λ. Consequently, the eigenfunctions of the adjoint equation constructed as in § 3.3 will not be orthogonal to those of the homogeneous form of (3.2). The procedure described above of developing the solution of the time-dependent inhomogeneous equation in terms of the eigenfunctions of the homogeneous equation is therefore no longer applicable. However, it appears that the number of each kind of fission fragment capable of emitting a delayed neutron is governed, as a function of time, by a first-order differential equation; this is seen from § 1.2.5, since the probability per unit time of β-ray emission is a constant for a given kind of nucleus or nuclear fragment in a given state of excitation. It follows that, if the more general approach described in § 2.3.4 is adopted (namely, to supplement the neutron distribution equation by those governing the distribution of fission fragments that emit delayed neutrons), then the Laplace transformation with respect to time will give a system of equations linear in the parameter. This system will have orthogonality relations similar to (3.11), and hence the expansion of the solution in terms of the eigenfunctions of the system of homogeneous equations will be possible. This treatment lies beyond our scope, however, and in what follows we shall confine ourselves to stationary problems alone.

PART II

THE CONSTANT CROSS-SECTION APPROXIMATION

IV

ONE-GROUP THEORY AND ITS RANGE OF APPLICATION

4.1. The assumptions underlying one-group theory

We turn now to the methods of solution of the equations discussed in Part I. Each available method is based on some simplifying assumption. Part II is concerned with the solution of (2.4′) under the following additional assumptions:

(i) the variation of the total mean free path $l_{\text{tot}}(v)$ with energy is neglected;

(ii) the relative probabilities of scattering, capture, and fission vary with energy in such a way that the mean number of secondaries per collision $c(v)$, given by equation (1.5), can be regarded as independent of energy;

(iii) the value of the integral

$$f_{v'}(\Omega' \to \Omega) = \int f(v'\Omega' \to v\Omega)\, dv \tag{4.1}$$

can be regarded as independent of v', the initial speed of the neutron.

Integrating (2.4′) over all v, using the above assumptions, and writing

$$\left.\begin{aligned} \psi(\mathbf{r},\Omega) &= \int vN(\mathbf{r},v\Omega)\, dv \\ \text{and} \qquad s(\mathbf{r},\Omega) &= \int S(\mathbf{r},v\Omega)\, dv, \end{aligned}\right\} \tag{4.2}$$

we obtain

$$\Omega.\mathbf{grad}\,\psi + \psi/l = (c/l)\iint \psi(\mathbf{r},\Omega')f(\Omega' \to \Omega)\, d\Omega' + s(\mathbf{r},\Omega). \tag{4.3}$$

This is the form of the transport equation in the *constant cross-section approximation*, or, as is it sometimes called, *one-velocity-group theory* o

simply *one-group theory*. The latter term,† though generally adopted, gives an incorrect idea of the range of applicability of the approximation, which is better shown by our name.

It is to be noticed that, if (2.8′), (2.9), and (2.16) are integrated over all v, we obtain respectively $\nu/4\pi$, $1/4\pi$, and (2.17), which are all independent of v'. The assumptions (i) to (iii) therefore follow if the relevant cross-sections are assumed independent of energy. Further, on integrating over all energies, the only possible difference between elastic and inelastic scattering is in the angular distributions of the neutrons after collision. However, if M is large, i.e. the scattering is by a heavy nucleus, (2.17) can be replaced by $1/4\pi$, that is, the same value as for inelastic scattering. If, on the other hand, M is small, inelastic scattering may be neglected. Thus elastic and inelastic scattering need not be distinguished, and the cross-sections which must be assumed constant are those for fission, capture, and total scattering (elastic plus inelastic) for each kind of nucleus present in the system. Though the chief attraction of the constant cross-section approximation is the simplicity of the transport equation compared with the general case, there are many systems for which the assumptions of this approximation are fairly well satisfied, and for these it will be a reasonably realistic treatment.

No reference has yet been made to the case of anisotropic scattering in the C system, i.e. when further terms of (1.1) have to be considered. However, we have shown in § 1.2.2 that these terms have coefficients $g_n(E)$ $(n \geqslant 1)$ which depend strongly on the energy, and so it would be unreasonable to apply the constant cross-section approximation to this case.

The reduction of time-dependent problems to stationary ones involves the introduction of a 'fictitious capture' varying as $1/v$, and an alteration in the number of secondaries per fission (see equation (3.2) and the discussion thereof). If the second change has a much greater effect than the former, or if the spread of neutron energies in the system is very small, the constant cross-section approximation may be applicable, despite the variation of the fictitious capture with energy. If the effect of the introduction of fictitious capture is important, however, and if the spread of neutron energies in the system is large, the variation of the fictitious capture as $1/v$ may make the constant cross-section approximation impossible of application to the time-dependent equation or to its Laplace transform (3.2), even when it can be used for the corresponding stationary problem.

† Due to the fact that the constant cross-section approximation is a particular case of *m-group theory*, discussed in Chapter XIX.

4.2. Terminology

Having indicated the chief limitations of the constant cross-section approximation, we return to equation (4.3). If all the nuclei in the medium are heavy (large M), then (2.17) can be approximately replaced by $1/4\pi$ for all scattering collisions, and hence $f(\Omega' \to \Omega)$ can be taken as constant. By the normalization condition (2.6) we then have

$$f(\Omega' \to \Omega) = 1/4\pi,$$

and the equation (4.3) becomes

$$\Omega.\mathbf{grad}\,\psi + \psi/l = (c/4\pi l) \iint \psi(\mathbf{r}, \Omega')\, d\Omega' + s(\mathbf{r}, \Omega). \tag{4.4}$$

The case where this equation is applicable is called *one-group theory with scattering isotropic in the laboratory system*, or simply *one-group theory with isotropic scattering*.

It is seen from equation (4.4) that the quantity

$$\rho(\mathbf{r}) = \iint \psi(\mathbf{r}, \Omega)\, d\Omega = \iiint vN(\mathbf{r}, v\Omega)\, dv d\Omega = \int vn(\mathbf{r}, v)\, dv \tag{4.5}$$

(which may be described physically as the track length per unit time per unit volume) will play an important part in neutron transport theory. It is generally called the *neutron flux*.† We also introduce two other appellations, the *neutron current*

$$\mathbf{j}(\mathbf{r}) = \iint \Omega\psi(\mathbf{r}, \Omega)\, d\Omega = \iiint v\Omega N(\mathbf{r}, v\Omega)\, dv d\Omega, \tag{4.6}$$

and the *neutron density*‡

$$n(\mathbf{r}) = \iiint N(\mathbf{r}, v\Omega)\, dv d\Omega. \tag{4.7}$$

It is to be noticed that the term *flux* is here used in a different sense from what is usual in physics, where it denotes a quantity analogous to (4.6). The terminology here introduced is for the sake of convenience, since 'track length per unit time per unit volume' is too long, while 'neutron current' is an equally satisfactory name for (4.6).

The importance of the quantity (4.5) arises in practice in the following way. Suppose that, in order to observe the neutron population at some point, an 'activation detector' (i.e. a thin foil, say, of absorbing material) is inserted. When the foil is withdrawn, the number of radioactive

† The term *neutron traffic* has lately been suggested to denote this quantity, but we prefer to retain the established name.

‡ It should be noticed that $n(\mathbf{r})$ (with one argument) is not the same function as $n(\mathbf{r}, v)$ (with two arguments), but equals $\int n(\mathbf{r}, v)\, dv$.

disintegrations of excited nuclei formed in the foil by neutron capture is counted, and so the number of neutrons captured by the foil while in the medium is found. The Boltzmann equation gives the number of neutrons captured by the foil per unit time as

$$\iiint_{\text{volume of foil}} dV \iiint \frac{vN(\mathbf{r}, v\boldsymbol{\Omega})}{l_{c,\text{foil}}(v)}\, dv d\boldsymbol{\Omega},$$

which is proportional to

$$\iiint \sigma_{c,\text{foil}}(v) vN(\mathbf{r}, v\boldsymbol{\Omega})\, dv d\boldsymbol{\Omega}.$$

Thus, if the capture cross-section in the detector follows the $1/v$ law (see § 1.3.2), the neutron density is measured by this method; but if this cross-section is constant (as in the present theory), the detector will measure the neutron flux. In the case of collimated, i.e. unidirectional, neutrons, the flux is the same as the current, and this is the origin of the term *flux* in the neutron transport sense.

The neutron flux (i.e. (4.5)) is sometimes denoted by $vn(\mathbf{r})$, where $n(\mathbf{r})$ is given by (4.7). In this case, however, v denotes the mean speed of the neutrons at the point $\mathbf{r}$, which is the ratio of (4.5) to (4.7).

The function $\psi(\mathbf{r}, \boldsymbol{\Omega})$ is called the *angular distribution of the neutron flux*, or simply the *neutron angular distribution*. The terms *neutron-energy flux*, *neutron-energy current*, and *neutron-energy density* are also sometimes used; they are defined similarly to (4.5), (4.6), and (4.7), with $N(\mathbf{r}, v\boldsymbol{\Omega})$ replaced by $\frac{1}{2}mv^2N(\mathbf{r}, v\boldsymbol{\Omega})$, where m is the neutron mass. Where the constant cross-section approximation is invalid, the *neutron collision density* is defined as

$$\iiint \frac{vN(\mathbf{r}, v\boldsymbol{\Omega})\, dv d\boldsymbol{\Omega}}{l_{\text{tot}}(v)}. \tag{4.8}$$

4.3. Other applications of the constant cross-section approximation

It has been shown in § 4.1 that, if the relevant cross-sections are all independent of energy, and if it is desired to find only $\psi(\mathbf{r}, \boldsymbol{\Omega})$, the angular distribution of the neutron flux, rather than $N(\mathbf{r}, v\boldsymbol{\Omega})$ itself, the equation (2.4′) can be replaced by the simpler form (4.3). If fission and inelastic scattering are absent and the cross-sections for capture and elastic scattering are again independent of energy, then the same method can be used to determine not only the neutron flux angular distribution $\psi(\mathbf{r}, \boldsymbol{\Omega})$, but also such quantities as the angular distribution of neutron

density and the angular distribution of neutron energy density, namely the quantities

$$\chi_n(\mathbf{r},\Omega) = \int_0^\infty v^n N(\mathbf{r}, v\Omega)\, dv. \tag{4.9}$$

For, if elastic scattering and capture are the only possible processes and the cross-sections for these processes are constant, then on substituting (2.7) and (2.16) into (2.4′) we have

$$v\Omega.\mathbf{grad}\, N + \frac{vN}{l_{\text{tot}}} = \iiint N(\mathbf{r}, v'\Omega') \sum_i \frac{f(v/v', \Omega.\Omega', M_i)}{l_{\text{el},i}}\, dv' d\Omega' + S, \tag{4.10}$$

where $f(v/v', \Omega.\Omega', M)$ denotes v' times the right side of (2.16). Multiplying (4.10) by v^{n-1}, integrating over all v, and noting that

$$\int_0^\infty v^{n-1} dv f(v/v', \Omega.\Omega', M_i) = v'^n \times \text{a function of } \Omega.\Omega' \text{ and } M_i \text{ only},$$

we clearly find an equation for $\chi_n(\mathbf{r},\Omega)$ which is of the same form as (4.3). As n increases, the relative importance of the low-energy neutrons decreases, and the lower parts of the neutron energy range may be disregarded. Thus, with a monochromatic source and cross-sections which vary slowly in the vicinity of the source energy, it may be possible to apply the constant cross-section approximation to determine $\chi_n(\mathbf{r},\Omega)$ for sufficiently large n, even if the cross-section varies considerably below the source energy.† It is sometimes possible to use this fact in investigating the biological effects of neutrons, which are approximately proportional to the neutron-energy flux of fast neutrons, i.e. to $\chi_3(\mathbf{r},\Omega)$.

4.4. A note regarding thermal neutrons‡

Hitherto the condition for the applicability of the constant cross-section approximation has been the constancy of the relevant cross-sections, whence the name; however, this theory is also applicable to an entirely different situation. Suppose that scattering is isotropic in the L system, while the energy changes in scattering are such that the neutron energy spectrum is independent of the position and direction of motion of the neutron; that is,

$$f(v'\Omega' \to v\Omega) = f(v' \to v)/4\pi, \tag{4.11}$$

and

$$N(\mathbf{r}, v\Omega) = \psi(\mathbf{r},\Omega).F(v). \tag{4.12}$$

† This remark is due to J. H. Tait.

‡ The considerations given in this section are due to J. P. Elliott.

Substituting these in (2.4′) and integrating over all v, we obtain, since by (2.6) $\int f(v' \to v)\, dv = 1$,

$$\Omega.\mathbf{grad}\,\psi(\mathbf{r},\Omega).\int_0^\infty vF(v)\,dv + \psi(\mathbf{r},\Omega).\int_0^\infty \frac{vF(v)\,dv}{l_{\mathrm{tot}}(v)}$$

$$= \frac{1}{4\pi}\iint \psi(\mathbf{r},\Omega')\,d\Omega'.\int_0^\infty \frac{c(v')v'F(v')\,dv'}{l_{\mathrm{tot}}(v')} + \int_0^\infty S(\mathbf{r},v\Omega)\,dv.$$

This equation is identical with what (4.4) becomes when l and c are replaced by l^* and c^* defined as follows:

$$\left.\begin{aligned} \frac{1}{l^*}\int_0^\infty vF(v)\,dv &= \int_0^\infty \frac{vF(v)\,dv}{l_{\mathrm{tot}}(v)}, \\ c^*\int_0^\infty \frac{vF(v)\,dv}{l_{\mathrm{tot}}(v)} &= \int_0^\infty \frac{vc(v)F(v)\,dv}{l_{\mathrm{tot}}(v)}, \end{aligned}\right\} \tag{4.13}$$

from which l^* and c^* can be calculated, provided that $F(v)$ and the variation of the cross-sections with energy are already known.

The situation here envisaged should be approximately realized in the case of thermal neutrons. For, if capture and escape from the system are neglected (which would imply omitting the sources also in a stationary problem), then the neutrons will be in strict thermal equilibrium with the medium. In this case, it is known from the kinetic theory of gases that, regardless of the scattering law and the variation of the mean free path with energy, the neutron spectrum will be Maxwellian, i.e. it will be given by

$$F(v) = \text{constant} \times v^2 e^{-mv^2/2kT} \tag{4.14}$$

(where m is the mass of the neutron, T the temperature of the medium, and k Boltzmann's constant), irrespective of the position or direction of motion. If the absorption of neutrons and their escape from the system are taken into account, then they will no longer be in strict thermal equilibrium with the medium, and equation (4.14) may require modification. However, if the absorption is large, the neutron population of the thermal energy region will be small and the exact determination of its distribution will be of little interest. If the absorption is small, on the other hand, the correction required to (4.14) will not be great. The escape of neutrons from the system can be regarded as a highly localized capture. Hence, except in the immediate vicinity of the boundaries, the effect of escape on the system will be like that of a weak absorber,

i.e. the correction to (4.14) on this account will be small. Near the boundaries the effect will be more important, and the actual magnitude of the necessary corrections to (4.12) and (4.14) will depend chiefly on the rate of approach to thermal equilibrium from a distribution in the thermal energy range. Although no exact results are available, it is hoped that (4.12) and (4.14) will be approximately applicable up to the boundaries. In this case, and if (4.11) is justified, it follows that the thermal neutrons can be treated by one-group theory, even when $l(v)$ and $c(v)$ vary appreciably with energy. Further, an exact knowledge of the behaviour of $f(v'\Omega' \to v\Omega)$ for thermal neutrons is not needed, except possibly in order to justify the use of (4.12) near the boundaries.

It is to be noticed that the effects which appear only in dealing with thermal neutrons—namely, the molecular binding and the thermal motion of the atomic nuclei—will both tend to make the scattering isotropic in the L system. This follows since the thermal motion is random and the effects of molecular binding can be expressed to a first approximation by ascribing to the nucleus a mass greater than its real mass; from (2.17), this decreases the anisotropy in the L system. Thus, for thermal neutrons, (4.11) should be valid for collisions with all nuclei except, perhaps, protons and deuterons.

We here conclude our survey of the possibilities of application of the constant cross-section approximation, and turn to the study of the appropriate transport equation.

4.5. The integral equation in the constant cross-section approximation

If the scattering is isotropic and the system consists of a single homogeneous non-re-entrant body (so that equation (2.30) holds), then, in the constant cross-section approximation, multiplying (2.30) by v, integrating over all v, and using the notation (4.5) we have

$$\rho(\mathbf{r}) = (c/4\pi l)\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/l}dV'/|\mathbf{r}-\mathbf{r}'|^2 + \\ + \iiint s(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/l}dV'/|\mathbf{r}-\mathbf{r}'|^2. \quad (4.15)$$

This equation could also have been obtained by starting with (4.4) and proceeding as in § 2.5. If this were done, we should obtain for the angular distribution

$$\psi(\mathbf{r},\Omega) = \int_0^\infty \left[\left(\frac{c}{4\pi l}\right)\rho(\mathbf{r}-R\Omega)+s(\mathbf{r}-R\Omega)\right]e^{-R/l}\,dR. \quad (4.16)$$

This relation, in turn, could have been got from (2.37) by multiplying by v, integrating over v, using the assumptions of the constant cross-section approximation, and noticing that, for isotropic scattering and isotropic sources,

$$\int Q(\mathbf{r}, v)\, dv = (c/4\pi l)\rho(\mathbf{r}) + s(\mathbf{r}).$$

Also, substituting (4.16) into (4.6) and putting $\mathbf{r} - R\Omega = \mathbf{r}'$, we find

$$\mathbf{j}(\mathbf{r}) = \iiint \left[\frac{c}{4\pi l}\rho(\mathbf{r}') + s(\mathbf{r}')\right] \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} e^{-|\mathbf{r}-\mathbf{r}'|/l}\, dV'. \tag{4.17}$$

If it is desired to work with the neutron emission density instead of the flux, then we integrate (2.32) over all v and write

$$q(\mathbf{r}) = \int Q(\mathbf{r}, v)\, dv, \tag{4.18}$$

obtaining

$$q(\mathbf{r}) = (c/4\pi l) \iiint q(\mathbf{r}') e^{-|\mathbf{r}-\mathbf{r}'|/l}\, dV'/|\mathbf{r}-\mathbf{r}'|^2 + s(\mathbf{r}). \tag{4.19}$$

Abandoning the assumption of a single homogeneous non-re-entrant body, but retaining those of constant cross-sections and isotropic scattering, makes it necessary to start with (2.30'). In that case (4.15) is replaced by

$$\rho(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{c(\mathbf{r}')}{l(\mathbf{r}')} \rho(\mathbf{r}') e^{-\tau(\mathbf{r},\mathbf{r}')} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + \iiint s(\mathbf{r}') e^{-\tau(\mathbf{r},\mathbf{r}')}\, dV'/|\mathbf{r}-\mathbf{r}'|^2, \tag{4.15'}$$

which could also be derived from (4.4), while (4.19) becomes

$$q(\mathbf{r}) = \frac{c(\mathbf{r})}{4\pi l(\mathbf{r})} \iiint q(\mathbf{r}') e^{-\tau(\mathbf{r},\mathbf{r}')} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + s(\mathbf{r}). \tag{4.19'}$$

Formula (4.16) becomes

$$\psi(\mathbf{r}, \Omega) = \int_0^\infty \left[\frac{c(\mathbf{r}-R\Omega)}{4\pi l(\mathbf{r}-R\Omega)} \rho(\mathbf{r}-R\Omega) + s(\mathbf{r}-R\Omega)\right] e^{-\tau(\mathbf{r},\mathbf{r}-R\Omega)}\, dR. \tag{4.16'}$$

If the anisotropy of scattering in the L system is not negligible, i.e. if the constant cross-section Boltzmann equation has to be taken as (4.3) instead of (4.4), then the neutron flux $\rho(\mathbf{r})$ and the neutron emission density $q(\mathbf{r})$ must be replaced by the neutron flux angular distribution $\psi(\mathbf{r}, \Omega)$ and the neutron emission angular distribution $q(\mathbf{r}, \Omega)$. Starting

from equation (4.3) and proceeding as in §§ 2.5 and 2.6, we obtain for these latter two quantities the integral equations

$$\psi(\mathbf{r},\Omega) = \int_0^\infty dR' e^{-\tau(\mathbf{r},\mathbf{r}-R'\Omega)} \times \times \left\{ \int\int \frac{c(\mathbf{r}-R'\Omega)}{l(\mathbf{r}-R'\Omega)} \psi(\mathbf{r}-R'\Omega,\Omega') f(\Omega' \to \Omega)\, d\Omega' + s(\mathbf{r}-R'\Omega,\Omega) \right\} \tag{4.20}$$

and

$$q(\mathbf{r},\Omega) = \{c(\mathbf{r})/l(\mathbf{r})\} \iiint q(\mathbf{r}-R'\Omega',\Omega') e^{-\tau(\mathbf{r},\mathbf{r}-R'\Omega')} f(\Omega' \to \Omega)\, dR' d\Omega' + s(\mathbf{r},\Omega).$$

Putting $\mathbf{r}-R'\Omega' = \mathbf{r}'$, we can rewrite the last equation as

$$q(\mathbf{r},\Omega) = \frac{c(\mathbf{r})}{l(\mathbf{r})} \iiint q\left(\mathbf{r}', \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right) e^{-\tau(\mathbf{r},\mathbf{r}')} f\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \to \Omega\right) \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + s(\mathbf{r},\Omega). \tag{4.21}$$

Equation (4.20) is, of course, the form of equation (2.40) in the constant cross-section approximation.

4.6. The adjoint integro-differential equation and the optical reciprocity theorem

The homogeneous equation under variable composition, corresponding to (4.3), is obtained by simply omitting the free term and replacing c by γc or, more generally, by $c_0+\gamma c_1$, where γ is a variable parameter and $c_0+c_1 = c$ (see (3.22)).‡ This equation we call (4.3′). The adjoint equation, similarly to (3.9), is

$$-\Omega.\mathbf{grad}\,\psi^\dagger + \psi^\dagger/l = \{(c_0+\gamma c_1)/l\} \int\int \psi^\dagger(\mathbf{r},\Omega') f(\Omega \to \Omega')\, d\Omega'. \tag{4.22}$$

On comparing this equation with (4.3′), and recalling that $f(\Omega' \to \Omega)$ depends only on $\Omega.\Omega'$, so that $f(\Omega \to \Omega') = f(\Omega' \to \Omega)$, we see that $\psi^\dagger(\mathbf{r},\Omega)$ must be proportional to $\psi(\mathbf{r},-\Omega)$. Thus, if $\psi(\mathbf{r},\Omega)$ and $\psi^\dagger(\mathbf{r},\Omega)$ are so normalized that for some particular $\mathbf{r}$ and Ω

$$\psi^\dagger(\mathbf{r},\Omega) = \psi(\mathbf{r},-\Omega), \tag{4.23}$$

then this relation will hold for every $\mathbf{r}$ and Ω. This at once gives the physical interpretation of the adjoint equation for the constant cross-section approximation, and confirms that in this case the eigenvalues of

‡ If the system consists of one medium only, we need not, of course, introduce γ but can treat c itself as a variable parameter.

the adjoint equation are the same as those of the original equation (cf. § 3.3), whether or not the eigenfunctions $\psi_{c,0}(\mathbf{r},\Omega)$, $\psi_{c,1}(\mathbf{r},\Omega)$, etc., form a complete set. If this is in fact the case, by the use of (4.23) the solution of an inhomogeneous equation can be expanded in terms of the eigenfunctions of (4.3′) (cf. § 3.5). In particular, for the case of a collimated point source of unit strength, we find

$$\psi(\mathbf{r}_0,\Omega_0\to\mathbf{r},\Omega)=\sum_n(1/\gamma_n)\psi_{c,n}(\mathbf{r}_0,-\Omega_0)\psi_{c,n}(\mathbf{r},\Omega) \tag{4.24}$$

(cf. equation (3.21)). From (4.24) there follows at once

$$\psi(\mathbf{r}_0,\Omega_0\to\mathbf{r},\Omega)=\psi(\mathbf{r},-\Omega\to\mathbf{r}_0,-\Omega_0); \tag{4.25}$$

that is, the number of neutrons at $\mathbf{r}$ travelling in the direction Ω due to a point source of unit strength at $\mathbf{r}_0$ emitting in the direction Ω_0 is equal to the number of neutrons at $\mathbf{r}_0$ travelling in the direction $-\Omega_0$ due to a source of unit strength at $\mathbf{r}$ emitting in the direction $-\Omega$.

This result is called the *optical reciprocity theorem*.† It has been proved above on the assumption that $\psi_{c,0}, \psi_{c,1}$, etc., form a complete set. Another proof does not depend on completeness, but assumes instead that the actual value of c is smaller than any of the eigenvalues c_0, c_1, etc. It may run as follows:

Let the integral equations for $\psi(\mathbf{r}_0, \Omega_0 \to \mathbf{r}, \Omega)$ and for

$$\psi(\mathbf{r}, -\Omega \to \mathbf{r}_0, -\Omega_0)$$

be written in the form (4.20). If the actual value of c is smaller than any of the eigenvalues of the corresponding homogeneous equation, these integral equations may be solved by means of Neumann expansions.‡ Using the facts that $f(\Omega' \to \Omega) = f(\Omega \to \Omega')$ and $\tau(\mathbf{r},\mathbf{r}') = \tau(\mathbf{r}',\mathbf{r})$, it is easy to show that the corresponding terms of the two Neumann series are equal, and hence (4.25) follows.

The assumption that c is smaller than any of the eigenvalues should always be satisfied in practice, for (as we have seen in § 3.2) a stationary solution is otherwise not a practical (though perhaps a theoretical) possibility in the presence of sources.

† The use of the word *optical* arises from the previous existence of a similar theorem in optics.

‡ This may be seen as follows. For finite systems, Fredholm's theory (52, pp. 215–17) is applicable, and shows that the solution as a function of c has singularities only at c equal to some c_j. The solution as a power series in c therefore converges for $|c| < \min|c_j|$, and this series is the Neumann series (52, pp. 221–2).

For infinite systems, the result as stated is valid only if there is no supply of neutrons from infinity. In this case we can take a large finite system, establish the convergence of the Neumann series as above, and pass to the limit as the system becomes infinite. We shall discuss this more closely in Chapter XV. If there are sources at infinity, they should be replaced by sources at a finite distance; we then apply the optical reciprocity theorem and again pass to the limit.

The formula (4.25) can be integrated over some variable to give a more particular form of the optical reciprocity theorem. Thus, integrating over all Ω and Ω_0, we obtain

$$\rho(\mathbf{r}_0 \to \mathbf{r}) = \rho(\mathbf{r} \to \mathbf{r}_0); \tag{4.26}$$

that is, the neutron flux at $\mathbf{r}$ due to an isotropic point source of unit strength at $\mathbf{r}_0$ is equal to the neutron flux at $\mathbf{r}_0$ due to an isotropic point source of unit strength at $\mathbf{r}$.

4.7. The adjoint integral equation

In the last section we considered the adjoint integro-differential equation in the constant cross-section approximation with isotropic scattering. We shall now consider the adjoint integral equation. If the homogeneous equation corresponding to (4.15′) is taken in the form

$$\rho_c(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{c_0(\mathbf{r}')+\gamma c_1(\mathbf{r}')}{l(\mathbf{r}')} \rho_c(\mathbf{r}') e^{-\tau(\mathbf{r}',\mathbf{r})} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{4.27}$$

then the adjoint integral equation is obviously

$$\rho_c^\dagger(\mathbf{r}) = \frac{1}{4\pi} \frac{c_0(\mathbf{r})+\gamma c_1(\mathbf{r})}{l(\mathbf{r})} \iiint \rho_c^\dagger(\mathbf{r}') e^{-\tau(\mathbf{r}',\mathbf{r})} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{4.28}$$

On comparing these equations with the homogeneous equation corresponding to (4.19′), we see that with suitable normalization their solutions are related by

$$\frac{4\pi}{\gamma_i} q_{c,i}(\mathbf{r}) = \rho_{c,i}^\dagger(\mathbf{r}) = \frac{c_0(\mathbf{r})/\gamma_i + c_1(\mathbf{r})}{l(\mathbf{r})} \rho_{c,i}(\mathbf{r}). \tag{4.29}$$

These relations will be utilized in Chapters XIV and XV.

We have not yet given the orthogonality relations between the various $\rho_{c,n}(\mathbf{r})$. It would be rather laborious to derive these from (3.20′), since that equation already involves integration over all v'. However, it is also possible to start from the equations satisfied by $\psi_{c,n}(\mathbf{r},\Omega)$ and $\psi_{c,n}^\dagger(\mathbf{r},\Omega)$, and proceed as in the derivation of (3.20′). This gives

$$(\gamma_i - \gamma_j) \iiint dV \frac{c_1(\mathbf{r})}{l(\mathbf{r})} \iint \psi_{c,i}(\mathbf{r},\Omega)\, d\Omega \iint f(\Omega \to \Omega') \psi_{c,j}^\dagger(\mathbf{r},\Omega')\, d\Omega' = 0.$$

Since the scattering is assumed isotropic in the L system, i.e. $f(\Omega \to \Omega') = 1/4\pi$, we can rewrite the above orthogonality relation, using (4.23) and (4.5), as

$$(\gamma_i - \gamma_j) \iiint dV \frac{c_1(\mathbf{r})}{l(\mathbf{r})} \rho_{c,i}(\mathbf{r}) \rho_{c,j}(\mathbf{r}) = 0. \tag{4.30}$$

If we do not use (3.22), but take the homogeneous equation under variable composition in its simplest form with $c_0(\mathbf{r}) = 0$, $c_1(\mathbf{r}) = c(\mathbf{r})$, we can use (4.29) to rewrite (4.30) in the more convenient form

$$(\gamma_i - \gamma_j) \iiint dV \rho_{c,i}(\mathbf{r}) \rho_{c,j}^{\dagger}(\mathbf{r}) = 0,$$

or, with the appropriate normalization,

$$\iiint dV \rho_{c,i}(\mathbf{r}) \rho_{c,j}^{\dagger}(\mathbf{r}) = \delta_{ij}, \tag{4.31}$$

where δ_{ij} is the Kronecker symbol.

Note added in proof. The argument given in the second part of § 4.4 to justify the application of (4.13) to thermal neutrons is inexact, and holds only under the further condition that the variation of $l_{\text{tot}}(v)$ over the peak of the Maxwellian distribution is small. If this condition is not met, the result (4.14) will hold only for the neutron density spectrum $n(\mathbf{r}, v)$.

V

EXACT SOLUTIONS FOR AN INFINITE MEDIUM WITH ISOTROPIC SCATTERING

5.1. The case of an infinite source-free medium

We now turn from the formulation of the equations governing the migration of neutrons to their solution. For clarity we deal with the simplest case first, namely the constant cross-section approximation with isotropic scattering, and this will occupy us for several chapters. The present chapter and the next are concerned with some cases where the equation for the neutron distribution can be solved exactly.

The simplest possible problem is that of a homogeneous source-free medium extending to infinity in all directions. For this, the equation (4.15) becomes

$$\rho(\mathbf{r}) = \frac{c}{4\pi l} \int\int\int_{\text{all space}} \rho(\mathbf{r}') e^{-|\mathbf{r}-\mathbf{r}'|/l} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{5.1}$$

The solution of this integral equation can be reduced to that of a differential equation, and we shall do this by three alternative methods.

5.1.1. *First method*

Since $\rho(\mathbf{r})$ must be everywhere regular, $\rho(\mathbf{r}')$ may be expanded in a Taylor series about $\mathbf{r}$, viz.

$$\rho(\mathbf{r}') = \rho(\mathbf{r}) + [(x'-x)\partial\rho/\partial x + (y'-y)\partial\rho/\partial y + (z'-z)\partial\rho/\partial z] + \tfrac{1}{2}[\ldots] + \ldots. \tag{5.2}$$

If we assume this series to converge everywhere, it may be substituted into (5.1) and the result integrated term by term. Since the resulting expression must be independent of the choice of Cartesian coordinates, it is invariant under rotation of the axes, and so can contain only the differential operators ∇^2, ∇^4, ∇^6, etc., where

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2, \quad \nabla^4 = (\nabla^2)^2, \quad \text{etc.}$$

Thus the result of substituting (5.2) into (5.1) should be of the form

$$\rho(\mathbf{r}) = c[\rho(\mathbf{r}) + A_1 \nabla^2\rho(\mathbf{r}) + A_2 \nabla^4\rho(\mathbf{r}) + \ldots].$$

The coefficients A_1, A_2, etc., which appear in this expression are easily found, giving

$$\rho(\mathbf{r}) = c[1 + \tfrac{1}{3}l^2\nabla^2 + \tfrac{1}{5}l^4\nabla^4 + \ldots]\rho(\mathbf{r}). \tag{5.3}$$

If a solution of the differential equation

$$\nabla^2\rho(\mathbf{r}) = \rho(\mathbf{r})/L^2 \tag{5.4}$$

is known which is regular throughout space, then clearly

$$\nabla^{2n}\rho(\mathbf{r}) = \rho(\mathbf{r})/L^{2n},$$

and, substituting this into (5.3), it is seen that this $\rho(\mathbf{r})$ will satisfy that equation also, provided that L is chosen so that

$$1 = c\left[1+\frac{1}{3}\frac{l^2}{L^2}+\frac{1}{5}\frac{l^4}{L^4}+\ldots\right],$$

that is,

$$\frac{1}{c} = \frac{L}{2l}\log\frac{L+l}{L-l}. \tag{5.5}$$

Thus, if L satisfies (5.5), any solution of (5.4) regular throughout space will satisfy (5.1). The converse is shown by the second method.

5.1.2. *Second method*

We attempt to find the solution of (5.1) in the form

$$\rho(\mathbf{r}) = \frac{1}{8\pi^3}\iiint F(\mathbf{p})e^{-i\mathbf{p}.\mathbf{r}}\,dV_p, \tag{5.6}$$

where $\mathbf{p}$ is the parameter of a three-dimensional Fourier transform, and dV_p is the volume element in $\mathbf{p}$-space. Since

$$\frac{1}{4\pi l}\iiint\frac{dV_r}{r^2}e^{-r/l+i\mathbf{p}.\mathbf{r}} = \frac{1}{2l}\int_0^\infty e^{-r/l}dr\int_{-1}^{1}e^{irp\mu}\,d\mu = \frac{1}{pl}\tan^{-1}pl \tag{5.7}$$

(where dV_r is the volume element in $\mathbf{r}$-space, $r = |\mathbf{r}|$, $p = |\mathbf{p}|$, and the principal branch of the inverse tangent is taken), we have by the Fourier transform inversion theorem

$$\frac{1}{4\pi lr^2}e^{-r/l} = \frac{1}{8\pi^3}\iiint\frac{1}{pl}\tan^{-1}pl\,e^{-i\mathbf{p}.\mathbf{r}}\,dV_p,$$

and then by the convolution theorem, using (5.6), we obtain

$$\frac{1}{4\pi l}\iiint_{\text{all space}}\rho(\mathbf{r}')\frac{dV_{r'}}{|\mathbf{r}-\mathbf{r}'|^2}e^{-|\mathbf{r}-\mathbf{r}'|/l} = \frac{1}{8\pi^3}\iiint F(\mathbf{p})\frac{1}{pl}\tan^{-1}pl\,e^{-i\mathbf{p}.\mathbf{r}}\,dV_p. \tag{5.8}$$

(The proof of the convolution theorem may be found, for instance, in Bochner (4), p. 189.) On comparing (5.8) with (5.6), we see that (5.1) will be satisfied if $F(\mathbf{p})$ is the solution of the equation

$$\{1-(c/pl)\tan^{-1}pl\}F(\mathbf{p}) = 0. \tag{5.9}$$

Since only the principal branch of the inverse tangent is involved, it is easily shown that the equation

$$1-(c/pl)\tan^{-1}pl = 0$$

has only two solutions, which in the notation of (5.5) are $p = \pm i/L$. Thus the most general permissible solution of (5.9) is

$$F(\mathbf{p}) = \delta(p-i/L)G_1(\mathbf{p})+\delta(p+i/L)G_2(\mathbf{p}), \tag{5.10}$$

where δ is Dirac's delta function and $G_1(\mathbf{p})$ and $G_2(\mathbf{p})$ are any functions. Substituting (5.10) into (5.6) and combining the contributions of the two terms of (5.10), we obtain

$$\rho(\mathbf{r}) = \iint G(\mathbf{k})e^{\mathbf{k}.\mathbf{r}/L}\,dk, \tag{5.11}$$

where $\mathbf{k}$ is a unit vector in an arbitrary direction and $G(\mathbf{k})$ is an arbitrary function. Thus (5.11) is the most general solution of (5.1) obtainable by this method. There exists, however, a general theorem (see Titchmarsh (49), pp. 305–7), according to which the procedure used here gives every solution of (5.1) which does not increase faster than $e^{r/l'}$ (where l' is any constant greater than l), provided that the most general solution of (5.9) is taken (as we have done). On the other hand, a solution which increases faster than this will clearly violate the condition at infinity given in § 2.3.3.

The most general solution of (5.1) that satisfies the condition at infinity is therefore of the form (5.11), and this obviously satisfies the differential equation (5.4).

5.1.3. *Third method*

In this case we start from the integro-differential equation (4.4), whose solution can always be constructed as a superposition of plane waves. In an infinite source-free medium, all directions are equivalent, and it is sufficient to find the solution which depends, as regards position, on one Cartesian coordinate, x say. The solution will then depend, as regards direction, only on the angle made with the x-axis, whose cosine is denoted by μ. Integrating (4.4) over the azimuthal angle, we get for the present case

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\frac{\psi(x,\mu)}{l} = \frac{c}{2l}\int_{-1}^{1}\psi(x,\mu')\,d\mu'. \tag{5.12}$$

This is a linear equation in terms of x with constant coefficients, and the solution is therefore of the form

$$\psi(x,\mu) = e^{sx}g(\mu). \tag{5.13}$$

Substituting this into (5.12) and dividing through by e^{sx}, we have

$$\left(\mu s+\frac{1}{l}\right)g(\mu)=\frac{c}{2l}\int_{-1}^{1}g(\mu')\,d\mu',$$

and if this equation has a non-trivial solution it follows that

$$1=\frac{c}{2}\int_{-1}^{1}\frac{d\mu}{1+\mu sl}=\frac{c}{2sl}\log\frac{1+sl}{1-sl}. \tag{5.14}$$

On comparing (5.14) with (5.5) we see at once that $s=\pm 1/L$, and so the neutron flux corresponding to (5.13) is given by

$$\rho(x)=\text{constant}\times e^{\pm x/L},$$

which is a solution of (5.4). The neutron flux associated with any other plane wave should similarly be a solution of (5.4), and so the general solution of (5.1), which is a superposition of such waves, is also a solution of (5.4).

5.2. The diffusion length

The quantity L introduced in § 5.1 and defined by (5.5) is of great mportance in neutron transport theory and, if real, is called the *diffusion length.*

If c, the number of secondaries per collision, is close to unity, L may be expanded in powers of $(1-c)$, and we easily find from the power series preceding (5.5):

$$L=\frac{l}{\sqrt{3(1-c)^{\frac{1}{2}}}}\{1+\tfrac{2}{5}(1-c)+O[(1-c)^2]\} \tag{5.15a}$$

or

$$L=\frac{l}{\sqrt{(3c)(1-c)^{\frac{1}{2}}}}\left\{1-\frac{1}{10}\left(\frac{1-c}{c}\right)+O\left[\left(\frac{1-c}{c}\right)^2\right]\right\}. \tag{5.15b}$$

The latter of these two formulae is more convenient, since the numerical coefficient of the second term in the series is smaller, and so the leading term is a good approximation over a rather wider range of values of $(1-c)$. However, if $(1-c)$ is not small compared with unity, then neither formula is reliable, and (5.5) itself must be solved.

If fission is absent, so that only capture and scattering are possible, then (5.15 b) can be rewritten in a somewhat different form. The formulae of § 1.3.1 become (since the present $l\equiv l_{\text{tot}}$)

$$1/l=1/l_s+1/l_c, \qquad c=l/l_s,$$

and in terms of l_s and l_c (5.15 b) becomes

$$L = \sqrt{\left(\frac{l_s l_c}{3}\right)}\left\{1 - \frac{1}{10}\frac{l_s}{l_c} + O\left[\left(\frac{l_s}{l_c}\right)^2\right]\right\}. \tag{5.16 b}$$

The formula (5.15 a) similarly becomes

$$L = \sqrt{\left(\frac{l l_c}{3}\right)}\left\{1 + \frac{2}{5}\frac{l}{l_c} + O\left[\left(\frac{l}{l_c}\right)^2\right]\right\}. \tag{5.16 a}$$

If a sufficiently large proportion of fissile material is present in the medium, the mean number of secondaries c will be greater than unity. In this case the solution of (5.5) is easily seen to be purely imaginary. Instead of L we then define

$$\kappa = i/L, \tag{5.17}$$

and rewrite (5.4) as

$$\nabla^2\rho(\mathbf{r}) + \kappa^2\rho(\mathbf{r}) = 0. \tag{5.4'}$$

5.3. An isotropic point source in an infinite homogeneous medium

We now turn to the case where sources are present in an infinite homogeneous medium, and consider first an isotropic point source. The solution need be found only for neutron fluxes $\rho(\mathbf{r})$ vanishing at infinity; for the solution with an arbitrary behaviour at infinity may, in view of the linearity of the equations, be represented as the sum of the solution for a point source and zero flux at infinity, and the solution for no sources and the required behaviour at infinity, which has already been discussed.

If the position of the point source is taken as origin and its total strength is $4\pi S_0$, then the source distribution in (4.15) is $s(\mathbf{r}) = S_0\delta(\mathbf{r})$, and (4.15) becomes

$$\rho(\mathbf{r}) = \frac{c}{4\pi l}\iiint_{\text{all space}} \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/l}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + S_0\frac{e^{-r/l}}{r^2}, \tag{5.18}$$

with the boundary condition

$$\rho(\infty) = 0. \tag{5.19}$$

In (5.18), dV' is the volume element in $\mathbf{r}'$-space, and $r = |\mathbf{r}|$. The number of secondaries per collision c is assumed not to exceed unity, since otherwise, by (5.4'), the infinite medium is already supercritical, and according to § 3.2 there is no direct physical significance in the examination of this medium in the presence of sources. We shall, in fact, exclude for the moment the case $c = 1$ also.

We shall solve (5.18) by using the Fourier transform technique as in § 5.1.2. The equation (5.18) is first subjected to a Fourier transformation,

i.e. is multiplied by $e^{i\mathbf{r}.\mathbf{p}}$ and integrated over all r-space, putting

$$F(\mathbf{p}) = \iiint_{\text{all space}} \rho(\mathbf{r}) e^{i\mathbf{r}.\mathbf{p}}\, dV_r. \tag{5.20}$$

The condition (5.19) ensures the convergence of this integral, provided that $c < 1$. Using (5.7) and the Fourier transform convolution theorem, we then find

$$F(\mathbf{p}) = (c/pl)\tan^{-1}pl\, F(\mathbf{p}) + (4\pi S_0/p)\tan^{-1}pl,$$

and hence

$$F(\mathbf{p}) = \frac{4\pi S_0 l \tan^{-1}pl}{pl - c\tan^{-1}pl}. \tag{5.21}$$

The Fourier transform inversion theorem then gives

$$\rho(\mathbf{r}) = \frac{S_0 l}{2\pi^2} \iiint_{\text{all space}} \frac{\tan^{-1}pl}{pl - c\tan^{-1}pl} e^{-i\mathbf{p}.\mathbf{r}}\, dV_p. \tag{5.22}$$

The integral in (5.22) does not converge in the ordinary sense, but oscillates; that is, if the integral is extended not over all p-space but over a large sphere of radius P, say, then the integral is an oscillating function of P as P tends to infinity. There is, however, a mean value about which the oscillation takes place, and by the Fourier inversion theorem this is the value to be used for the integral. Such a procedure is called interpreting the integral *in the Cesàro sense*.

Thus, integrating (5.22) over all angles in p-space, we have

$$\rho(\mathbf{r}) = \frac{2S_0}{\pi r} \int_0^\infty \frac{\tan^{-1}pl}{1 - (c/pl)\tan^{-1}pl} \sin pr\, dp,$$

or, equivalently,

$$\rho(\mathbf{r}) = \frac{S_0}{i\pi r} \int_{-\infty}^{\infty} \frac{\tan^{-1}pl}{1 - (c/pl)\tan^{-1}pl} e^{ipr}\, dp, \tag{5.23}$$

where the integrali taken in the Cesàro sense, i.e. it is evaluated between the limits $-P$ and P, where P is large, and the average over P is taken. The evaluation is effected by deforming the path of integration. The singularities of the integrand in (5.23) are

(i) the poles at the two roots of

$$1 - (c/pl)\tan^{-1}pl = 0,$$

i.e. at the points

$$p = \pm i/L, \tag{5.24}$$

where L is defined by (5.5). Since c is assumed less than unity, L is real,

and therefore the poles (5.24) are on the imaginary axis, one in the upper and one in the lower half-plane;

(ii) the cuts along the imaginary axis from $p = i/l$ to $p = i\infty$ and from $p = -i/l$ to $p = -i\infty$, with the values of $\tan^{-1}pl$ on the two sides of the cut differing by π.

Thus the integral (5.23) taken along the real axis from $-P$ to P will be equal to the integral taken along the following path (see Fig. 3): from $p = -P$ along the arc $p = Pe^{i\theta}$ to $p = iP$; from $p = iP$ along the imaginary axis, left of the cut, to $p = i/l$; from $p = i/l$ along the imaginary axis, right of the cut, to $p = iP$; from $p = iP$ along the arc $p = Pe^{i\theta}$ to $p = P$, plus the integral anticlockwise round a small circle about the point $p = i/L$. The integrals along the two arcs of $p = Pe^{i\theta}$ are oscillating functions of P, but it is easy to see that their sum oscillates about a mean value tending to zero as P tends to infinity. Their contribution to the integral (5.23) taken in the Cesàro sense is therefore zero. On passing to the limit $P \to \infty$ and calculating the residue at $p = i/L$ and the integrals along the cut, we find

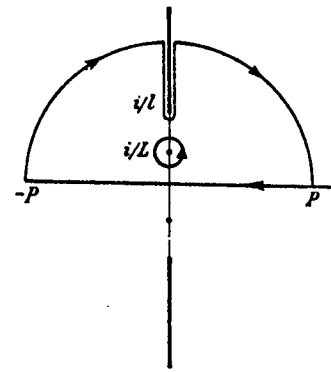


FIG. 3.

$$\rho(r) = \frac{S_0}{rl}\left\{\frac{2l^2(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}e^{-r/L} + \right.$$
$$\left. + \int_1^\infty e^{-rs/l}\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}. \quad (5.25)$$

The case $c = 1$ is dealt with by passing to the limit as $c \to 1$ and using (5.15); thus

$$[\rho(r)]_{c=1} = \frac{S_0}{rl}\left\{3+\int_1^\infty e^{-rs/l}\left[1+\frac{1}{s}\log\frac{s-1}{s+1}+\frac{1}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}. \quad (5.26)$$

It may be noticed here that the use of Cesàro summation could have been avoided, at the cost of more laborious algebra, by putting

$$\rho(\mathbf{r}) = (S_0/r^2)e^{-r/l}+\rho_1(\mathbf{r}),$$

and applying the Fourier transform to the result of substituting this into

(5.18). Instead of (5.22) we should then have

$$\rho(\mathbf{r}) = \frac{S_0}{r^2}e^{-r/l} + \frac{S_0 cl}{2\pi^2}\iiint_{\text{all space}} \frac{(\tan^{-1}pl)^2}{pl(pl - c\tan^{-1}pl)}e^{-i\mathbf{p}.\mathbf{r}}\,dV_p,$$

which leads to (5.25) as before.

The values of $\rho(r)$ derived from these formulae have been given by Bothe (5), and Fig. 4 shows his values of $r\rho_1(r)\times\pi l/2cS_0$.

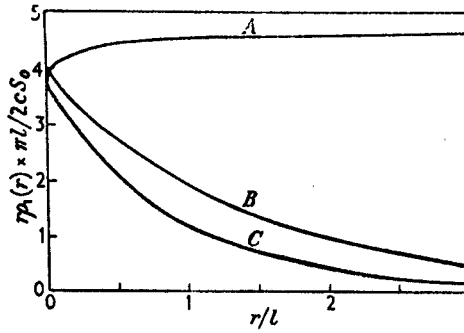


FIG. 4. *A*, *B*, and *C* are for $c = 1$, 0·81, and 0·5 respectively.

Using (5.5) and/or (5.15), we see that, except for $c \ll 1$ (or the physically impossible case $c \gg 1$), the coefficient

$$\frac{2l^2(L^2 - l^2)}{cL^2(cL^2 + l^2 - L^2)}$$

is comparable with unity. Thus the first term of (5.25) is of the order of $e^{-r/L}$, while for $r \gg l$ the second term is seen to be of the order of $e^{-r/l}$. Hence, for $r \gg l$, the second term in (5.25) and (5.26) can be neglected in the first approximation, and the remaining term is, for $r \neq 0$, a solution of (5.4). This means that $\rho(r)$ can be represented in the form†

$$\rho(r) = \rho_{as}(r) + \rho_{tr}(r), \tag{5.27}$$

† It must be remarked, however, that the distinction between $\rho_{as}(r)$ and $\rho_{tr}(r)$ is significant only if c is comparatively close to unity, since only in this case does $\rho_{as}(r)$ become predominant at reasonably small distances. If c is small, $\rho_{as}(r)$ does not become predominant except at very large distances. Case, de Hoffmann, and Placzek (8, p. 106) have made a comparison of $\rho_{as}(r)$ and $\rho_{tr}(r)$ for various values of c and r/l:

$c = 1$	$r/l = 1$	$\rho_{tr}(r)/\rho_{as}(r) = 0{\cdot}085$
	2·5	0·007
0·5	1	0·62
	12	0·10
	20	0·043
0·3	1	0·93
	10	0·65
	20	0·54
0·1	20	5×10^{-4}

where $\rho_{as}(r)$ is, for $r \neq 0$, a solution of (5.4) and

$$\rho_{tr}(r) = O(e^{-r/l}). \tag{5.28}$$

Since the equation (5.4) is identical in form with the equation of diffusion, $\rho_{as}(r)$ is sometimes denoted by $\rho_{diff}(r)$.

5.4. An anisotropic point source in an infinite homogeneous medium

The treatment given above can easily be extended to the case of an anisotropic point source. We shall not work out this case in detail, but we shall outline two possible methods of solution and then prove that the formula (5.27) is applicable in this case also. The first method is the more useful in practice, but the second makes easier the proof of (5.27).

In the first method, the neutron flux due to the source is represented as the sum of two terms, the flux of neutrons coming directly from the source and the flux of neutrons that have had at least one collision. The former is given by

$$(1/r^2)e^{-r/l}S_0(\mathbf{r}/r),$$

where $S_0(\Omega)\,d\Omega$ is the flux of neutrons emitted by the anisotropic point source into the solid angle $d\Omega$ about Ω, while the latter term can be regarded as the neutron flux due to a distributed system of isotropic sources, the source strength per unit volume and solid angle being given by

$$(c/4\pi r^2 l)e^{-r/l}S_0(\mathbf{r}/r).$$

Using the results of § 5.3 and combining the two contributions, we have

$$\rho(\mathbf{r}) = \iiint_{\text{all space}} \frac{dV_0}{r_0^2} e^{-r_0/l} S_0\left(\frac{\mathbf{r}_0}{r_0}\right)\left\{\delta(\mathbf{r}-\mathbf{r}_0)+\frac{c}{4\pi l}\rho_{iu}(|\mathbf{r}-\mathbf{r}_0|)\right\}, \tag{5.29}$$

where dV_0 is the volume element in $\mathbf{r}_0$-space; ρ_{iu} is the flux due to an isotropic point source of unit strength, and is given by (5.25) with S_0 equal to unity.

In the second method, we start from the integral equation for the neutron flux in an infinite homogeneous medium, with isotropic scattering and an anisotropic point source at the origin emitting a flux $S_0(\Omega)\,d\Omega$ into the solid angle $d\Omega$ about Ω; this is

$$\rho(\mathbf{r}) = \frac{c}{4\pi l}\iiint_{\text{all space}} \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/l}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + \frac{1}{r^2}e^{-r/l}S_0\left(\frac{\mathbf{r}}{r}\right). \tag{5.30}$$

This can be seen either from (5.18), since the equations for isotropic source and anisotropic source can differ only in the term due to neutrons

coming directly from the source, which is the free term; or from (4.20), which for isotropic scattering and an infinite homogeneous medium becomes

$$\psi(\mathbf{r},\Omega) = \int_0^\infty dR e^{-|\mathbf{r}-(\mathbf{r}-R\Omega)|/l}\left\{\frac{c}{4\pi l}\int\int \psi(\mathbf{r}-R\Omega,\Omega')\,d\Omega' + s(\mathbf{r}-R\Omega,\Omega)\right\}.$$

On integrating this over all Ω, putting $\mathbf{r}' = \mathbf{r}-R\Omega$ and using

$$s(\mathbf{r},\Omega) = \delta(\mathbf{r})S_0(\Omega),$$

we obtain (5.30).

As before, we assume the boundary condition (5.19) satisfied, and $c < 1$. If now $S_0(\Omega)$ is expanded in spherical harmonics, so that $S_0(\Omega) = \sum S_{0m}(\Omega)$, say, where S_{0m} is a spherical harmonic of order m, then the solution of (5.30) will be $\rho(\mathbf{r}) = \sum \rho_m(\mathbf{r})$, where $\rho_m(\mathbf{r})$ is the solution of (5.30) with $S_0(\mathbf{r}/r)$ replaced by $S_{0m}(\mathbf{r}/r)$. It is therefore necessary to solve (5.30) only when S_0 is a spherical harmonic S_{0m} of some order m, so that

$$S_{0m}(\mathbf{r}/r) = r^{-m}p_m(x,y,z),$$

where $p_m(x,y,z)$ is a polynomial of order m satisfying the equation $\nabla^2 p = 0$. It is known that, if $T(r)$ is any m times differentiable function of r only, then

$$p_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)T(r) = p_m(x,y,z)\frac{d^m T(r)}{d(\tfrac{1}{2}r^2)^m}, \tag{5.31}$$

where $p_m(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ is obtained from $p_m(x,y,z)$ by replacing x by $\partial/\partial x$, etc. This formula is proved in (23), p. 127.

We take for $T_m(r)$ that solution of the differential equation

$$\frac{d^m T_m(r)}{d(\tfrac{1}{2}r^2)^m} = \frac{e^{-r/l}}{r^{m+2}} \tag{5.32}$$

which satisfies the boundary condition

$$\lim_{r\to\infty}[e^{r/l}T_m(r)] = 0, \tag{5.33}$$

so that the free term of (5.30) becomes

$$\frac{1}{r^2}e^{-r/l}S_{0m}\left(\frac{\mathbf{r}}{r}\right) = p_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)T_m(r);$$

and we consider the auxiliary equation

$$g_m(r) = \frac{c}{4\pi l}\iiint_{\text{all space}} g_m(r')e^{-|\mathbf{r}-\mathbf{r}'|/l}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + T_m(r). \tag{5.34}$$

This can be solved in the same way as (5.18), since, though the functions $T_m(r)$ for $m \geqslant 1$ are not themselves elementary, their Fourier transforms are.† The solution of (5.34) therefore offers no new difficulties compared with that of (5.18). But substituting in (5.34) $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$ and operating on the equation with $p_m(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ shows that

$$p_m(\partial/\partial x, \partial/\partial y, \partial/\partial z) g_m(r)$$

gives the solution of (5.30) with $S_0(\mathbf{r}/r)$ replaced by $S_{0m}(\mathbf{r}/r)$. The solution of (5.30) is therefore

$$\rho(\mathbf{r}) = \sum_m p_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) g_m(r). \tag{5.35}$$

We now prove from (5.34) that the result (5.27) holds also for the case of an anisotropic source. The derivation of (5.27) rested primarily on the kernel of (5.18), which is the same as that of (5.34); the only two properties of the free term of (5.18) used in the derivation of (5.27) were:

(i) it depends only on the radial coordinate r, so that its Fourier transform depends only on p, and is an even function of p;

(ii) for large r it behaves as $e^{-r/l}$, so that its Fourier transform is regular in the strip $-1/l < \operatorname{im} p < 1/l$.

The free term of (5.34) possesses both these properties, the second being ensured by the boundary condition (5.33). It follows that the solution of (5.34) can be represented, similarly to (5.27), in the form

$$g_m(r) = g_{m,as}(r) + g_{m,tr}(r), \tag{5.36}$$

where $g_{m,as}$ satisfies the equation $(\nabla^2 - 1/L^2) g_{m,as} = 0$, while $g_{m,tr}(r)$ is of the order of $e^{-r/l}$. Operating on (5.36) with $p_m(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ and using (5.35), we see that (5.27) holds also in the case of an anisotropic source, provided that the series $\sum S_{0m}(\Omega)$ converges, i.e. $S_0(\Omega)$ does not contain a delta function.

From this extension of (5.27) we can draw some conclusions about the neutron distribution in finite media. Let a homogeneous medium extend over a volume V bounded by a surface S, while outside S the medium has different properties or is absent altogether. Let $\psi(\mathbf{r},\Omega)$ be the distribution of neutrons actually present in the system, and let $\psi'(\mathbf{r},\Omega)$ be a possible angular distribution of neutrons in an infinite medium with

† This is proved by integrating the expression for the Fourier transform of T_m m times by parts, using (5.32), expanding the result in powers of the transform variable, and using the expressions for the moments of Legendre functions over the range 0 to 1, $\int_0^1 \mu^n P_m(\mu)\, d\mu$.

the same values of l and c as are found in V. Let $\epsilon(\mathbf{r},\Omega)$ be defined by

$$\epsilon(\mathbf{r},\Omega) = \begin{cases} 1 \text{ for } \mathbf{r} \text{ on } S \text{ and } \Omega \text{ pointing into } V, \\ -1 \text{ for } \mathbf{r} \text{ on } S \text{ and } \Omega \text{ pointing out of } V, \\ 0 \text{ otherwise,} \end{cases}$$

and consider the angular distribution of neutrons in the infinite homogeneous medium, satisfying the same conditions at infinity as $\psi'(\mathbf{r},\Omega)$, but in the presence of independent neutron sources given by

$$s(\mathbf{r},\Omega) = [\psi(\mathbf{r},\Omega) - \psi'(\mathbf{r},\Omega)]\epsilon(\mathbf{r},\Omega). \tag{5.37}$$

It is readily seen that the resulting angular distribution $\psi''(\mathbf{r},\Omega)$ will coincide with $\psi(\mathbf{r},\Omega)$ inside V and with $\psi'(\mathbf{r},\Omega)$ outside V; for with $\psi'' = \psi'$ outside V, the number of neutrons entering V from outside, added to the contribution of the sources (5.37), will equal that entering V in the actual case of a finite medium; while if $\psi'' = \psi$ inside V, the number of neutrons leaving V, added to the source contribution, will equal that leaving V when the sources are absent and the distribution in V is ψ'.

Thus the neutron angular distribution, and therefore also the flux, in any finite region bounded by a surface S, are equal to those in an infinite region when a suitable system of anisotropic sources is placed on the surface S. Further, by (5.37), for fixed $\mathbf{r}$ these sources are not delta functions of Ω, and so the expansion of $s(\mathbf{r},\Omega)$ in spherical harmonics in Ω will converge, and the formula (5.27) is applicable. Therefore, in any region, finite or not, in which l and c are constant, the neutron flux in the constant cross-section approximation can be represented in the form

$$\rho(\mathbf{r}) = \rho_{\text{diff}}(\mathbf{r}) + \rho_{\text{tr}}(\mathbf{r}), \tag{5.38}$$

where $\rho_{\text{diff}}(\mathbf{r})$ satisfies (5.4) and

$$\rho_{\text{tr}}(\mathbf{r}) = O(e^{-d/l}), \tag{5.39}$$

where d is the distance from $\mathbf{r}$ to the nearest source or boundary.

The first term in (5.38) we call $\rho_{\text{diff}}(\mathbf{r})$ rather than $\rho_{\text{as}}(\mathbf{r})$, since for finite media one cannot properly speak of asymptotic solutions.

5.5. An isotropic line or plane source in an infinite homogeneous medium

5.5.1. *The line source*

Using the availability of the point source solution, it is easy to construct solutions for line, plane, and distributed sources. We first consider a line source of unit strength per unit length and suppose for the moment that $c < 1$. Let r now denote the distance of $\mathbf{r}$ from the line source,

$\rho_l(r)$ be the neutron flux at distance r from the line, and z be measured along the line. Then clearly

$$\rho_l(r) = \int_{-\infty}^{\infty} \rho_{pt}(R)\, dz, \tag{5.40}$$

where R is the distance from the point r to the line element dz, i.e. with a suitable choice of origin for z, $r^2+z^2 = R^2$, and $\rho_{pt}(r)$ is the isotropic point source solution determined in § 5.3. Since the strength of the line source is unity per unit length when integrated over all directions, in $\rho_{pt}(r)$ we must take $S_0 = 1/4\pi$. Substituting (5.25) into (5.40) and using R as the variable of integration, we find

$$\rho_l(r) = \frac{1}{2\pi l}\int_r^{\infty} \frac{dR}{\sqrt{(R^2-r^2)}}\left\{\frac{2l^2(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}\, e^{-R/L} + \right.$$

$$\left. + \int_1^{\infty} e^{-Rs/l}\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}.$$

Putting $R = xr$ and using the result

$$\int_1^{\infty} \frac{e^{-xy}}{\sqrt{(x^2-1)}}\, dx = K_0(y),$$

where K_0 is the modified Bessel function of the second kind of zero order, we have

$$\rho_l(r) = \frac{1}{2\pi l}\left\{\frac{2l^2(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}\, K_0\left(\frac{r}{L}\right) + \right.$$

$$\left. + \int_1^{\infty} K_0(rs/l)\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}. \tag{5.41}$$

The solution for $c = 1$ cannot be obtained by proceeding to the limit in (5.41), since for $c \to 1$, $L \to \infty$ and $K_0(r/L)$ becomes infinite. However, if the boundary condition $\rho_l(\infty) = 0$ is replaced by the less strict condition that $\rho_l(r)$ is a function only of r, then, since $I_0(r/L)$ is a solution of $\nabla^2\rho = \rho/L^2$ regular in all space,

$$\tilde{\rho}_l(r) = \frac{1}{2\pi l}\left\{\frac{2l^2(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}\left[K_0\left(\frac{r}{L}\right) - AI_0\left(\frac{r}{L}\right)\right] + \right.$$

$$\left. + \int_1^{\infty} K_0(rs/l)\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}, \tag{5.42}$$

where A is a constant, is also a possible solution for a line source of unit strength for $c < 1$.

For small x,

$$K_0(x) = \log(2/x) - \gamma + O(x^2 \log x)$$

and

$$I_0(x) = 1 + O(x^2),$$

where γ is Euler's constant. Thus, if we choose

$$A = \log(2L/l) - \gamma - A',$$

where A' is another constant, independent of c, substitute in (5.42) and pass to the limit $c \to 1$, we obtain

$$[\rho_l(r)]_{c=1} = \frac{1}{2\pi l}\left\{3\log\frac{l}{r} + A' + \right.$$

$$\left. + \int_1^\infty K_0(rs/l)\left[1 + \frac{1}{s}\log\frac{s-1}{s+1} + \frac{1}{4s^2}\left(\pi^2 + \log^2\frac{s-1}{s+1}\right)\right]^{-1} ds\right\}. \quad (5.43)$$

It is evident that, whatever the value of A', if it is finite, (5.43) becomes negative for r sufficiently large. A solution positive for $r < a$ (say) can, however, always be achieved by putting $A' = 3\log(a/l)$.

This shows that a stationary neutron distribution is impossible in an infinite non-capturing medium containing an infinitely long line source of constant strength per unit length. In practice, however, the system will be finite, the infinite line being introduced as an approximation. In this case, the flux beyond the boundary of the system will be irrelevant, and a stationary solution is possible.

5.5.2. *The plane source*

We now consider a plane source of unit strength per unit area and again assume that $c < 1$. Let $\rho_{pl}(x)$ be the neutron flux at distance x from the plane, R the distance from the point of measurement to some element of area of the plane, and r the projection of R on the plane. Similarly to (5.40) we have†

$$\rho_{pl}(x) = 2\pi \int_0^\infty r\, dr\, \rho_{pl}(R), \quad (5.44)$$

where $R^2 = r^2 + x^2$.

† The converse of (5.44) is

$$\rho_{pl}(r) = -\frac{1}{2\pi r}\left[\frac{\partial \rho_{pl}(x)}{\partial x}\right]_{x=r}.$$

Thus, if the flux distribution due to either a point source or a plane source is known, the other can be found. These formulae are valid generally, and not only for the constant cross-section approximation.

Again substituting (5.25) with $S_0 = 1/4\pi$ for $\rho_{pl}(r)$, and using R as the variable of integration, we find

$$\rho_{pl}(x) = \frac{1}{2l}\left\{\frac{2l^2(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}\int_{|x|}^{\infty}\frac{R\,dR}{R}e^{-R/L} + \right.$$

$$\left. + \int_1^{\infty}\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1}ds\int_{|x|}^{\infty}\frac{R\,dR}{R}e^{-Rs/l}\right\},$$

that is,

$$\rho_{pl}(x) = \frac{1}{2}\left\{\frac{2l(L^2-l^2)}{cL(cL^2+l^2-L^2)}e^{-|x|/L} + \right.$$

$$\left. + \int_1^{\infty}\frac{e^{-|x|s/l}}{s}\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1}ds\right\}. \quad (5.45)$$

To deal with the case $c = 1$ it is again necessary to replace the condition $\rho_{pl}(\pm\infty) = 0$ by the less strict requirement that $\rho_{pl}(x)$ is a function of $|x|$ only. In this case

$$\tilde{\rho}_{pl}(x) = \frac{1}{2}\left\{\frac{2l(L^2-l^2)}{cL(cL^2+l^2-L^2)}\left(e^{-|x|/L}-A\cosh\frac{x}{L}\right) + \right.$$

$$\left. + \int_1^{\infty}\frac{e^{-|x|s/l}}{s}\left[1+\frac{c}{s}\log\frac{s-1}{s+1}+\frac{c^2}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1}ds\right\} \quad (5.46)$$

is also a permissible solution, and choosing $A = 1-A'l/L$, where A' is independent of c, substituting in (5.46) and passing to the limit $c \to 1$ and thus $L \to \infty$, we have

$$[\rho_{pl}(x)]_{c=1} = \frac{1}{2}\left\{3\left(A'-\frac{|x|}{l}\right) + \right.$$

$$\left. + \int_1^{\infty}\frac{e^{-|x|s/l}}{s}\left[1+\frac{1}{s}\log\frac{s-1}{s+1}+\frac{1}{4s^2}\left(\pi^2+\log^2\frac{s-1}{s+1}\right)\right]^{-1}ds\right\}. \quad (5.47)$$

As with (5.43), this expression will become negative for sufficiently large $|x|$, whatever the value of A'. The consequences of this are the same as were stated regarding the equation (5.43).

On comparing (5.45) with (5.25), we see that, whereas in the latter the relative error involved in replacing $\rho(\mathbf{r})$ by $\rho_{\text{diff}}(\mathbf{r})$ was of the order of $e^{-r/l}$ for $r \gg l$, the corresponding error for a plane source is of the order of $(l/L)e^{-|x|/l}$, so that $\rho(\mathbf{r})$ can be justifiably replaced by $\rho_{\text{diff}}(\mathbf{r})$ much

nearer to a plane source than to a point source. The same will clearly be true for an anisotropic plane source also, and consequently for plane boundaries; and the method of derivation shows that, even for any boundary of considerable lateral extent and small curvature, $\rho(\mathbf{r})$ may be replaced by $\rho_{\text{diff}}(\mathbf{r})$ at quite small distances. Furthermore, it can be shown that, if there is a system of isotropic sources distributed uniformly in a plane layer of finite thickness $2a$, say, then provided that

$$L/l \gg \log(l/a)$$

(that is, unless a is very small), the error involved in replacing $\rho(\mathbf{r})$ by $\rho_{\text{diff}}(\mathbf{r})$ will be negligible not only far from the source layer, but also up to and even inside it.

VI

EXACT SOLUTIONS FOR AN INFINITE HALF-SPACE

6.1. Milne's problem

THE theorem proved at the end of § 5.4, together with the discussion at the end of the last section, shows that in dealing with a domain whose dimensions are very much larger than the mean free path we shall obtain an approximate solution of (4.15) (without the source term) by solving instead the differential equation (5.4) with the appropriate boundary conditions. These conditions, however, cannot be found in an elementary manner; that is, they will not be the boundary conditions that would be imposed on (5.4) if it were valid up to the boundary. In order to find out something about the necessary boundary conditions, we shall now solve some problems which involve various kinds of boundary but are otherwise simple enough to permit an exact solution.

We first consider *Milne's problem*. This consists in determining the neutron flux in a homogeneous source-free medium occupying the half-space $x \geqslant 0$, the half-space $x < 0$ being a vacuum and $x = 0$ being a *free surface* (see § 2.3.2); that is, no neutrons fall on this surface from outside the medium. The flux is assumed to depend only on the coordinate x which measures the distance normally into the medium. For this case, the source-free form of equation (4.15) is

$$\rho(\mathbf{r}) = \frac{c}{4\pi l} \int\int\int_{x'>0} \rho(\mathbf{r}') e^{-|\mathbf{r}-\mathbf{r}'|/l} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2},$$

or, putting $\rho(\mathbf{r}) = \rho(x)$ and integrating over the planes $x' =$ constant,

$$\rho(x) = \frac{c}{2l} \int_0^\infty \rho(x') E_1\left(\frac{|x-x'|}{l}\right) dx', \tag{6.1}$$

where $E_1(z)$ is defined by

$$E_1(z) = \int_1^\infty \frac{e^{-zt}\, dt}{t}. \tag{6.2}$$

We shall now solve (6.1) by the Wiener–Hopf method of Fourier transforms (see Titchmarsh (49), p. 339). For this purpose we extend the definition of $\rho(x)$ into the region of negative x, by assuming the formula (6.1) to be valid for x either positive or negative. The $\rho(x)$ so defined has, of course, no relation to the flux actually found in $x < 0$ in

the conditions of the actual problem. Its physical significance is found by replacing the vacuum in $x < 0$ by a medium with $c = 0$ (i.e. all collisions in it result in capture) and the same l as for $x \geqslant 0$. The solution for $x \geqslant 0$ is unaltered, while equation (6.1) now holds for all x.

We now introduce two Fourier transforms,

$$F^+(p) = \int_0^\infty e^{ipx}\rho(x)\,dx, \qquad F^-(p) = \int_{-\infty}^0 e^{ipx}\rho(x)\,dx; \tag{6.3}$$

that is, $F^+(p)$ is the Fourier transform of a function which equals $\rho(x)$ for $x \geqslant 0$ and vanishes for $x < 0$, while $F^-(p)$ is that of a function which equals $\rho(x)$ for $x \leqslant 0$ and vanishes for $x > 0$. The equation is multiplied by e^{ipx} and integrated over all x. After simple transformations resembling those of Chapter V, we find

$$F^+(p)+F^-(p) = (c/pl)\tan^{-1}pl\,F^+(p),$$

or

$$F^+(p)\left(1-c\frac{\tan^{-1}pl}{pl}\right) = -F^-(p). \tag{6.4}$$

The equation (6.4) involves two unknown functions, but there are further conditions to be imposed on them. It follows from (6.1) that, for $x \to -\infty$, $\rho(x)$ should behave approximately as $e^{-|x|/l}$. The integral (6.3) defining $F^-(p)$ will then converge for all $\operatorname{im} p < 1/l$, including the whole lower half-plane, and will converge uniformly for all $\operatorname{im} p < (1-\epsilon)/l$, where ϵ is any real positive number. Thus $F^-(p)$ is regular in the half-plane $\operatorname{im} p < 1/l$. Similarly, it follows from the discussion in Chapter V that, for $x \to +\infty$, $\rho(x)$ cannot increase faster than $e^{x/L}$, and therefore $F^+(p)$ should be regular in the half-plane $\operatorname{im} p > \operatorname{re} 1/L$. Further, by a general property of Fourier transforms, $F^+(p)$ and $F^-(p)$ should be quadratically summable along paths parallel to the real axis in the half-planes in which they are regular. That is, the integrals

$$\int_{-\infty+is_-}^{\infty+is_-} |F^-(p)|^2\,dp \quad \text{and} \quad \int_{-\infty+is_+}^{\infty+is_+} |F^+(p)|^2\,dp \tag{6.5}$$

should converge for any $s_- < 1/l$ and $s_+ > \operatorname{re} 1/L$.

The problem is therefore reduced to that of representing

$$1-(c/pl)\tan^{-1}pl$$

as a ratio of two functions satisfying the above regularity conditions. It is known that the representation of a function as the sum or difference of two functions, one regular in each half-plane, can be effected by using the Cauchy integral. Our problem, however, is nothing more than representing the logarithm of a function as the difference of the logarithms

of two such functions. There is therefore no difficulty, apart from the point that at a zero of a function its logarithm is singular, even though the function itself is regular. It has been pointed out in § 5.1.2 that $1-(c/pl)\tan^{-1}pl$ has only two zeros in the complex p-plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l$ and from $p = i/l$ to $p = i\infty$; these are at $p = \pm i/L$, where L is the solution of (5.5). Thus

$$\frac{1}{p^2+1/L^2}\left(1-c\frac{\tan^{-1}pl}{pl}\right)$$

has no zeros for finite p in the cut plane. For $p \to \infty$ this expression is zero, but this can be dealt with by multiplying it by (p^2+1/l^2); the logarithm of the result is

$$\log\left\{\frac{p^2+1/l^2}{p^2+1/L^2}\left(1-c\frac{\tan^{-1}pl}{pl}\right)\right\},$$

and this has no zeros in the cut plane and tends to zero for $p \to \infty$. Putting then

$$\frac{p^2+1/l^2}{p^2+1/L^2}\left(1-c\frac{\tan^{-1}pl}{pl}\right) = \frac{h^+(pl)}{h^-(pl)}, \tag{6.6}$$

where $h^+(pl)$ is regular and non-zero for $\operatorname{im} p > 0$, and $h^-(pl)$ is regular and non-zero for $\operatorname{im} p < 0$, and both tend to unity when $|p| \to \infty$, the functions h^+ and h^- can be determined at once by means of the Cauchy integral.

For this, we consider the integral

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dt}{t-p}\log\left[\frac{t^2+1/l^2}{t^2+1/L^2}\left(1-c\frac{\tan^{-1}tl}{tl}\right)\right], \tag{6.7}$$

where the integration is along the real axis. By (6.6), this equals

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dt}{t-p}\log h^+(tl)+\frac{1}{2\pi i}\int_{\infty}^{-\infty}\frac{dt}{t-p}\log h^-(tl). \tag{6.8}$$

According to the assumed properties of $h^+(tl)$, the integral

$$\int_{T}^{-T}\frac{dt}{t-p}\log h^+(tl),$$

taken along the arc $t = Te^{i\theta}$ for $0 < \theta < \pi$, tends to zero as $T \to \infty$. The first integral in (6.8) can therefore be regarded as taken along the closed contour shown in Fig. 5*a*, and similarly the second integral can be regarded as taken along the closed contour shown in Fig. 5*b*.

If now $\operatorname{im} p > 0$, the integrand in the first term has just one singularity inside the contour of Fig. 5 *a*, namely a simple pole at $t = p$, so that the first term in (6.8) is equal to $\log h^+(pl)$. The integrand in the second term is regular inside the contour of Fig. 5 *b*, so that the second term in (6.8) vanishes. Thus, for $\operatorname{im} p > 0$, (6.8) is equal to $\log h^+(pl)$. Similarly, for

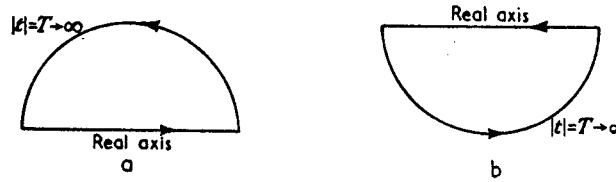


FIG. 5.

$\operatorname{im} p < 0$ it equals $\log h^-(pl)$. Since (6.6) is regular in the cut plane, it follows by deforming the path of integration that

$$\frac{1}{2\pi i}\int_{|t|=\infty,\,\mathrm{re}\,t<0}^{|t|=\infty,\,\mathrm{re}\,t>0} \frac{dt}{t-p}\log\left[\frac{t^2+1/l^2}{t^2+1/L^2}\left(1-c\frac{\tan^{-1}tl}{tl}\right)\right] = \begin{cases}\log h^+(pl) \text{ if the path is below } t=p,\\ \log h^-(pl) \text{ if the path is above } t=p,\end{cases} \tag{6.9}$$

for any path of integration that does not intersect itself or the cuts. This result shows that $h^+(pl)$ is regular in the entire plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l$, and $h^-(pl)$ is regular in the entire plane cut from $p = i/l$ to $p = i\infty$.

By substituting $-t$ for t in (6.9) we find the relation

$$h^-(pl) = 1/h^+(-pl), \tag{6.10}$$

and combining this with (6.6) we obtain also

$$\frac{p^2+1/l^2}{p^2+1/L^2}\left(1-c\frac{\tan^{-1}pl}{pl}\right) = h^+(pl)h^+(-pl), \tag{6.11}$$

a relation which will be of use later.

We now continue with the determination of $F^+(p)$. Substituting (6.6) into (6.4) and eliminating the denominators, we have

$$F^+(p)h^+(pl)(p^2+1/L^2) = -F^-(p)h^-(pl)(p^2+1/l^2). \tag{6.12}$$

Each factor on the right of (6.12) is regular for $\operatorname{im} p < 1/l$, and each factor on the left is regular for $\operatorname{im} p > \operatorname{re} 1/L$, where $\operatorname{re} 1/L < 1/l$, as follows from (5.5). Both sides of (6.12) are therefore regular everywhere.

Further, $F^-(p)$ and $h^-(pl)$ are regular at $p = -i/l$, hence the right side of (6.12) is zero there; the left side is therefore also zero, and so

$$\frac{p^2+1/L^2}{p+i/l} F^+(p)h^+(pl) \quad \text{and} \quad -F^-(p)h^-(pl)(p-i/l) \tag{6.13}$$

are themselves regular everywhere. Since $F^+(p)$ is quadratically summable in the upper half-plane, it tends to zero as $|p| \to \infty$ with $\operatorname{im} p > \operatorname{re} 1/L$; $h^+(p)$ tends to unity and $(p^2+1/L^2)/(p+i/l)$ behaves as p, for $|p| \to \infty$. Thus, for $\operatorname{im} p > \operatorname{re} 1/L$, the first expression (6.13) increases more slowly than $|p|$, if at all. Similarly, for $|p| \to \infty$ with $\operatorname{im} p < 1/l$, the second expression (6.13) increases more slowly than $|p|$, if at all. The two expressions being equal, it follows that for any $\operatorname{im} p$ they increase slower than $|p|$, if at all. They are therefore equal to a constant, by Liouville's theorem. Therefore

$$F^+(p) = \frac{A(p+i/l)}{p^2+1/L^2} \frac{1}{h^+(pl)},$$

where A is some constant. To determine A, we take $p = iM$ and let M tend to infinity. Since $h^+(i\infty) = 1$, the above formula gives

$$F^+(iM) = A/iM + \text{smaller terms},$$

while from the definition (6.3) we have

$$F^+(iM) = \rho(0)/M + \text{smaller terms}.$$

We shall see later that the 'smaller terms' are of the order of $(\log M)/M^2$, since the derivative of $\rho(x)$ has a logarithmic singularity at $x = 0$.

Thus $A = i\rho(0)$, and the expression for $F^+(p)$ becomes

$$F^+(p) = \frac{i\rho(0)(p+i/l)}{p^2+1/L^2} \frac{1}{h^+(pl)}. \tag{6.14}$$

Taking the inverse Fourier transform, we obtain the neutron flux:

$$\rho(x) = \frac{i\rho(0)L^2}{2\pi l} \int_{-\infty+it}^{\infty+it} \frac{pl+i}{p^2L^2+1} e^{-ipx} \frac{dp}{h^+(pl)} \quad (x > 0), \tag{6.15}$$

where $t > \operatorname{re} 1/L$. Mark (34) has utilized this result to evaluate $\rho(x)$ for small x.

The angular distribution also is often of interest, and in particular the emergent distribution $\psi(0, \mu)$. For this, comparing (6.3) and (4.16), we find, since there are no sources present,

$$\psi(0,\mu) = \frac{c}{4\pi l} \int_0^\infty \rho(-R\mu)e^{-R/l}\, dR = \frac{c}{4\pi l|\mu|} F^+(i/|\mu|l) \quad (\mu < 0), \tag{6.16}$$

and this gives the physical interpretation of $F^+(p)$. Combining (6.16) and (6.14) we also have

$$\psi(0,\mu) = \frac{c\rho(0)}{4\pi}\,\frac{1-\mu}{1-l^2\mu^2/L^2}\,\frac{1}{h^+(i/|\mu|)} \quad (\mu<0). \tag{6.17}$$

6.2. The asymptotic behaviour of the neutron flux in Milne's problem

We shall now put the above results into a more convenient form. On deforming the path of integration in (6.15), calculating the contributions from the poles $p = \pm i/L$ by the method of residues, and substituting in the remaining integral $p = -is$, we obtain

$$\rho(x) = \tfrac{1}{2}\rho(0)\left\{\left(\frac{L}{l}+1\right)\frac{1}{h^+(il/L)}e^{x/L} - \left(\frac{L}{l}-1\right)\frac{1}{h^+(-il/L)}e^{-x/L} + \right.$$
$$\left. + \frac{i}{\pi}\int_{1/l}^{\infty}\frac{s-1/l}{s^2-1/L^2}\left[\frac{1}{h^+(-isl+)} - \frac{1}{h^+(-isl-)}\right]e^{-xs}\,ds\right\},$$

where $h^+(-isl+)$ and $h^+(-isl-)$ are the values of $h^+(pl)$ immediately right and left of the cut at $p = -is$. Transforming the last term by means of (6.10) and (6.6), we finally have

$$\rho(x) = \tfrac{1}{2}\rho(0)\left\{\left(\frac{L}{l}+1\right)\frac{1}{h^+(il/L)}e^{x/L} - \left(\frac{L}{l}-1\right)\frac{1}{h^+(-il/L)}e^{-x/L} - \right.$$
$$\left. - c\int_{1/l}^{\infty}\frac{h^+(isl)e^{-xs}}{s(ls+1)}\left[1+\frac{c}{ls}\log\frac{ls-1}{ls+1} + \frac{c^2}{4l^2s^2}\left(\pi^2+\log^2\frac{ls-1}{ls+1}\right)\right]^{-1}ds\right\}. \tag{6.18}$$

The first two terms in (6.18) represent the $\rho_{as}(x)$ or $\rho_{diff}(x)$ of § 5.4, and the last term is $\rho_{tr}(x)$. If $x \gg l$, the last term can be neglected, while the other two terms can be conveniently written by introducing the quantity z_0, defined by

$$z_0 = \tfrac{1}{2}L\log\left\{\frac{L+l}{L-l}\,\frac{h^+(-il/L)}{h^+(il/L)}\right\}. \tag{6.19}$$

For, using (6.11), we get as in (5.25)

$$h^+\left(\frac{il}{L}\right)h^+\left(-\frac{il}{L}\right) = \frac{cL^2-L^2+l^2}{2l^2}, \tag{6.20}$$

and then by (6.19) and (6.20) the formula (6.18) can be rewritten

$$\rho(x) = \rho(0)\left(\frac{2(L^2-l^2)}{cL^2-L^2+l^2}\right)^{\frac{1}{2}}\sinh\frac{x+z_0}{L} + O(e^{-x/l}). \tag{6.21}$$

In particular, for the case of no capture ($c = 1$ and so $L = \infty$), by passing to the limit $c \to 1$ in (6.21) we find

$$[\rho(x)]_{c=1} = \sqrt{3}\,\rho(0)(x+z_0)/l + O(e^{-x/l}). \tag{6.22}$$

The quantity z_0 defined by (6.19) has a simple interpretation, since (6.21) implies that

$$\rho_{as}(-z_0) = 0; \tag{6.23}$$

that is, z_0 is the distance beyond the surface of the medium at which the asymptotic neutron flux vanishes, and it is therefore known as the *extrapolated end-point*. The equation (6.23) may be regarded as the boundary condition satisfied by $\rho_{as}(\mathbf{r})$ in Milne's problem.

The results which we have obtained for the asymptotic neutron flux can alternatively be expressed by means of the quantity

$$\lambda = \rho_{as}(0)/\rho'_{as}(0), \tag{6.24}$$

where the dash denotes differentiation with respect to x. This is called the *linear extrapolation length*. Using (6.21), it is easy to verify that the linear extrapolation length is connected with the extrapolated end-point by the relation

$$\lambda = L\tanh(z_0/L). \tag{6.25}$$

This formula also shows that, for the case of no capture, λ and z_0 are the same, which could have been derived directly from (6.22).

The numerical evaluation of z_0 and λ may be carried out as follows: (6.19) is combined with (6.9), choosing in the latter a path of integration starting at $t = -i\infty$ just left of the cut, passing up the cut to $t = -i/l$ and then back to $t = -i\infty$ just right of the cut. Putting $t = -is$, we easily find

$$z_0 = \tfrac{1}{2}L\log\frac{L+l}{L-l} - \frac{1}{2\pi}\int_{1/l}^{\infty}\frac{ds}{s^2-1/L^2}\lim_{\epsilon\to 0+}\left\{\log\left[1-c\,\frac{\tan^{-1}(-isl+\epsilon)}{-isl+\epsilon}\right] - \right.$$
$$\left. - \log\left[1-c\,\frac{\tan^{-1}(-isl-\epsilon)}{-isl-\epsilon}\right]\right\},$$

and since in the cut plane

$$\lim_{\epsilon\to 0}\tan^{-1}[-isl+\epsilon] = \tfrac{1}{2}\pi - \tfrac{1}{2}i\log\left(\frac{sl+1}{sl-1}\right),$$

$$\lim_{\epsilon\to 0}\tan^{-1}[-isl-\epsilon] = -\tfrac{1}{2}\pi - \tfrac{1}{2}i\log\left(\frac{sl+1}{sl-1}\right),$$

this gives

$$z_0 = \tfrac{1}{2}L\log\frac{L+l}{L-l} - \frac{1}{\pi}\int_{1/l}^{\infty}\frac{ds}{s^2-1/L^2}\tan^{-1}\left\{\frac{c\pi}{2sl-2c\log\{(sl+1)/(sl-1)\}}\right\}, \tag{6.26}$$

which can be evaluated numerically. Equation (6.25) then gives λ. In the case of no capture ($c = 1$), (6.26) gives

$$z_0/l = 0{\cdot}7104..., \tag{6.27}$$

so that (6.22) becomes

$$[\rho(x)]_{c=1} = \sqrt{3}\,\rho(0)[x/l + 0{\cdot}7104... + O(e^{-x/l})]. \tag{6.28}$$

The tabulation of z_0 from (6.26) for other values of c has shown that, over a wide range of c, z_0 is well approximated by

$$z_0 = 0{\cdot}7104l/c, \tag{6.29}$$

which gives z_0 to within 0·7 per cent. for $0{\cdot}6 \leqslant c \leqslant 2$. From (6.29) and (6.25), together with (5.15), it follows that, for c fairly close to unity,

$$\lambda \cong 0{\cdot}7104l/\sqrt{c}, \tag{6.30}$$

but the range of validity of this expression is smaller than that of (6.29).

6.3. The neutron current in Milne's problem

6.3.1. *The neutron current in the deep interior*

The neutron current $\mathbf{j}(\mathbf{r})$ was defined in (4.6). This quantity is of frequent occurrence, and it is therefore of interest to find its value in Milne's problem. In a homogeneous medium with constant cross-sections, isotropic scattering, and no anisotropic sources, it is given in terms of $\rho(\mathbf{r})$ by (4.17). In the absence of sources, the latter is

$$\mathbf{j}(\mathbf{r}) = \frac{c}{4\pi l}\iiint \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} e^{-|\mathbf{r}-\mathbf{r}'|/l}\rho(\mathbf{r}')\,dV', \tag{4.17'}$$

and if $\rho(\mathbf{r})$ depends only on x, the y and z components of $\mathbf{j}(\mathbf{r})$ will vanish, while the x component, which we call $j(x)$ simply, is given, on integrating (4.17') over y and z, by

$$j(x) = \frac{c}{2l}\int_0^\infty \rho(x')E_2\left(\frac{|x-x'|}{l}\right)\operatorname{sign}(x-x')\,dx', \tag{6.31}$$

where $E_2(z)$ and $\operatorname{sign} z$ are defined by

$$\left.\begin{aligned} E_2(z) &= \int_1^\infty \frac{e^{-zt}}{t^2}\,dt \\ \operatorname{sign} z &= +1 \text{ for } z > 0, \quad = -1 \text{ for } z < 0 \end{aligned}\right\}. \tag{6.32}$$

The limits of integration in (6.31) correspond to the case where the medium occupies the space $x > 0$. If $j(x) > 0$, the net current is in the direction of increasing x.

An alternative expression for the neutron current in this case may be derived as follows. From the definition (4.6), the net excess of neutrons flowing out through a closed surface A per unit time is

$$\iint_A \mathbf{j}(\mathbf{r}).\mathbf{dA}, \tag{6.33}$$

where $\mathbf{dA}$ is a vector element of area directed outwards. If V is the volume inside A, the number of neutrons undergoing collisions in V per unit time is, in the constant cross-section approximation,

$$\iiint \rho(\mathbf{r})\, dV/l(\mathbf{r}),$$

and the number of secondary neutrons thereby produced is

$$\iiint c(\mathbf{r})\rho(\mathbf{r})\, dV/l(\mathbf{r}).$$

Equating the excess production to the net escape from V we have

$$\iint_A \mathbf{j}(\mathbf{r}).\mathbf{dA} = \iiint_V \left[\frac{c(\mathbf{r})-1}{l(\mathbf{r})}\rho(\mathbf{r})+s(\mathbf{r})\right] dV, \tag{6.34}$$

where $s(\mathbf{r})$ is the contribution due to independent neutron sources.

In the case of no sources, $c(\mathbf{r})$ and $l(\mathbf{r})$ independent of position and $\rho(\mathbf{r})$ depending only on x, we get

$$j(x)-j(x_0) = \frac{c-1}{l}\int_{x_0}^{x} \rho(x')\, dx', \tag{6.35}$$

which could also be derived from (6.31) by differentiating and using (6.1).

For the asymptotic region in Milne's problem, i.e. for $x \gg l$, we find by substituting (6.21) into (6.31), extending the lower limit of integration to $-\infty$, and writing

$$B = \rho(0)\left(\frac{2(L^2-l^2)}{cL^2-L^2+l^2}\right)^{\frac{1}{2}}, \tag{6.36}$$

the result

$$j(x) = \frac{cB}{2l}\int_{-\infty}^{\infty} \sinh\frac{x'+z_0}{L} E_2\left(\frac{|x-x'|}{l}\right)\mathrm{sign}(x-x')\, dx' \,[1+O(e^{-x/l})]$$

$$= -\frac{cB}{l}\cosh\frac{x+z_0}{L}\int_0^{\infty} \sinh\frac{y}{L} E_2\left(\frac{y}{l}\right) dy\,[1+O(e^{-x/l})].$$

On effecting the integration and using (6.21) and (5.5), we find

$$j(x) = \frac{L^2(c-1)}{l}\rho'(x)[1+O(e^{-x/l})]. \tag{6.37}$$

In particular, for the case of no capture, (6.37) gives in the limit $c \to 1$

$$[j(x)]_{c=1} = -\tfrac{1}{3}l[\rho'(x)]_{c=1}[1+O(e^{-x/l})],$$

and, since in this case $j(x)$ is a constant by (6.35),

$$[j(x)]_{c=1} = -\tfrac{1}{3}l[\rho'(\infty)]_{c=1}. \tag{6.38}$$

6.3.2. *The neutron current at the free surface*

We now consider the value of the neutron current at the free surface, which we know already for $c = 1$, from (6.38). This could be obtained by substituting (6.18) into (6.31), but we shall find it by an alternative method, which does not use the solutions found in §§ 6.1 and 6.2. We introduce the function

$$Y(x) = \rho'(x) - \frac{1}{L^2}\int_0^\infty \rho(x')\,dx' + \frac{lj(0)}{L^2(1-c)}, \tag{6.39}$$

and the abbreviation

$$\Lambda\phi(x) = \frac{c}{2l}\int_0^\infty \phi(x')E_1\left(\frac{|x-x'|}{l}\right)dx', \tag{6.40}$$

where $\phi(x)$ is any function such that (6.40) converges. With this notation, the equation (6.1) can be written $(1-\Lambda)\rho(x) = 0$. We shall also use the relations

$$\frac{d}{dx}\Lambda\phi(x) - \Lambda\frac{d\phi(x)}{dx} = \frac{c}{2l}\phi(0)E_1\left(\frac{x}{l}\right), \tag{6.41}$$

$$\int_0^x \Lambda\phi(u)\,du - \Lambda\int_0^x \phi(u)\,du = -\tfrac{1}{2}c\int_0^\infty \phi(u)E_2\left(\frac{u}{l}\right)du, \tag{6.42}$$

$$\int_x^\infty \Lambda\phi(u)\,du - \Lambda\int_x^\infty \phi(u)\,du = \tfrac{1}{2}c\,E_2\left(\frac{x}{l}\right)\int_0^\infty \phi(u)\,du. \tag{6.43}$$

The first of these is obtained by differentiating (6.40), and the third is then obtained by replacing $\phi(x)$ in (6.41) by $\int_x^\infty \phi(u)\,du$ and integrating from x to infinity. The left side of (6.42) is a constant by a similar procedure; to evaluate the constant we put $x = 0$ and use the fact that

$$\left[\Lambda\int_0^x \phi(u)\,du\right]_{x=0} = \frac{c}{2l}\int_0^\infty E_1\left(\frac{x'}{l}\right)dx'\int_0^{x'}\phi(u)\,du$$

$$= \tfrac{1}{2}c\int_0^\infty \phi(u)\,du\int_u^\infty E_1\left(\frac{x'}{l}\right)\frac{dx'}{l}.$$

According to (6.31),

$$j(0) = -\frac{c}{2l}\int_0^\infty \rho(u)E_2(u/l)\,du. \tag{6.44}$$

Using (6.1), (6.41), and (6.42) it can be seen that $Y(x)$ satisfies the integral equation

$$(1-\Lambda)Y(x) = \frac{c}{2l}\rho(0)E_1(x/l) + \frac{cl}{2L^2(1-c)}j(0)E_2(x/l). \tag{6.45}$$

Combining (6.35), (6.37), and (6.39), we find that, for $x \gg l$, $Y(x)$ is of the order of $e^{-x/l}$, and hence the integral of $Y(x)\rho(x)$ over all positive x converges. On multiplying both sides of (6.45) by $\rho(x)$ and integrating, using the result

$$\int_0^\infty \rho(x)(1-\Lambda)Y(x)\,dx = \int_0^\infty Y(x)(1-\Lambda)\rho(x)\,dx = 0$$

(which follows from (6.1) since Λ is symmetrical and the integrals are absolutely convergent), we find

$$\frac{c}{2l}\rho(0)\int_0^\infty \rho(x)E_1(x/l)\,dx + \frac{clj(0)}{2L^2(1-c)}\int_0^\infty \rho(x)E_2(x/l)\,dx = 0.$$

Substituting for the first integral from (6.1) and for the second from (6.44) gives

$$\rho^2(0) - \{l^2/L^2(1-c)\}j^2(0) = 0,$$

or, since with the definitions used $j(x)$ is negative,

$$j(0) = \frac{-L\sqrt{(1-c)}}{l}\rho(0). \tag{6.46}$$

The combination of this with (6.21) gives the value of $j(0)$ in terms of the asymptotic expression for $\rho(x)$.

The passage to the limit $c \to 1$ in (6.46) gives

$$[j(0)]_{c=1} = -\frac{1}{\sqrt{3}}[\rho(0)]_{c=1}. \tag{6.47}$$

This is the same as would have been obtained by combining (6.38) and (6.22).

6.4. Milne's problem in the presence of sources

We now consider the case where there are sources at a finite distance inside or outside the medium. There are three simple situations which we shall discuss.

(i) *The case of constant production* (§ 6.5), where isotropic neutron sources are distributed through the half-space $x > 0$ at constant strength per unit volume:

$$s(\mathbf{r},\Omega) = S, \text{ a constant.} \tag{6.48}$$

(ii) *The albedo problem* (§ 6.6), where a plane source of uniform strength is situated in the half-space $x < 0$ and emits neutrons which enter the plane $x = 0$ at some given angle. The *albedo* is the ratio of the current of neutrons returning across the plane $x = 0$ to that entering this plane. In the albedo problem, the number of neutrons per unit time per unit volume having their first collisions at a distance x into the medium is clearly $(s/\mu_0 l)\exp(-x/\mu_0 l)$, where s is the current falling on the medium from the source outside it, and μ_0 is the cosine of the angle between the direction of the source neutrons and the x-axis. The flux of neutrons that have had at least one collision is therefore the same as that due to a system of isotropic sources in the medium whose strength per unit volume and solid angle at depth x is

$$s(x) = (sc/4\pi\mu_0 l)e^{-x/\mu_0 l}. \tag{6.49}$$

(iii) *The isotropic plane source case* (§ 6.7), where an isotropic plane source of constant strength per unit area lies inside the medium. In this case

$$s(x) = (S/4\pi)\delta(x-x_0) \quad (x_0 \geqslant 0). \tag{6.50}$$

Any system of isotropic distributed sources whose strength depends only on x can be obtained by superposing sources of the type (6.50). It is, however, much simpler to consider source distributions of types (i) and (ii) directly.

In each of these three cases it is obviously sufficient to consider the solution which remains bounded as $x \to +\infty$. The more general solution, in which there is a prescribed supply of neutrons from infinity, is found by adding to this a solution of Milne's problem found in § 6.1.

Such solutions could be obtained by means of a slightly modified form of the Wiener–Hopf method, in which the source term appears in the integral on the right of (6.1), and its Fourier transform is divided into two functions, one regular in the upper and the other in the lower half-plane. This procedure would, however, mean re-doing much of the work involved in the solution of the source-free Milne's problem. We therefore try to express the solution of the inhomogeneous equation, with a source term of the form (6.48), (6.49), (6.50), in terms of the solution of the homogeneous equation, discussed in § 6.1.

We shall denote by $\rho(x)$ and $\psi(x,\mu)$ without suffixes or additional arguments the neutron flux and angular distribution for the source-free case.

6.5. The case of constant production

For isotropic sources of unit strength uniformly distributed in the medium, the equation (6.1) is replaced by

$$\rho_q(x) = \frac{c}{2l}\int_0^\infty \left[\rho_q(x') + \frac{l}{c}\right] E_1\left(\frac{|x-x'|}{l}\right) dx', \tag{6.51}$$

where $\rho_q(x)$ denotes the neutron flux in this case. With the notation (6.40) this is

$$(1-\Lambda)[\rho_q(x)+l/c] = l/c. \tag{6.52}$$

Differentiating and using (6.41), we obtain

$$(1-\Lambda)\frac{d\rho_q(x)}{dx} = \left[\frac{c}{2l}\rho_q(0)+\tfrac{1}{2}\right] E_1\left(\frac{x}{l}\right). \tag{6.53}$$

The same transformations applied to (6.1) give

$$(1-\Lambda)\frac{d\rho(x)}{dx} = \frac{c}{2l}\rho(0)E_1\left(\frac{x}{l}\right). \tag{6.54}$$

Combining (6.53) and (6.54), we find

$$(1-\Lambda)\left\{\left[\rho_q(0)+\frac{l}{c}\right]^{-1}\frac{d\rho_q(x)}{dx} - \frac{1}{\rho(0)}\frac{d\rho(x)}{dx}\right\} = 0; \tag{6.55}$$

that is, the expression in braces satisfies (6.1), and is therefore equal to a multiple of $\rho(x)$. Integrating this equality from 0 to x, we obtain

$$\frac{c\rho_q(x)+l}{c\rho_q(0)+l} = \frac{\rho(x)}{\rho(0)} + g\int_0^x \rho(u)\,du, \tag{6.56}$$

where g is some constant, which is determined by operating on (6.56) with $(1-\Lambda)$ and using (6.1), (6.42), (6.44), and (6.51). This gives

$$\frac{l}{c\rho_q(0)+l} = glj(0), \tag{6.57}$$

where $j(x)$ is the source-free current. Combining (6.56) and (6.57) and using (6.46) and the fact that $j(0)$ is negative (the current being in the direction of x decreasing), we then find

$$\rho_q(x) = \frac{1}{c|j(0)|}\left\{\frac{L\sqrt{(1-c)}}{l}[c\rho_q(0)+l]\rho(x) - \int_0^x \rho(u)\,du\right\} - \frac{l}{c}. \tag{6.58}$$

The condition that no neutrons come from infinity has not yet been used. It implies that $\rho_q(\infty)$ is finite, and this allows us to determine $\rho_q(0)$. Substituting $\rho(x)$ from (6.21) into (6.58) and using the abbreviation (6.36), we have

$$\rho_q(x) = \frac{B}{c|j(0)|}\left\{\frac{L\sqrt{(1-c)}}{l}[c\rho_q(0)+l]\sinh\frac{x+z_0}{L} - L\cosh\frac{x+z_0}{L}\right\} + \\ + \text{constant} + O(e^{-x/l}). \quad (6.59)$$

If this remains finite for $x \to \infty$ we must have

$$\rho_q(0) = \frac{l}{c}\left\{\frac{1}{(1-c)^{\frac{1}{2}}} - 1\right\}, \quad (6.60)$$

and with this value of $\rho_q(0)$, (6.58) becomes

$$\rho_q(x) = \frac{1}{c|j(0)|}\left\{L\rho(x) - \int_0^\infty \rho(u)\,du - l|j(0)|\right\}. \quad (6.61)$$

The asymptotic behaviour of $\rho_q(x)$ as $x \to \infty$ is also easily obtained. The limiting value itself, assumed finite and therefore constant, is from (6.51)

$$\rho_q(\infty) = l/(1-c).$$

Combining (6.36) and (6.46), we have also

$$\frac{B}{c|j(0)|} = \frac{l}{cL}\left\{\frac{2(L^2-l^2)}{(1-c)(cL^2-L^2+l^2)}\right\}^{\frac{1}{2}},$$

and using (6.59) and (6.60) we get, for $x \gg l$,

$$\rho_q(x) = \frac{l}{1-c}\left[1 - \frac{1}{c}\left\{\frac{2(1-c)(L^2-l^2)}{cL^2-L^2+l^2}\right\}^{\frac{1}{2}} e^{-(x+z_0)/L} + O(e^{-x/l})\right]. \quad (6.62)$$

The angular distribution $\psi_q(0,\mu)$ of the emerging neutrons is obtained as follows. The formula (4.16) gives

$$\psi_q(0,\mu) = \frac{c}{4\pi l}\int_0^\infty \left[\rho_q(-R\mu) + \frac{l}{c}\right] e^{-R/l}\,dR$$

$$= \frac{1}{4\pi l|\mu|}\int_0^\infty [c\rho_q(x)+l]e^{-x/|\mu|l}\,dx \quad (\mu < 0). \quad (6.63)$$

Substituting from (6.61) and eliminating the double integral through integration by parts yields

$$\psi_q(0,\mu) = \frac{L-|\mu|l}{4\pi l|\mu j(0)|}\int_0^\infty \rho(x)e^{-x/|\mu|l}\,dx,$$

and comparing this with $\psi(0,\mu)$ for no sources we finally have

$$\psi_2(0,\mu) = \frac{L+l\mu}{c|j(0)|}\psi(0,\mu) = \frac{L+l\mu}{2\pi c}\frac{\psi(0,\mu)}{\int_{-1}^{0}|\mu'|\psi(0,\mu')\,d\mu'} \quad (\mu<0). \quad (6.64)$$

6.6. The albedo problem

We now consider the albedo problem. Let one neutron per unit area per unit time enter the free surface from outside at an angle $\cos^{-1}\mu_0$ with the inward normal. If $\rho_s(x,\mu_0)$ is the resulting neutron flux, it satisfies the equation

$$(1-\Lambda)\rho_s(x,\mu_0) = (1/\mu_0)e^{-x/\mu_0 l}. \quad (6.65)$$

In this, $\rho_s(x,\mu_0)$ includes all neutrons, whether or not they have undergone collisions. If only scattered neutrons are included, then the free term of (6.65) becomes $(1/\mu_0)\Lambda e^{-x/\mu_0 l}$.

We first determine the albedo γ. Since the current vanishes at infinity (there being no supply of neutrons from infinity), the net current at $x=0$ is by (6.35)

$$j_s(0,\mu_0) = \frac{1-c}{l}\int_0^\infty \rho_s(x,\mu_0)\,dx,$$

while the inward current at $x=0$ is by the normalization adopted

$$j_{si}(0,\mu_0) = 1.$$

The albedo is therefore

$$\gamma = \frac{|j_s(0,\mu_0)-j_{si}(0,\mu_0)|}{j_{si}(0,\mu_0)} = 1-\frac{1-c}{l}\int_0^\infty \rho_s(x,\mu_0)\,dx. \quad (6.66)$$

It is to be noticed that the argument μ_0 of j refers to the direction of the incoming neutrons, not to that of the emergent neutrons.

To evaluate the integral in (6.66) we require the following

LEMMA. *If $\phi_1(x)$ and $\phi_2(x)$ are the solutions bounded at infinity of the equations*

$$(1-\Lambda)\phi_1(x) = q_1(x), \qquad (1-\Lambda)\phi_2(x) = q_2(x), \quad (6.67)$$

where $q_1(x)$ and $q_2(x)$ are non-negative and c (in Λ) $\leqslant 1$, then

$$\int_0^\infty \phi_1(x)q_2(x)\,dx = \int_0^\infty \phi_2(x)q_1(x)\,dx. \quad (6.68)$$

The proof of this lemma is very simple if the integral $\int_0^\infty \phi_1(x)\phi_2(x)\,dx$ exists. If it does not exist, the proof given by Hopf (25) runs as follows. If $q_1(x)$, $q_2(x)$, $\phi_1(x)$, and $\phi_2(x)$ are all non-negative, then by iterating (6.67) it is found that their solution as a power series in c converges, and

since for $c \leqslant 1$ the corresponding homogeneous equation has no bounded solution, the solution of (6.67) bounded at infinity will be that given by the Neumann series, i.e. the power series in c. The result (6.68) is then derived by substituting this solution for $\phi_1(x)$ and changing the order of integration in each term of the series.

Alternatively, (6.69) below can be derived from the optical reciprocity theorem (4.26).

Putting in (6.68) $\phi_1(x) = \rho_s(x, \mu_0)$, $\phi_2(x) = \rho_Q(x) + l/c$, and using (6.52), we obtain

$$\frac{l}{c}\int_0^\infty \rho_s(x,\mu_0)\,dx = \frac{1}{\mu_0}\int_0^\infty e^{-x/\mu_0 l}\left[\rho_Q(x)+\frac{l}{c}\right]dx, \tag{6.69}$$

so that (6.63), (6.64), and (6.66) give

$$\gamma = 1-\frac{2}{cl}(1-c)(L-l\mu_0)\frac{\psi(0,-\mu_0)}{\int_{-1}^{0}|\mu|\psi(0,\mu)\,d\mu}. \tag{6.70}$$

We now determine $\rho_s(x, \mu_0)$. Proceeding as in § 6.5, we derive from (6.51) and (6.65) the equation

$$(1-\Lambda)\left\{[\rho_s(0,\mu_0)]^{-1}\left(\frac{d}{dx}+\frac{1}{\mu_0 l}\right)\rho_s(x,\mu_0) - \left[\rho_Q(0)+\frac{l}{c}\right]^{-1}\frac{d}{dx}\rho_Q(x)\right\} = 0. \tag{6.71}$$

Since $\rho_s(x, \mu_0)$ and $\rho_Q(x)$ are both bounded at $x = \infty$, so is the quantity in braces in (6.71), and since the homogeneous equation $(1-\Lambda)\rho(x) = 0$ has for $c \leqslant 1$ no non-trivial solution bounded at infinity, that quantity must therefore vanish identically. Integrating the resulting ordinary differential equation and using (6.60) and (6.61) gives

$$\rho_s(x,\mu_0) = \rho_s(0,\mu_0)\left\{e^{-x/\mu_0 l} + \frac{L\sqrt{(1-c)}}{l}\frac{1}{|j(0)|}\int_0^x\left[\frac{d\rho(u)}{du}-\frac{\rho(u)}{L}\right]e^{-(x-u)/\mu_0 l}\,du\right\},$$

or, integrating by parts and using (6.46),

$$\rho_s(x,\mu_0) = \frac{\rho_s(0,\mu_0)}{\rho(0)}\left\{\rho(x) - \left[\frac{1}{\mu_0 l}+\frac{1}{L}\right]\int_0^x \rho(u)e^{-(x-u)/\mu_0 l}\,du\right\}. \tag{6.72}$$

In order to find $\rho_s(0, \mu_0)$, we again use (6.68), taking this time

$$\phi_1(x) = (d/dx + 1/\mu_0 l)\rho_s(x,\mu_0) \quad \text{and} \quad \phi_2(x) = \rho_Q(x) + l/c.$$

Since, as can easily be shown by the use of (6.41), this $\phi_1(x)$ satisfies

$$(1-\Lambda)\phi_1(x) = (c/2l)\rho_s(0,\mu_0)E_1(x/l),$$

this gives

$$\frac{c}{2l}\rho_s(0,\mu_0)\int_0^\infty E_1\left(\frac{x}{l}\right)\left[\rho_Q(x)+\frac{l}{c}\right]dx=\frac{l}{c}\int_0^\infty\left(\frac{d}{dx}+\frac{1}{\mu_0 l}\right)\rho_s(x,\mu_0)\,dx,$$

which in view of (6.51) and (6.60) can be written

$$\rho_s(0,\mu_0)=\frac{(1-c)^{\frac{1}{2}}}{\mu_0 l}\int_0^\infty \rho_s(x,\mu_0)\,dx.$$

This last integral has already been met in the derivation of (6.70), and using that result together with (6.46) we obtain

$$\rho_s(0,\mu_0)=\frac{4\pi}{c\mu_0}\left(1-\frac{l\mu_0}{L}\right)\frac{\psi(0,-\mu_0)}{\rho(0)}. \tag{6.73}$$

The final expression for $\rho_s(x,\mu_0)$ is obtained by combining (6.72) and (6.73). The asymptotic expression for $\rho_s(x,\mu_0)$ is easily found by combining (6.21) and (6.72). It is

$$\frac{\rho_s(x,\mu_0)}{\rho_s(0,\mu_0)}=\frac{\mu_0 l}{L-\mu_0 l}\left(\frac{2(L^2-l^2)}{cL^2-L^2+l^2}\right)^{\frac{1}{2}}e^{-(x+s_0)/L}+O(e^{-x/l}). \tag{6.74}$$

For the angular distribution of emergent neutrons, which we call $\psi_s(0,\mu_0\to\mu)$, we have from (4.16)

$$\psi_s(0,\mu_0\to\mu)=\frac{c}{4\pi l|\mu|}\int_0^\infty \rho_s(x,\mu_0)e^{-x/l|\mu|}\,dx \quad (\mu<0).$$

Substituting from (6.72), eliminating the double integral by integration by parts and using (6.16), we find

$$\psi_s(0,\mu_0\to\mu)=\frac{\mu_0}{L}\frac{L+\mu l}{\mu_0-\mu}\frac{\rho_s(0,\mu_0)}{\rho(0)}\psi(0,\mu),$$

and then from (6.73)

$$\psi_s(0,\mu_0\to\mu)=\frac{4\pi}{cL^2}\frac{(L+\mu l)(L-\mu_0 l)}{\mu_0-\mu}\frac{\psi(0,\mu)\psi(0,-\mu_0)}{\rho^2(0)} \quad (\mu<0). \tag{6.75}$$

It is to be noticed that (6.75) is such that

$$\psi_s(0,\mu_0\to\mu)=\psi_s(0,-\mu\to-\mu_0).$$

This is to be expected from the optical reciprocity theorem (4.25).

6.7. The isotropic plane source case

In this case we again normalize the strength of the source to unity, i.e. one neutron emitted in all directions per unit time per unit area. Let $x=x_0$ be the position of the source and $\rho_g(x_0\to x)$ the resulting neutron

flux. As we have seen in § 6.4, if the quantity $\rho_Q(x_0 \to x)$ is known, a problem with any other source distribution depending only on x can be solved.† The equation satisfied by $\rho_Q(x_0 \to x)$ is clearly

$$\rho_Q(x_0 \to x) = \frac{c}{2l}\int_0^\infty \rho_Q(x_0 \to x')E_1\left(\frac{|x-x'|}{l}\right)dx' + \tfrac{1}{2}E_1\left(\frac{|x-x_0|}{l}\right), \quad (6.76)$$

while according to the optical reciprocity theorem

$$\rho_Q(x_0 \to x) = \rho_Q(x \to x_0), \quad (6.77)$$

so that it is sufficient to solve (6.76) for $x > x_0$, for instance. First, let $x_0 = 0$, i.e. the plane source coincides with the free surface. On comparing (6.76) with $x_0 = 0$ with (6.53) and using (6.60), we see that $\rho_Q(0 \to x)$ satisfies the same equation as $(1-c)^{\frac{1}{2}}d\rho_Q(x)/dx$, and since both these quantities are bounded at infinity

$$\rho_Q(0 \to x) = (1-c)^{\frac{1}{2}}d\rho_Q(x)/dx. \quad (6.78)$$

If now $x_0 > 0$ and $x > x_0$, the flux can be written as

$$\rho_Q(x_0 \to x) = \rho_Q(0 \to x-x_0) - \int_0^{x_0} \frac{\partial}{\partial u}[\rho_Q(x_0-u \to x-u)]\,du. \quad (6.79)$$

The equation (6.76) gives

$$\rho_Q(x_0-u \to x-u) = \frac{c}{2l}\int_u^\infty \rho_Q(x_0-u \to x'-u)E_1\left(\frac{|x-x'|}{l}\right)dx' + \tfrac{1}{2}E_1\left(\frac{|x-x_0|}{l}\right),$$

and differentiating this with respect to u we get

$$\frac{\partial}{\partial u}[\rho_Q(x_0-u \to x-u)] = \frac{c}{2l}\int_u^\infty \frac{\partial}{\partial u}[\rho_Q(x_0-u \to x'-u)]E_1\left(\frac{|x-x'|}{l}\right)dx' - \frac{c}{2l}\rho_Q(x_0-u \to 0)E_1\left(\frac{|x-u|}{l}\right),$$

which shows that the quantity

$$-\frac{l}{c}\frac{1}{\rho_Q(x_0-u \to 0)}\frac{\partial}{\partial u}[\rho_Q(x_0-u \to x-u)]$$

satisfies the same equation, and the same condition at infinity, as

† That is, $\rho_Q(x_0 \to x)$ is the Green's function for Milne's problem with sources.

$\rho_g(0 \to x-u)$. These two quantities are therefore equal. On this account (6.79) gives

$$\rho_g(x_0 \to x) = \rho_g(0 \to x-x_0) + \\ + \frac{c}{l} \int_0^{x_0} \rho_g(x_0-u \to 0)\rho_g(0 \to x-u)\, du \quad (x > x_0), \quad (6.80)$$

and combining (6.77), (6.78), and (6.80) we have finally

$$\rho_g(x_0 \to x) = (1-c)^{\frac{1}{2}}\rho_a'(|x-x_0|) + \frac{c(1-c)}{l} \int_0^{\min(x,x_0)} \rho_a'(y)\rho_a'(y+|x-x_0|)\, dy, \quad (6.81)$$

where y has been substituted for x_0-u, and the formula has been rewritten so as to be applicable whether or not $x > x_0$.

The result (6.81) is due to Placzek (15), but the proof given here is by Elliott (19). The quantity $-(\partial/\partial u)[\rho_g(x_0-u \to x-u)]\, du$ may be interpreted as the contribution to $\rho_g(x_0 \to x)$ from neutrons which, in migrating from the source to the point x, at some time reached a distance u from the surface, but came no nearer to it. The proof is therefore equivalent to the calculation and summation of such contributions.

The angular distribution of emergent neutrons $\psi_g(x_0 \to 0, \mu)$ in this case is obtained from (4.16), (6.65), and (6.76). We then have

$$(1-\Lambda_{x_0})\psi_g(x_0 \to 0, \mu) \\ = \frac{c}{4\pi l|\mu|}(1-\Lambda_{x_0}) \int_0^{\infty} \left[\rho_g(x_0 \to x) + \frac{l}{c}\delta(x-x_0)\right] e^{-x/|\mu|l}\, dx \\ = \frac{1}{4\pi|\mu|} e^{-x_0/|\mu|l} \quad (\mu < 0),$$

and hence $$\psi_g(x_0 \to 0, \mu) = (1/4\pi)\rho_e(x_0, |\mu|) \quad (\mu < 0). \quad (6.82)$$

6.8. The isotropic point source case

The sources considered hitherto have been of strengths which depended only on the distance from the boundary of the medium. We now consider a point source; the solution for any source whose strength depends on the lateral distance can be obtained by superposition if the point source solution is known.

The general solution is difficult, but Elliott has carried out the analysis for an isotropic point source on the free surface of a non-capturing half-space. We shall give the main results here; further information may be found in Elliott's report (17).

The method employed by Elliott was to use the polar coordinates x, r, θ, where $x = 0$ is the free surface, the x-axis passes through the source, r is measured from the x-axis, and θ is the azimuthal angle, the solution required being independent of θ. The Bessel transform of (4.15) is taken, putting

$$F(x, s) = \int_0^\infty \rho(x, r) J_0(rs) r \, dr, \tag{6.83}$$

where $J_0(rs)$ is the Bessel function. This leads, for every s, to an equation for $F(x, s)$ as a function of x which is of the same form as (6.76), and can be solved by the same methods. The transform (6.83) is then inverted. The result for $c = 1$ (no capture), $x_0 = 0$ (the point source on the free surface), and $x^2 + r^2 \gg l^2$ (i.e. for large distances from the source), is

$$\rho_{pl}(x, r) = \frac{\sqrt{3}}{2\pi} q \left\{ \frac{P_1(\cos\omega)}{R^2} + 2{\cdot}152 \frac{l^3 P_4(\cos\omega)}{R^5} + O\left(\frac{l^4}{R^6}\right) + \right.$$
$$\left. + \frac{[\rho(x) - (x + z_0)\rho'(\infty)]}{r^3 \rho'(\infty)} + O(l^3 r^{-5} e^{-x/l}) \right\}, \tag{6.84}$$

where $R^2 = r^2 + (x + z_0)^2$, $\omega = \tan^{-1}\{r/(x + z_0)\}$, P_1 and P_4 are Legendre polynomials, $\rho(x)$ is, as before, the solution of the source-free Milne's problem (§§ 6.1, 6.2), $z_0 = 0{\cdot}7104l$ is the extrapolated end-point, and q is the number of neutrons emitted by the source per unit time.

The neutron current at the free surface, and the extensions of (6.84) for small capture and the source inside the medium, are also given by Elliott, but we shall not quote these results.

VII

THE CASE OF TWO ADJACENT HALF-SPACES

7.1. The extension of the Wiener–Hopf method

THE analysis given above for the case of a medium occupying a half-space when the other half-space is a vacuum can easily be extended to the case of two adjacent half-spaces, provided that in both media the scattering is isotropic and the constant cross-section approximation is justified. In this case the properties of each medium are specified by the values of l, the mean free path, and c, the mean number of secondaries. We use the subscript 1 for the medium in the half-space $x > 0$, and 2 for that in $x < 0$.

We first consider the case of no sources at a finite distance. Since $l(\mathbf{r})$ now depends on the position, we should start from (4.15′) rather than (4.15). However, if the quantity ξ is introduced, defined by

$$\xi = x/l_1 \ (x > 0), \quad = x/l_2 \ (x < 0), \tag{7.1}$$

then, for the particular dependence of $l(\mathbf{r})$ on $\mathbf{r}$ in question,

$$\frac{\tau(\mathbf{r}, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\xi-\xi'}{x-x'}, \tag{7.2}$$

and so, assuming again that $\rho(\mathbf{r})$ depends only on x, we can integrate (4.15′) over the planes $x' =$ constant, using ξ as the variable of integration, and obtain

$$\rho(\xi) = \frac{c_1}{2}\int_0^\infty \rho(\xi')E_1(|\xi-\xi'|)\,d\xi' + \frac{c_2}{2}\int_{-\infty}^0 \rho(\xi')E_1(|\xi-\xi'|)\,d\xi'. \tag{7.3}$$

The quantity ξ defined by (7.1) is called† the *optical depth*. More generally, when the mean free path depends only on one Cartesian co-ordinate (x, say), the optical depth of $\mathbf{r}$ is defined as the optical distance (see (2.39)) between $\mathbf{r}$ and its projection on a fixed plane $x =$ constant, with the appropriate sign.

The form of (7.3) shows that in the *plane case* (when all quantities depend on x only), the manner of dependence of l on x does not affect the solution as a function of optical depth.

† Cf. footnote in § 2.6.

The solution of (7.3) can be obtained in the same way as that of (6.1). Putting

$$F_{12}^{+}(p) = \int_0^\infty e^{ipx}\rho(\xi)\,d\xi, \qquad F_{12}^{-}(p) = \int_{-\infty}^0 e^{ipx}\rho(\xi)\,d\xi, \tag{7.4}$$

then as in (6.4) we find

$$F_{12}^{+}(p)\left(1-c_1\frac{\tan^{-1}p}{p}\right) = -F_{12}^{-}(p)\left(1-c_2\frac{\tan^{-1}p}{p}\right), \tag{7.5}$$

and then put as in (6.6)

$$\frac{p^2+1}{p^2+l_1^2/L_1^2}\left(1-c_1\frac{\tan^{-1}p}{p}\right) = \frac{h_1^{+}(p)}{h_1^{-}(p)}, \qquad \frac{p^2+1}{p^2+l_2^2/L_2^2}\left(1-c_2\frac{\tan^{-1}p}{p}\right) = \frac{h_2^{+}(p)}{h_2^{-}(p)}. \tag{7.6}$$

Here L_1 and L_2 are the diffusion lengths in media 1 and 2; $h_1^{+}(p)$ and $h_2^{+}(p)$ are regular and non-zero for $\operatorname{im} p > 0$, and tend to unity as $|p|$ tends to infinity in this half-plane, while $h_1^{-}(p)$ and $h_2^{-}(p)$ have the same properties for $\operatorname{im} p < 0$. The expressions for $h_1^{+}(p)$, etc., have been given in (6.9), apart from the units in which p is measured. Continuing as in the derivation of (6.14) we find

$$F_{12}^{+}(p) = i\frac{h_2^{+}(p)}{h_1^{+}(p)}\frac{\rho(0)(p-p_0)}{p^2+l_1^2/L_1^2}, \qquad F_{12}^{-}(p) = -i\frac{h_2^{-}(p)}{h_1^{-}(p)}\frac{\rho(0)(p-p_0)}{p^2+l_2^2/L_2^2}, \tag{7.7}$$

where p_0 is a purely imaginary constant. If $\rho(-\infty) = 0$ but $\rho(\infty) \neq 0$ (that is, if all the neutrons come from $x = +\infty$), then $p_0 = -il_2/L_2$. In the converse case, $p_0 = il_1/L_1$. Finally, $\rho(\xi)$ is obtained by applying the Fourier transform inversion formula to (7.7).

7.2. The asymptotic behaviour of the neutron flux

The most important application of the analysis of this section is to establish a relation between the asymptotic behaviours of $\rho(\xi)$ when $\xi \to +\infty$ and $\xi \to -\infty$. If, as in (5.38), we put

$$\rho(\xi) = \rho_{as}(\xi) + O(e^{-|\xi|}), \tag{7.8}$$

where $\rho_{as}(\xi)$ satisfies

$$\frac{d^2\rho_{as}(\xi)}{d\xi^2} = \begin{cases} (l_1^2/L_1^2)\cdot\rho_{as}(\xi) & (\xi > 0) \\ (l_2^2/L_2^2)\cdot\rho_{as}(\xi) & (\xi < 0) \end{cases}, \tag{7.8'}$$

then this relation can be conveniently expressed as

$$\begin{aligned} \rho_{as}(0-) &= G_{00}\rho_{as}(0+) + G_{01}[d\rho_{as}(\xi)/d\xi]_{\xi=0+} \\ [d\rho_{as}(\xi)/d\xi]_{\xi=0-} &= G_{10}\rho_{as}(0+) + G_{11}[d\rho_{as}(\xi)/d\xi]_{\xi=0+} \end{aligned}, \tag{7.9}$$

where the coefficients G_{00}, etc., may be determined as follows. Let θ be the proportion of the flux at the interface, $\rho(0)$, which is due to the supply

of neutrons from $\xi = +\infty$ and $1-\theta$ that due to the supply from $\xi = -\infty$. Then the constant p_0 in (7.7) is equal to

$$i[(1-\theta)(l_1/L_1)-\theta(l_2/L_2)],$$

and continuing as in the derivation of (6.18) we have for $\xi > 0$

$$\rho_{as}(\xi) = \rho(0)\Big\{\tfrac{1}{2}\theta\Big(1+\frac{L_1 l_2}{L_2 l_1}\Big)\frac{h_2^+(il_1/L_1)}{h_1^+(il_1/L_1)}e^{\xi l_1/L_1}+ \\ +\Big[1-\tfrac{1}{2}\theta\Big(1+\frac{L_1 l_2}{L_2 l_1}\Big)\Big]\frac{h_2^+(-il_1/L_1)}{h_1^+(-il_1/L_1)}e^{-\xi l_1/L_1}\Big\}, \quad (7.10)$$

and a similar expression for $\xi < 0$. The quantities $h_2^+(il_1/L_1)$, etc., can be found from (6.9), and then the coefficients G_{00}, etc., are known. For the case when both c_1 and c_2 are close to unity, the result is, putting (7.9) in a more symmetrical form,

$$W_2\rho_{as}(0-) = W_1\Big\{\rho_{as}(0+)-0{\cdot}204(c_1-c_2)^2\Big[\frac{d\rho_{as}(\xi)}{d\xi}\Big]_{\xi=0+}\Big\}\times \\ \times[1+O(\langle 1-c_1\rangle^3)+O(\langle 1-c_2\rangle^3)],$$

$$W_2\Big[\frac{d\rho_{as}}{d\xi}\Big]_{\xi=0-} = W_1\Big\{\Big[\frac{d\rho_{as}}{d\xi}\Big]_{\xi=0+}-0{\cdot}510(c_1-c_2)^2\rho_{as}(0+)\Big\}\times \\ \times[1+O(\langle 1-c_1\rangle^3)+O(\langle 1-c_2\rangle^3)], \quad (7.11)$$

where for brevity we have put

$$W_j = \frac{L_j}{l_j}\Big[\frac{c_j(c_j L_j^2+l_j^2-L_j^2)}{6(L_j^2-l_j^2)}\Big]^{\frac{1}{2}} \quad (j = 1, 2). \quad (7.12)$$

Since by (5.5) these W_j are given by

$$W_j = \frac{L_j^2}{l_j^2}(1-c_j)\{1+0{\cdot}309(1-c_j)^2+O[(1-c_j)^3]\}, \quad (7.13)$$

on neglecting terms of order $(1-c_1)^2$ and $(1-c_2)^2$ we can rewrite (7.11) as

$$(L_2^2/l_2^2)(1-c_2)\rho_{as}(0-) \\ = (L_1^2/l_1^2)(1-c_1)\rho_{as}(0+)[1+O(\langle 1-c_1\rangle^2)+O(\langle 1-c_2\rangle^2)],$$

$$\frac{L_2^2}{l_2^2}(1-c_2)\Big[\frac{d\rho_{as}(\xi)}{d\xi}\Big]_{\xi=0-} \\ = \frac{L_1^2}{l_1^2}(1-c_1)\Big[\frac{d\rho_{as}(\xi)}{d\xi}\Big]_{\xi=0+}[1+O(\langle 1-c_1\rangle^2)+O(\langle 1-c_2\rangle^2)]. \quad (7.14)$$

Using (6.37) and defining $j_{as}(\xi)$ separately in each medium, as was done for $\rho_{as}(\xi)$ in (7.8′), the second of (7.14) can be put in the form

$$j_{as}(0-) = j_{as}(0+)[1+O(\langle 1-c_1\rangle^2)+O(\langle 1-c_2\rangle^2)]. \quad (7.15)$$

The discontinuity of $j_{as}(\xi)$ is therefore a small quantity, of the order of $1-c$, compared with that of $\rho_{as}(\xi)$. This could have been foreseen as follows: the actual neutron current $j(\xi)$ is, like the actual flux $\rho(\xi)$, continuous at $\xi = 0$. If c_1 and c_2 are close to unity, we see from (6.35) that the relative deviation of $j(\xi)$ from $j_{as}(\xi)$ is much smaller than that of $\rho(\xi)$ from $\rho_{as}(\xi)$, and (7.15) therefore follows. Also, by (6.35), $dj(\xi)/d\xi$ is continuous at $\xi = 0$, while from (7.3) $d\rho(\xi)/d\xi$ has a logarithmic singularity there if $c_1 \neq c_2$.

7.3. The angular distribution, flux, and current at the interface

We now consider the angular distribution, flux, and current of neutrons at the interface $x = 0$ between the two media. These will naturally depend on the asymptotic behaviour of $\rho(\xi)$ as $\xi \to +\infty$ and $\xi \to -\infty$. We restrict ourselves to the case when no neutrons are supplied from $x = -\infty$, i.e. $\rho(-\infty) = 0$, and first determine for this case the relation between $\rho_{as}(\xi)$ for $\xi > 0$ and $\rho(0)$. This is directly obtained from (7.10), since by combining (6.11) and (5.5) we derive

$$h_2^+\left(\frac{il_1}{L_1}\right)h_2^+\left(-\frac{il_1}{L_1}\right) = L_2^2\frac{c_1-c_2}{c_1}\frac{L_1^2-l_1^2}{L_1^2l_2^2-L_2^2l_1^2}.$$

Assuming $c_1 > c_2$, and using (6.20) and the fact that $\rho(-\infty) = 0$, $\rho(+\infty) \neq 0$ (i.e. $\theta = 1$), we find

$$\rho_{as}(\xi) = \rho(0)\left\{\frac{c_1-c_2}{c_1}\frac{2(L_1^2-l_1^2)}{c_1L_1^2-L_1^2+l_1^2}\right\}^{\frac{1}{2}}\sinh\frac{\xi l_1+z_1}{L_1} \quad (\xi > 0), \quad (7.16)$$

where

$$z_1 = \frac{L_1}{2}\log\left\{\frac{L_1l_2+L_2l_1}{L_1l_2-L_2l_1}\frac{h_2^+(il_1/L_1)}{h_2^+(-il_1/L_1)}\frac{h_1^+(-il_1/L_1)}{h_1^+(il_1/L_1)}\right\}. \quad (7.17)$$

If $c_2 > c_1$, then in order to avoid complex quantities in the final result (7.16) should be rewritten

$$\rho_{as}(\xi) = \rho(0)\left\{\frac{c_2-c_1}{c_1}\frac{2(L_1^2-l_1^2)}{c_1L_1^2-L_1^2+l_1^2}\right\}^{\frac{1}{2}}\cosh\frac{\xi l_1+\tilde{z}_1}{L_1} \quad (\xi > 0), \quad (7.16')$$

where the definition of $\tilde{z}_1$ is obvious. For $c_2 < c_1$, the quantity z_1 can be described as the *extrapolated end-point of medium* 1 *in the presence of reflection by an infinite half-space of medium* 2. The calculation of z_1 can be carried out like that of z_0 (see formula (6.26)).

By a slight modification of the argument of § 6.3, assuming again that $\rho(-\infty) = 0$, we find that the current at the interface is

$$j(0) = -\frac{L_1(1-c_1)^{\frac{1}{2}}(1-c_2)^{\frac{1}{2}}}{l_1}\rho(0). \quad (7.18)$$

The modification in question is that the operator Λ is now defined by

$$\Lambda\phi(\xi) = \frac{c_1}{2}\int_0^\infty \phi(\eta)E_1(|\xi-\eta|)\,d\eta + \frac{c_2}{2}\int_{-\infty}^0 \phi(\eta)E_1(|\xi-\eta|)\,d\eta, \quad (7.19)$$

and since this operator is not symmetrical, whereas $c(\mathbf{r})\Lambda$ is, the equation corresponding to (6.45) should be multiplied by $c(\mathbf{r})\rho(\xi)$, and not by $\rho(\xi)$ only. Otherwise the proof is as before.

We denote the angular distribution of the neutrons at the interface by $\psi_{12}(0,\mu)$. An inspection of the formulae (7.7) and (6.17) suggests that this could be expressed in terms of the emerging angular distributions in the ordinary Milne's problem for media 1 and 2. Let $\psi_j(0,\mu)$ be the emergent angular distribution ($\mu < 0$) when medium j occupies the half-space $x > 0$ and the other medium is absent. Using (4.16), (7.4), and (7.7), and assuming that $\rho(-\infty) = 0$, we find as for (6.17)

$$\left.\begin{aligned} \psi_{12}(0,\mu) &= \frac{c_2\rho(0)}{4\pi}\frac{h_1^+(i/\mu)}{h_2^+(i/\mu)}\frac{1}{1+\mu l_2/L_2} \qquad (\mu > 0) \\ \text{and} \\ \psi_{12}(0,\mu) &= \frac{c_1\rho(0)}{4\pi}\frac{h_2^+(-i/\mu)}{h_1^+(-i/\mu)}\frac{1-\mu l_2/L_2}{1-\mu^2 l_1^2/L_1^2} \quad (\mu < 0) \end{aligned}\right\}. \quad (7.20)$$

By direct application of (6.17), we have for $\psi_1(0,\mu)$ and $\psi_2(0,\mu)$

$$\psi_j(0,\mu) = \frac{c_j\rho_j(0)}{4\pi}\frac{1}{h_j^+(-i/\mu)}\frac{1-\mu}{1-\mu^2 l_j^2/L_j^2} \quad (\mu < 0;\, j = 1,2), \quad (7.21)$$

where $\rho_j(0)$ is the flux corresponding to the angular distribution $\psi_j(0,\mu)$. The combination of (7.20) and (7.21) gives

$$\left.\begin{aligned} \psi_{12}(0,\mu) &= \frac{\rho(0)\rho_1(0)}{\rho_2(0)}\frac{c_1}{4\pi}\frac{1-\mu l_2/L_2}{1-\mu^2 l_1^2/L_1^2}\frac{\psi_2(0,-\mu)}{\psi_1(0,-\mu)} \quad (\mu > 0) \\ \text{and} \\ \psi_{12}(0,\mu) &= \frac{\rho(0)\rho_2(0)}{\rho_1(0)}\frac{c_2}{4\pi}\frac{1}{1+\mu l_2/L_2}\frac{\psi_1(0,\mu)}{\psi_2(0,\mu)} \qquad (\mu < 0) \end{aligned}\right\}. \quad (7.22)$$

The expressions for $\rho(0)$, $j(0)$, and $\psi_{12}(0,\mu)$ for the case when the neutron supply comes from $\xi = -\infty$ [$\rho(-\infty) \neq 0$, $\rho(\infty) = 0$] are obtained similarly, and the solution for the general case is then obtained by superposition.

7.4. The effects of sources

We finally examine the effects of sources in a system of two semi-infinite media. We shall, however, consider only two types of source: constant production in one medium, and a plane source parallel to the

interface. The analogue of the albedo problem has no direct physical significance here, while the problem of a point source has not yet actually been solved, although the principle of the solution is obvious.

For the case of constant production in medium 1 ($\xi > 0$), we normalize the source strength per unit volume to unity as before, and then the equation satisfied by the neutron flux is

$$\rho_q(\xi) = \frac{c_1}{2}\int_0^\infty \left[\rho_q(\eta)+\frac{l_1}{c_1}\right]E_1(|\xi-\eta|)\,d\eta + \frac{c_2}{2}\int_{-\infty}^0 \rho_q(\eta)E_1(|\xi-\eta|)\,d\eta,$$

or, using the operator Λ as defined in (7.19),

$$(1-\Lambda)\rho_q(\xi) = \begin{cases} l_1[1-\frac{1}{2}E_2(\xi)] & (\xi > 0) \\ l_1\cdot\frac{1}{2}E_2(-\xi) & (\xi < 0) \end{cases}. \tag{7.23}$$

The factor l_1 is explained by the fact that the unit of volume in x-measure is $1/l_1$ times the unit volume in ξ-measure. It will not, however, appear in the case of a plane source (see equation (7.28) below), as can alternatively be found from the transformation properties of the delta function.

With Λ given by (7.19), the formulae (6.41) and (6.42) become

$$\frac{d}{d\xi}\Lambda\phi(\xi) - \Lambda\frac{d\phi(\xi)}{d\xi} = \frac{c_1-c_2}{2}\phi(0)E_1(|\xi|),$$

and

$$\int_0^\xi [\Lambda\phi(\eta)]\,d\eta - \Lambda\int_0^\xi \phi(\eta)\,d\eta$$

$$= -\frac{c_1}{2}\int_0^\infty \phi(\eta)E_2(\eta)\,d\eta + \frac{c_2}{2}\int_{-\infty}^0 \phi(\eta)E_2(-\eta)\,d\eta. \tag{7.24}$$

Proceeding now as in the derivation of (6.56), i.e. comparing $(1-\Lambda)\rho_q'(\xi)$ with $(1-\Lambda)\rho'(\xi)$, where $\rho(\xi)$ is the solution of the source-free equation (7.3) which vanishes for $\xi = -\infty$, we find

$$\frac{(c_1-c_2)\rho_q(\xi)+l_1}{(c_1-c_2)\rho_q(0)+l_1} = \frac{1}{\rho(0)}\left\{\rho(\xi)+g\int_0^\xi \rho(\eta)\,d\eta\right\}, \tag{7.25}$$

and operating on (7.25) with $(1-\Lambda)$, we obtain

$$g = \frac{\rho(0)}{j(0)}\,\frac{l_1(1-c_2)}{(c_1-c_2)\rho_q(0)+l_1}, \tag{7.26}$$

since the current at the interface in the source-free case is

$$j(0) = -\frac{c_1}{2}\int_0^\infty \rho(\eta)E_2(\eta)\,d\eta + \frac{c_2}{2}\int_{-\infty}^0 \rho(\eta)E_2(-\eta)\,d\eta.$$

Since, if $\rho_g(\xi)$ is bounded for $\xi \to +\infty$, $g = -l_1/L_1$, we have by (7.18)

$$\rho_g(0) = \frac{l_1}{c_1 - c_2}\left[\left(\frac{1-c_2}{1-c_1}\right)^{\frac{1}{2}} - 1\right]. \tag{7.27}$$

All the constants in (7.25) are therefore known, which gives $\rho_g(\xi)$; $j_g(0)$ and $\psi_g(0,\mu)$ can then be determined as before. This completes the solution for the case of constant production.

In the case of a plane source of unit strength per unit area, the equation is

$$\rho_g(\xi_0 \to \xi) = \tfrac{1}{2}\int_{-\infty}^\infty c(\xi')\rho_g(\xi_0 \to \xi')E_1(|\xi - \xi'|)\,d\xi' + \tfrac{1}{2}E_1(|\xi_0 - \xi|), \tag{7.28}$$

where $c(\xi') = c_1$ for $\xi' > 0$ and $= c_2$ for $\xi' < 0$; by the optical reciprocity theorem,

$$\rho_g(\xi_0 \to \xi) = \rho_g(\xi \to \xi_0). \tag{7.29}$$

For $\xi_0 = 0$, we obtain by comparing (7.28) with the equation for $d\rho_g(\xi)/d\xi$ (see (7.23), etc.)

$$\rho_g(0 \to \xi) = \frac{\rho_g'(\xi)}{l_1}\left(\frac{1-c_1}{1-c_2}\right)^{\frac{1}{2}}. \tag{7.30}$$

The coefficient of $\rho_g'(\xi)$ is not symmetrical in the suffixes 1 and 2 because the definition of $\rho_g(\xi)$ is not symmetrical in them.

Rewriting (7.28) as an equation for $\rho_g(\xi_0 - \eta \to \xi - \eta)$ and differentiating with respect to η, we find as in the derivation of (6.80)

$$\frac{\partial}{\partial\eta}[\rho_g(\xi_0 - \eta \to \xi - \eta)] = \frac{1}{2}\int_{-\infty}^\infty c(\xi' - \eta)\frac{\partial}{\partial\eta}[\rho_g(\xi_0 - \eta \to \xi' - \eta)]E_1(|\xi - \xi'|)d\xi' -$$

$$-\frac{(c_1 - c_2)}{2}\rho_g(\xi_0 - \eta \to 0)E_1(|\xi - \eta|), \tag{7.31}$$

and therefore

$$\frac{\partial}{\partial\eta}[\rho_g(\xi_0 - \eta \to \xi - \eta)] = -(c_1 - c_2)\rho_g(\xi_0 - \eta \to 0)\rho_g(0 \to \xi - \eta). \tag{7.32}$$

Integrating (7.32) and using (7.30) we obtain $\rho_g(\xi_0 \to \xi)$ for $|\xi| > |\xi_0|$, and on using (7.29) this formula can be immediately extended to $|\xi_0| > |\xi|$.

VIII

THE DIFFUSION APPROXIMATION

8.1. The basic ideas of diffusion theory

In the last three chapters we have discussed the exact solutions of some simple problems. In more complex cases, the exact solution, even if it could be obtained at all, would be too complicated to be of any use. It is therefore necessary to consider now approximate methods of solving problems of neutron transport.

The simplest of these methods is based on the conclusion of § 5.4, that in a uniform isotropically scattering medium in which the constant cross-section approximation is applicable, the exact solution for the neutron flux can always be represented in the form

$$\rho(\mathbf{r}) = \rho_{as}(\mathbf{r}) + \rho_{tr}(\mathbf{r}), \tag{8.1}$$

where $\rho_{as}(\mathbf{r})$ satisfies the equation

$$\nabla^2 \rho_{as}(\mathbf{r}) = \rho_{as}(\mathbf{r})/L^2, \tag{8.2}$$

or

$$\nabla^2 \rho_{as}(\mathbf{r}) + \kappa^2 \rho_{as}(\mathbf{r}) = 0,$$

where $\kappa = 1/iL$ and is used when L is imaginary, and

$$\rho_{tr}(\mathbf{r}) = O(e^{-d/l}), \tag{8.3}$$

where d is the distance of the point $\mathbf{r}$ from the nearest source or boundary (see (5.38), (5.39), and (5.4)). (Cf. also the footnote at the end of § 5.3.)

If now the geometrical dimensions involved are large compared with the mean free path, then except in the neighbourhood of sources and boundaries $\rho_{tr}(\mathbf{r})$ in (8.1) can be neglected in comparison with $\rho_{as}(\mathbf{r})$. That is, the solution of the integral equation (4.15′) can be replaced by that of the differential equation (8.2). Because of the similarity of equation (8.2) to that of thermal or chemical diffusion, this approximation is called the *diffusion approximation*.

In order that the solution of (8.2) shall be determinate, it is necessary to specify the boundary conditions, and since at the actual boundary of the system $\rho_{tr}(\mathbf{r})$ cannot be neglected and equation (8.2) does not hold for the actual flux, this requires some care.

Suppose, for instance, that it is desired to determine the critical thickness of a bare infinite plane slab of slightly multiplying material, i.e. $c > 1$, $c-1 \ll 1$ and so $1/\kappa \gg l$. If the critical thickness is $2a$ and

the midplane of the slab contains the origin, then from (8.1)–(8.3) we can write

$$\rho(x) = A\cos\kappa x + O(e^{-(a-x)/l}) + O(e^{-(a+x)/l}). \tag{8.4}$$

Using for the moment a crude boundary condition such as equating the first term of (8.4) to zero, we see that a will be comparable with $1/\kappa$, and since $1/\kappa \gg l$, $e^{-2a/l}$ will be extremely small. In the neighbourhood of the boundary $x = a$, the third term in (8.4) can therefore be neglected, and this is equivalent to replacing the slab by an infinite medium $x < a$, which contains a neutron flux stationary in time and asymptotically periodic in space. The latter problem has been considered in §§ 6.1 and 6.2. (The argument in those sections related to the case $c < 1$, but this restriction is inessential and was made only to avoid the appearance of negative neutron densities in what was there part of the system.)

It is known that, in this infinite medium problem, $\rho_{as}(x)$ satisfies the conditions $\rho_{as}(a+z_0) = 0$ and $\rho_{as}(a) = -\lambda\rho'_{as}(a)$; see (6.23) and (6.24). Thus we can say that in the slab $\rho_{diff}(x)$ (as we shall henceforward call the solution of (8.2)) should approximately satisfy the boundary condition

$$\rho_{diff}(a+z_0) = 0, \tag{8.5}$$

or, what is the same thing,

$$\rho_{diff}(a) = -\lambda\rho'_{diff}(a); \tag{8.5'}$$

and similarly, for $x = -a$, the condition

$$\rho_{diff}(-a-z_0) = 0 \quad \text{or} \quad \rho_{diff}(-a) = \lambda\rho'_{diff}(-a).$$

The critical half-thickness is therefore given by $a = \pi/2\kappa - z_0$.

The conditions here derived are approximate because the mutual interference of the two boundaries has been neglected, though if $a \gg l$ this neglect of the interference is perfectly justified.

The general procedure envisaged in the diffusion approximation is this: for each kind of boundary, the simplest possible problem involving that kind of boundary is solved as exactly as possible, and at least without using the diffusion approximation; the conditions actually satisfied by $\rho_{as}(r)$ at the boundary in question are thus found. In more complex problems, the interference between different boundaries is neglected, and in solving (8.2) the boundary conditions determined from these simplified problems are used. Of course, cases arise where this plan cannot be strictly followed, and then the boundary condition must be guessed.

It is also necessary to remark that, though the value of c in the system does not enter directly the criteria of applicability of the diffusion approximation, in practice the dimensions of the system will usually be

of the order of L or $1/\kappa$, and this effectively limits the application of the theory to systems where $|1-c|$ is small.

8.2. The problem of spherical systems

The analysis of Chapters VI and VII is adequate for the determination of boundary conditions at plane surfaces. In the case of spherical boundaries, a very convenient simplification is afforded by a property of spherically symmetrical systems; namely, that if in such a system the mean free path is independent of position and the scattering is isotropic, then the determination of the spherically symmetrical neutron distribution in the system can be reduced to the determination of that in a certain system with plane symmetry. This result is rigorously true, whether or not the constant cross-section approximation is applicable, but for simplicity we shall prove it only in the latter case.

Let r be the radial coordinate, $c(r)$ the mean number of secondaries per collision at distance r from the centre, and $\rho(r)$ the neutron flux. The equation (4.15) can be written

$$\rho(\mathbf{r}) = \frac{1}{4\pi l}\iiint_{r'<a} c(r')\rho(r')e^{-|\mathbf{r}-\mathbf{r}'|/l}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{8.6}$$

where a is the outer radius of the system, finite or infinite. On integrating over angles, since $dV' = r'^2\,dr'd\mu d\phi$, say, where $\mu = (\mathbf{r}.\mathbf{r}')/rr'$, so that

$$d\mu/|\mathbf{r}-\mathbf{r}'| = d|\mathbf{r}-\mathbf{r}'|/rr', \tag{8.7}$$

we find

$$\rho(r) = \frac{1}{2lr}\int_0^a c(r')r'\rho(r')\left[E_1\left(\frac{|r-r'|}{l}\right) - E_1\left(\frac{r+r'}{l}\right)\right]dr',$$

or, extending the definition of $c(r)$ and $\rho(r)$ to $r < 0$ by putting $c(-r) = c(r)$, $\rho(-r) = \rho(r)$,

$$r\rho(r) = \frac{1}{2l}\int_{-a}^{a} c(r')r'\rho(r')E_1\left(\frac{|r-r'|}{l}\right)dr'. \tag{8.8}$$

This can be identified at once as the equation governing the neutron distribution in an infinite slab of thickness $2a$, where $c(r)$ is the number of secondaries per collision at distance r from the midplane, and $r\rho(r)$ is the neutron flux. This completes the proof. Unfortunately, the result is no longer valid for anisotropic scattering or mean free path varying with position.

In the application of this result to find spherically symmetric solutions of (8.6), it is necessary to notice that, with our definition of $\rho(r)$ for

$r < 0$, $r\rho(r)$ must be an odd function. Only the eigenfunctions of (8.8) which are of odd parity will therefore be relevant; the lowest eigenfunction of (8.6) will be given by the second one of (8.8), the second of (8.6) by the fourth of (8.8), and so on.

The above result makes it possible in many cases to formulate the boundary conditions to be used in the diffusion treatment of problems of spherical symmetry. For instance, for the case of a bare homogeneous sphere they can be found from those for a bare homogeneous slab, which we already know, and we also know the order of magnitude of the error committed in using them. The boundary conditions at the free surface of such a sphere become

$$\rho_{\text{diff}}(a+z_0) = 0 \tag{8.9}$$

if we use (8.5), or

$$a\rho_{\text{diff}}(a) = -\lambda\left[\frac{d\{r\rho_{\text{diff}}(r)\}}{dr}\right]_{r=a} \tag{8.10}$$

if we use (8.5′); in either case, the error will be of the order of $e^{-2a/l}$, which is completely negligible if $a \gg l$.

In the case where the sphere consists of an inner homogeneous core of radius b surrounded by a *reflector* of thickness $a-b$, both having the same mean free path, the same arguments as before will lead to the same boundary condition (8.9) or (8.10) at the outer surface, but now the error will be of the order of $e^{-(a-b)/l}$. This in fact holds whether or not the mean free path is the same in core and reflector, since it is obvious physically that a change in the core material cannot affect the conditions near the surface of the reflector except by quantities of the order of $e^{-(a-b)/l}$. This conclusion can be verified mathematically by calculating the analogue of (8.8) for $b < r < a$; the result is identical with that for a slab containing sources whose strength is of the order $\exp[-(r-b)/l]$, and these sources cannot affect the solution near $r = a$ by more than the order of $\exp[-(a-b)/l]$.

8.3. The boundary conditions at a free surface

We shall now summarize the boundary conditions which are at present believed to be the correct ones for use in the diffusion approximation. Some of these will be the result of direct application of the above calculations, others will be guesses that are only indirectly supported by calculation.

If the boundary condition at the free surface is taken in the form (8.5) for the plane case and (8.9) for the spherical case, we notice that these are of the same form. In fact, the form of (8.9) is independent of

the radius of the sphere, and this leads us to suppose that the extrapolated end-point for any shape of surface will be independent of the actual shape; that is, for any non-re-entrant surface the boundary condition can, to a reasonable approximation, be put in the form:

$\rho_{\text{diff}}(r)$ *vanishes at a distance* z_0 *outside the actual surface of the medium.* (8.11)

The form (8.10) of the boundary condition, on the other hand, involves the curvature of the surface, and consequently the extension to arbitrary shapes of boundary surface is more uncertain.

The question sometimes arises as to the proper form of the boundary condition at a sharp edge, such as that of a cube or a finite cylinder. In this case, it is usual to form a larger cube or cylinder whose dimensions each exceed those of the actual body by $2z_0$ (that is, z_0 at each side). At the sharp edges, $\rho_{\text{diff}}(r)$ will then vanish at a distance $\sqrt{2}z_0$ from the surface, rather than z_0, and this may appear to contradict (8.11). There are, however, two considerations to be borne in mind. Firstly, at the sharp edges $\rho_{\text{diff}}(r)$, defined as above, will behave as the square of the distance from the edge, not as the first power, and so the effect of the deviation from (8.11) will be negligible if the dimensions are $\gg z_0$ (and otherwise the diffusion approximation is unjustified). Secondly, (8.11) is based on the study of plane and spherical systems, and cannot be regarded as necessarily accurate for sharp edges.

8.4. The boundary conditions at an interface

In the case of a plane boundary between two media, where the flux does not vary laterally, the formulae (7.14) and (7.15) show that, whatever the mean free paths in the two media, provided that the values of c are not very different, the best boundary conditions available are

$$(L^2/l^2)(1-c)\rho_{\text{diff}}(\mathbf{r}) \text{ continuous} \tag{8.12}$$

and

$$\mathbf{j}_{\text{diff}}(\mathbf{r}) \text{ continuous.} \tag{8.13}$$

These conditions are, however, not the result of exact calculation, but only an approximation to it. To derive the best possible boundary conditions would require a tabulation of the G_{ij} in (7.9) for a wide range of c_1 and c_2, and this is not yet available.

The arguments of § 8.2 show that the same conditions can be applied to the spherical case, provided that the mean free path is the same in the two media. The diffusion approximation for the current is now to be taken as

$$\mathbf{j}_{\text{diff}}(\mathbf{r}) = (L^2/l)(c-1)\mathbf{grad}\,\rho_{\text{diff}}(\mathbf{r}), \tag{8.14}$$

which is a direct generalization† of (6.37) and is, of course, evaluated separately for each medium.

Since, when the mean free paths are the same in both media and the mean numbers of secondaries nearly so, the conditions (8.12) and (8.13) are applicable to both plane and spherical interfaces, we may suppose that the same is true for other shapes of interface.

No exact calculations are available for the case of differing mean free paths, and so one would normally continue to use (8.12) and (8.13). If c is close to unity in both media, then (6.34) shows that (8.13) at least cannot be seriously in error.

The presence of a lateral variation of the flux, which has been neglected hitherto, will not influence the boundary conditions to any great extent, since the tangential components of $\mathbf{grad}\,\rho$ cannot appear, by symmetry, and the second derivatives of ρ in tangential directions will be at most of the order of $|c_1 l_1^2/a^2 - c_2 l_2^2/a^2|$, where a is some characteristic geometrical dimension, and this is commensurate with the likely error in (8.12). This also implies that, in the case of lateral variation of the flux, the condition (8.13) must be taken to imply the continuity only of the normal component of $\mathbf{j}_{\text{diff}}(\mathbf{r})$. The continuity of the tangential component will be ensured by (8.12) if the mean free paths in the two media are the same, while, if they are different, there is no reason why the tangential current should be continuous.

If only the critical size of the system is of interest, and the constant cross-section approximation is used, then (8.14) shows that the conditions (8.12) and (8.13) are equivalent to the simpler ones

$$\left.\begin{array}{r} \rho_{\text{diff}}(\mathbf{r})\ \textit{continuous} \\ l\,\mathbf{grad}\,\rho_{\text{diff}}(\mathbf{r})\ \textit{continuous} \end{array}\right\}. \tag{8.15}$$

These can, of course, be used only in the conditions stated, for, unless c is the same for both media, the second of (8.15) means that more neutrons come to the interface than leave it, or vice versa, in some direction. It is impossible to use (8.15) when there are sources in one region and the

† The derivation of (8.14) is as follows: in (6.34), $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are replaced by their diffusion approximations, and the result is applied to a source-free region in one of the media, giving

$$\iint_A \mathbf{j}_{\text{diff}}(\mathbf{r}).\mathbf{dA} = \frac{c-1}{l} \iiint_V \rho_{\text{diff}}(\mathbf{r})\, dV.$$

Substituting from (5.4), this is

$$\iint_A \mathbf{j}_{\text{diff}}(\mathbf{r}).\mathbf{dA} = \frac{L^2(c-1)}{l} \iiint_V \text{div}\,\mathbf{grad}\,\rho_{\text{diff}}(\mathbf{r})\, dV$$

since $\nabla^2 = \text{div}\,\mathbf{grad}$, and (8.14) follows by Gauss's theorem.

number of neutrons escaping from or captured in another region is required. This is particularly so when the strength of the sources in a region depends on the capture in another region, as often happens in multi-group theory (Chapter XIX).

8.5. Black bodies

In dealing with thermal neutrons, it may happen that a very strong absorber, such as cadmium, has been introduced into the medium, and that the chance of any neutron entering the absorber and escaping capture in it can be neglected. Such absorbers are called *black bodies*. The effects of such bodies on the neutron distribution, and in particular the boundary conditions to be applied at their surfaces in using the diffusion approximation, have been studied in detail, and we give here the principal results.

If the extent of the black body is very large compared with the mean free path in the surrounding medium, then its surface may clearly be regarded as a free surface, and the condition (8.9) or (8.10) may be used. If, on the other hand, its dimensions are small compared with the mean free path, then the boundary conditions can be derived by the following argument.

Suppose for simplicity that the medium is infinite and non-capturing and has no sinks at infinity. In the absence of the black body, the only possible solution for the neutron flux would be $\rho(\mathbf{r}) = \text{constant} = \rho(\infty)$, and so the angular distribution would be

$$\psi(\mathbf{r},\Omega) = \text{constant} = \rho(\infty)/4\pi. \tag{8.16}$$

If the dimensions of the black body are small compared with the mean free path, its effects on the angular distribution will extend over only a fraction of a mean free path. A neutron entering the black body, however, will have come from a distance of the order of one mean free path. The angular distribution of entering neutrons will therefore still be approximately given by (8.16), and so, if the black body is convex without sharp edges, the current at a point a of its surface will be

$$\mathbf{j}(\mathbf{a}) = \iint_{\mu<0} \frac{\rho(\infty)}{4\pi}\,\Omega\,d\Omega = \frac{\rho(\infty)}{4\pi}\int_{-1}^{0}\mu\,d\mu\int_{0}^{2\pi}d\phi = -\frac{\rho(\infty)}{4}. \tag{8.17}$$

We now consider some volume V enclosing the black body, and so large that the diffusion approximation is valid near its surface. Since the medium is assumed non-capturing, the net current of neutrons into V (i.e. the current integrated over the outer surface of V) will equal the

total current into the black body, whether we use the exact solution or the diffusion approximation. In dealing with a non-capturing medium, therefore, the diffusion approximation to the current at the boundary can always be taken as identical with the correct current there. This is always valid on the average, and the more symmetrical the body, the more accurate it will be. The formula (8.17) thus gives $\mathbf{j}_{\mathrm{diff}}(\mathbf{a})$ also, and combining this with (8.14) we have

$$l\,\mathbf{grad}\,\rho_{\mathrm{diff}}(\mathbf{r}) = \tfrac{3}{4}\rho(\infty) \quad \text{for} \quad \mathbf{r} = \mathbf{a}, \tag{8.18}$$

since $L^2(1-c)/l^2 = -\frac{1}{3}$ when $c = 1$.

It remains to make (8.18) suitable for application by eliminating $\rho(\infty)$ in favour of $\rho_{\mathrm{diff}}(\mathbf{a})$. For a non-capturing medium the equation for $\rho_{\mathrm{diff}}(\mathbf{r})$ is $\nabla^2\rho_{\mathrm{diff}}(\mathbf{r}) = 0$, and the solution of this equation possessing the required value at infinity and satisfying the boundary condition (8.18) is clearly

$$\rho_{\mathrm{diff}}(\mathbf{r}) = \rho(\infty)[1+O(a^2/rl)].$$

We take the value of this expression for $\mathbf{r} = \mathbf{a}$ and combine it with (8.18), using the definition of the linear extrapolation length (6.24) and the assumption $a \ll l$, obtaining finally

$$\lambda = \tfrac{2}{3}l \qquad \text{for } a \ll l. \tag{8.19}$$

It is to be noticed that this derivation does not use the true value of the neutron flux at the surface of the black body; in fact this is not possible, as there is no simple relation between $\rho(\mathbf{a})$ and $\rho_{\mathrm{diff}}(\mathbf{a})$. However, as we have seen, (8.18) in any case contains all the information that the diffusion theory is capable of utilizing.

The above derivation presupposes that the black body is of small extent in at least two directions, and finite in all three—the last condition is needed to ensure the finiteness of $\rho(\infty)$, as may be seen from the discussion of a line source in § 5.5.1. However, supposing the black body to be small, for instance, in the directions of x and y, and of finite extent h in the z direction, we see that h does not enter (8.19) explicitly, and so as $h \to \infty$ the result (8.19) should be applicable also to such bodies as an infinite cylinder of small radius. Further, although we have assumed that $c = 1$, it is clear that these results will hold also for $c \neq 1$, provided that $|c-1| \ll 1$.

We thus know the value of the linear extrapolation length for very large (8.10) and very small (8.19) black bodies. Since λ must vary monotonically with the size of the body, it follows that for any black body λ will lie within the limits

$$0{\cdot}7104l \leqslant \lambda \leqslant \tfrac{2}{3}l;$$

the actual laws of the variation of λ/l with a/l for a black sphere and a black cylinder have been examined for a non-capturing medium by Marshak (35) and Davison (12). The results are shown below. We shall later describe some of the methods used by these authors, though not in application to this particular problem.

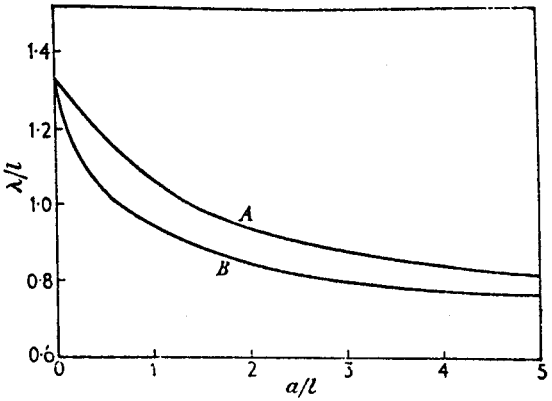


FIG. 6. A, black sphere; B, black cylinder.

8.6. Cavities and gaps

8.6.1. *Cases where the presence of cavities has no effect*

We finally consider the boundary conditions at cavities and gaps. These will include the surfaces of re-entrant bodies, and those of sub-systems which are exposed to neutrons from other sub-systems.

In two situations, the presence of a cavity or gap is without effect. Firstly, if there is a cavity of any shape in a non-capturing medium, and the conditions at the outer surface or at infinity are such that $\rho(\mathbf{r}) = \text{constant}$ would be a solution in the absence of the cavity, then it is a solution in the presence of the cavity also; for in this case the expression for the angular distribution becomes

$$\psi(\mathbf{r},\Omega) = \frac{\rho}{4\pi}\int_0^\infty e^{-\tau(\mathbf{r},\,\mathbf{r}-R\Omega)}\,dR/l,$$

and the numerical value of this is unaffected by the presence of the cavity.

Secondly, if there is plane symmetry (in which case the gap must take the form of a plane slit), the solution, if dependent only on x, can be expressed in terms of the optical depth alone, eliminating the geometrical

coordinates. Since the optical thickness of the gap is zero, the result follows.

8.6.2. *The case of a plane slit*

We next consider the case of a plane slit when there is lateral variation of the neutron flux (that is, over the surface of the slit). It is assumed that the number of secondaries per collision in the media on either side of the slit is close to unity, so that the diffusion lengths are large compared with the respective mean free paths, and it is also assumed that the lateral variation of the neutron flux over one mean free path is small. This latter assumption presupposes that there are no localized sources or other irregularities near the slit.

Let the coordinates be chosen so that the surfaces of the slit are $z = 0$ and $z = h$ (say), and let the components of Ω in the directions of x, y, and z be respectively $(1-\mu^2)^{\frac{1}{2}}\cos\phi$, $(1-\mu^2)^{\frac{1}{2}}\sin\phi$, and μ; with this convention, we now write $\psi(\mathbf{r},\Omega)$ as $\psi(x,y,z,\mu,\phi)$. We first evaluate the z-component of the current $j_z(x,y,0)$ at a point $(x,y,0)$ on the lower surface of the slit. The direction of a neutron is given for $\mu > 0$ by stating the values of μ and ϕ, and for $\mu < 0$ by stating the coordinates x' and y' of the point where it meets the upper surface of the slit. Within the slit the Boltzmann equation is $\Omega.\mathbf{grad}\,\psi = 0$, and hence, so long as $\mathbf{r}-R\Omega$ is within the slit, $\psi(\mathbf{r}-R\Omega,\Omega)$ is independent of R. Since $d\Omega = d\mu d\phi$, while for neutrons crossing the slit

$$\mu = -h/p$$

and $\quad d\mu d\phi = h\,dx'dy'/p^3, \quad$ where $p^2 \equiv h^2+(x-x')^2+(y-y')^2$,

we find from (4.6)

$$j_z(x,y,0) = \int_0^1 \mu\,d\mu \int_0^{2\pi} \psi(x,y,0,\mu,\phi)\,d\phi - \\ -\int\int \frac{h^3\,dx'dy'}{p^4}\,\psi\left(x',y',h,-\frac{h}{p},\tan^{-1}\frac{y'-y}{x'-x}\right). \quad (8.20)$$

This formula is so far exact. To apply it to the diffusion approximation one usually proceeds as follows: $j_z(x,y,0)$ is identified with $\mathbf{j}_{\text{diff}}(x,y,0)$ and the quantity $\psi_{\text{diff}}(\mathbf{r},\Omega)$ is introduced, defined by

$$\psi_{\text{diff}}(\mathbf{r},\Omega) = \frac{1}{4\pi}[\rho_{\text{diff}}(\mathbf{r})+3\Omega.\mathbf{j}_{\text{diff}}(\mathbf{r})], \quad (8.21)$$

and the values of $\psi(\mathbf{r},\Omega)$ appearing in (8.20) are approximated by

$\psi_{\text{diff}}(\mathbf{r},\Omega)$. On performing the integrations over μ and ϕ and rearranging, we have

$$\tfrac{1}{4}\rho_{\text{diff}}(x,y,0)-\tfrac{1}{2}j_{\text{diff}}(x,y,0)$$
$$=\frac{h^2}{4\pi}\int\int\frac{\rho_{\text{diff}}(x',y',h)\,dx'dy'}{p^4}-\frac{3h^3}{4\pi}\int\int\frac{j_{\text{diff}}(x',y',h)\,dx'dy'}{p^5}. \quad (8.22)$$

A similar expression is obtained for the point (x,y,h); this we call (8.22′). In the above expressions, j_{diff} is understood to be the z-component of $\mathbf{j}_{\text{diff}}$, and the other two components are assumed to be zero.

The approximations used in deriving (8.22) may be justified as follows, assuming that the variation of the flux in a lateral direction is negligible over one mean free path. The dependence on the azimuthal angle of the angular distribution of neutrons entering the slit should be negligible, and so should the lateral components of the current up to a few mean free paths into either medium; the lateral components of $\mathbf{j}_{\text{diff}}(\mathbf{r})$ at the surfaces of the slit can therefore also be neglected. The neglect of the difference between the normal components of $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}_{\text{diff}}(\mathbf{r})$ may be justified by arguments similar to those used in connexion with (7.15).

The main error in (8.22) therefore arises from approximating $\psi(\mathbf{r},\mu)$, which equals $(1/2\pi)\int\psi(\mathbf{r},\mu,\phi)\,d\phi$, by $\psi_{\text{diff}}(\mathbf{r},\mu)$ at the surfaces of the slit. The magnitude of the error can be estimated from the fact that, when (8.22) and (8.22′) are applied to the case of plane symmetry (i.e. where there is no lateral variation and the slit is of infinite lateral extent), one obtains the conditions $\rho_{\text{diff}}(z)$ and $j_{\text{diff}}(z)$ continuous, instead of the more accurate conditions (7.14).

In view of this, it might be supposed that the boundary conditions at the surface of the slit could be improved, while retaining the diffusion approximation, as follows. Let us define, instead of (8.21), the quantity

$$\bar{\psi}_{\text{diff}}(\mathbf{r},\Omega)=\frac{3}{4\pi}\left[\frac{L^2(1-c)}{l^2}\rho_{\text{diff}}(\mathbf{r})+\Omega.\mathbf{j}_{\text{diff}}(\mathbf{r})\right], \quad (8.23)$$

and approximate the angular distribution of neutrons entering the slit, that is, $\psi(\mathbf{r},\Omega)$ in (8.20), by $\bar{\psi}_{\text{diff}}(\mathbf{r},\Omega)$ instead of $\psi_{\text{diff}}(\mathbf{r},\Omega)$. We then have in place of (8.22)

$$\tfrac{3}{4}(L_1^2/l_1^2)(1-c_1)\rho_{\text{diff}}(x,y,0)-\tfrac{1}{2}j_{\text{diff}}(x,y,0)$$
$$=\frac{3h^2}{4\pi}\frac{L_2^2}{l_2^2}(1-c_2)\int\int\frac{\rho_{\text{diff}}(x',y',h)\,dx'dy'}{p^4}-\frac{3h^3}{4\pi}\int\int\frac{j_{\text{diff}}(x',y',h)\,dx'dy'}{p^5}, \quad (8.24)$$

where the subscript 1 refers to the medium in $z<0$, and 2 to that in $z>h$, and a corresponding formula (8.24′).

However, though these latter conditions are certainly superior to (8.22) and (8.22′) when $\rho(\mathbf{r})$ and $j(\mathbf{r})$ vary very slowly in the lateral direction, nothing can be said as to their respective merits when this is no longer the case.

8.6.3. *Gaps and cavities of other shapes*

The analysis given above for the case of a plane slit can be extended at once to gaps and cavities of other shapes. We shall not give details, but only some general remarks on the form of the extensions.

Firstly, the factor $-h^2/p^4$ which accompanies the angular distribution $\psi(x', y', h)$ in the integrand of (8.20) can be written as

$$\cos\theta \cos\theta'/R^2, \tag{8.25}$$

where R is the distance of the area element dA, where the current is evaluated, from the area element dA' where the neutrons are emitted, θ is the angle between the normal to dA and the direction of travel of the neutrons, and similarly for θ'. Since $\cos\theta$ arises from the definition of the normal component of the current, while $\cos\theta'|dA'|/R^2$ is the solid angle subtended by dA' at a point on dA, it appears that the factor multiplying the angular distribution at the emitting surface in the expression for the current at dA will always be of the form (8.25), whether or not the gap has the form of a plane slit. The integration will, of course, extend over all area elements dA' which are 'visible' from dA.

Secondly, in cases of spherical or cylindrical symmetry, where the flux depends only on the radial coordinate, although the existence of the slit cannot be ignored as it can in the case of a plane slit in similar conditions, the following simplification occurs. No neutrons can be created or destroyed in the gap, and since, in the symmetry conditions assumed, the normal component of the neutron current should be a constant on either face of the gap, it follows that it must be inversely proportional to the area of the corresponding face. Identifying $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}_{\text{diff}}(\mathbf{r})$, we then have

$$\left.\begin{aligned} a^2 j_{\text{diff}}(a) &= b^2 j_{\text{diff}}(b) \quad \text{for the spherical case} \\ a j_{\text{diff}}(a) &= b j_{\text{diff}}(b) \quad \text{for the cylindrical case} \end{aligned}\right\}, \tag{8.26}$$

where a and b are the inner and outer radii of the gap.

It is evident that the boundary condition (8.26) will be a consequence of the analogues of (8.24) and (8.24′) (or of (8.22) and (8.22′)) for this case. In dealing with gaps in a system where the neutron flux has spherical or cylindrical symmetry, it is therefore possible to use (8.26) and the analogue of either (8.24) or (8.24′).

8.7. The effects of sources

It is necessary to decide also how the presence of sources is to be taken account of in the diffusion approximation. We distinguish two cases, one where the sources are situated at isolated points and the flux is considered at distances from them of many mean free paths, and the other where the sources are distributed over a volume V at a strength slowly varying with position, and the flux is considered throughout V.

In the first case, (5.25) shows that the diffusion approximation to $\rho(\mathbf{r})$ represents the solution of the equation

$$\left(\nabla^2 - \frac{1}{L^2}\right)\rho_{\text{diff}}(\mathbf{r}) = -\frac{2l(L^2-l^2)}{cL^2(cL^2+l^2-L^2)}S(\mathbf{r}), \tag{8.27}$$

where $S(\mathbf{r})$ is the number of neutrons emitted per unit time per unit volume by the sources at $\mathbf{r}$.

In the second case, however, the analysis of Chapter V is not directly applicable. Instead, we at first suppose the sources uniformly distributed in V with strength S_0 per unit volume per unit time. Let the neutron flux be represented as

$$\rho(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_2(\mathbf{r}),$$

where $\rho_1(\mathbf{r})$ is what the flux would be if V and its sources were infinite in extent, and $\rho_2(\mathbf{r})$ is the deviation of $\rho(\mathbf{r})$ from this. Evidently $\rho_1(\mathbf{r})$ is a constant, whose value is given by the condition that the number of neutrons produced by the sources is equal to the number captured, and hence

$$(1-c)\rho_1/l = S_0. \tag{8.28}$$

On the other hand, $\rho_2(\mathbf{r})$ may be regarded as the distribution in a source-free medium, and according to the results of Chapter V it may be divided into $\rho_{2,\text{diff}}(\mathbf{r})$ and $\rho_{2,\text{tr}}(\mathbf{r})$. To extend the diffusion approximation to the present case, we should identify $\rho_{\text{tr}}(\mathbf{r})$ with $\rho_{2,\text{tr}}(\mathbf{r})$; that is, $\rho_{\text{diff}}(\mathbf{r})$ is defined by

$$\rho_{\text{diff}}(\mathbf{r}) = \rho_1(\mathbf{r}) + \rho_{2,\text{diff}}(\mathbf{r}). \tag{8.29}$$

On combining (8.28), (8.29), and (5.4) for $\rho_{2,\text{diff}}(\mathbf{r})$, we have then

$$\left(\nabla^2 - \frac{1}{L^2}\right)\rho_{\text{diff}}(\mathbf{r}) = -\frac{lS_0}{L^2(1-c)}. \tag{8.30}$$

The same argument can be applied when the source strength per unit volume is not a constant but is a linear function of the Cartesian co-ordinates of $\mathbf{r}$. Assuming that the second and higher derivatives of the source strength are too small to have effect, the equation for $\rho_{\text{diff}}(\mathbf{r})$ in

the region occupied by the sources becomes

$$\left(\nabla^2-\frac{1}{L^2}\right)\rho_{\text{diff}}(\mathbf{r}) = -\frac{lS(\mathbf{r})}{L^2(1-c)}. \tag{8.31}$$

The appearance of the factor $l/L^2(1-c)$ in the free term of (8.31) can be understood on the grounds that the output of the sources is measured in terms of $\mathbf{j}(\mathbf{r})$ and not of $\mathbf{grad}\,\rho(\mathbf{r})$, and these two quantities are connected by (6.37). The free term of (8.27) contains a further factor

$$\frac{2(1-c)(L^2-l^2)}{c(cL^2+l^2-L^2)} \cong 1-\tfrac{1}{3}(1-c)+\ldots; \tag{8.32}$$

this arises because only the fraction (8.32) of the source neutrons is associated with the generation of $\rho_{\text{diff}}(\mathbf{r})$, i.e. can travel a distance of the order of L before capture, and only this fraction is therefore of interest in the first case discussed above, whose solution is (8.27).

8.8. A remark on orthogonality

We conclude the present chapter by considering a rather different aspect of the diffusion approximation. We have seen in § 4.7 (see equation (4.30)) that, if the number of secondaries per collision, c, is regarded as a variable parameter, then the exact solutions of the problem for different eigenvalues are mutually orthogonal. The question arises whether this is true in the diffusion approximation also. However, it may be seen from simple examples that this is not in general the case, when the diffusion theory is used with the best boundary conditions available.

The mathematical reason for the lack of orthogonality is that (supposing for simplicity that the system is bare, homogeneous, and non-reentrant) the differential equation of the diffusion approximation will involve κ^2, while the boundary conditions (which may be taken in the form (8.11)) will involve z_0. Both these quantities depend on c, but the relation between them is not linear, and it is known that in problems nonlinear in the parameter the different eigenfunctions are not in general orthogonal.

The physical reason for the lack of orthogonality is as follows. Let suffixes denote the successive eigenfunctions. It is known that the exact solution $\rho_1(\mathbf{r})$ is orthogonal to $\rho_2(\mathbf{r})$. If $\rho_{1,\text{diff}}(\mathbf{r})$ were also orthogonal to $\rho_{2,\text{diff}}(\mathbf{r})$, this would imply that

$$\iiint [\rho_{1,\text{tr}}(\mathbf{r})\rho_{2,\text{diff}}(\mathbf{r})+\rho_{1,\text{diff}}(\mathbf{r})\rho_{2,\text{tr}}(\mathbf{r})+\rho_{1,\text{tr}}(\mathbf{r})\rho_{2,\text{tr}}(\mathbf{r})]\,dV = 0, \tag{8.33}$$

and there is no reason why this should be so.

It is of interest to remark that if the most accurate boundary conditions available are *not* used, but instead z_0 is taken as $0{\cdot}7104l$ regardless of the value of c (cf. (6.27), (6.29)), then the problem becomes linear in κ^2 and the successive eigenfunctions will again be orthogonal. We have seen, however, from (8.33) that this circumstance has no particular significance. Of course, the above remarks apply to a system containing two or more media only if c varies in just one medium.

IX

THE SERBER–WILSON METHOD

9.1. Serber's formulation of the method

We have seen from the discussions of Chapters V and VIII that the error involved in the use of the diffusion approximation is due, not so much to the replacement of $\rho(\mathbf{r})$ by a solution of the differential equation (8.2), as to the attempt to specify the solution of (8.2) by means of boundary conditions, while the boundaries are the very places where the approximation of $\rho(\mathbf{r})$ by a solution of (8.2) is least justified. This suggests that the quality of the approximation could be considerably improved if the solution of (8.2) were specified not by boundary conditions but by some integral-type condition.

A method of this nature was suggested by Wilson (55), and an essentially similar one independently by Serber (39). The method can be applied only to the determination of the critical size of spherically symmetrical systems, but in some cases it shows a noticeable improvement in accuracy over the diffusion approximation. We first give Serber's formulation of the method.

The best test of the accuracy of a particular solution of (8.2) is to substitute it into the exact integral equation for $\rho(\mathbf{r})$ and find the difference between the two sides. In dealing with spherically symmetric systems, it is clearly simplest to apply this test at the centre, since the integration need then be effected only over the radial coordinate, the integration over angles being replaced by multiplication by 4π. Serber accordingly stipulated that

> *The integral equation* (4.15′) *shall be exactly satisfied at the centre of the system.* (9.1)

In finding the critical radius of a bare homogeneous sphere, there is only one region in which (8.2) is to be solved; the solution must be spherically symmetrical (since only this can give the lowest eigenvalue which determines the critical configuration) and regular in the sphere. There is only one such solution, apart from an arbitrary factor, namely

$$\rho(0)\frac{\sin \kappa r}{\kappa r}. \tag{9.2}$$

The only parameter to be determined is the critical radius itself, and for this the single condition (9.1) is naturally sufficient.

In finding the critical radius of a homogeneous spherical core in an infinite homogeneous *reflector* (assuming that for the core $c > 1$ and for the reflector $c < 1$), the relevant solution of (8.2) is again (9.2) for the core, while that for the reflector is

$$(A^-/r)e^{-r/L_2}, \tag{9.3}$$

where now $\kappa = i/L_1$ and L_1 and L_2 are the solutions of (5.5) for the core and reflector respectively; A^- is an arbitrary constant. The other spherically symmetrical solution of (8.2) for the reflector, namely

$$(A^+/r)e^{+r/L_2}, \tag{9.4}$$

is omitted, since it gives $\rho(\infty) = \pm\infty$, and this means there is a supply of neutrons to or from infinity, which is assumed not to be the case in problems of critical size. We therefore have two parameters to determine, the critical radius a and the ratio $\rho(0)/A^-$. To do this, two conditions are required, i.e. (9.1) must be supplemented by another condition. This is easily done in the present case, since in a stationary problem the number of neutrons in the system must be conserved, that is,

The excess number of neutrons produced in the core = the number captured in the reflector + the number escaping from the system. (9.5)

If the reflector is infinite in extent, the second term on the right is zero, and the other terms of (9.5) can be evaluated by substituting the relevant solutions of (8.2) for $\rho(r)$. This provides the second condition to supplement (9.1).

9.2. Wilson's formulation of the method

In more involved cases, such as those where the reflector is of finite thickness only, or where it consists of several layers of different properties, a larger number of conditions will be required. However, it has not been possible to find any further conditions which are as easy to apply and as convincing as (9.1) and (9.5). There is, nevertheless, a fairly plausible *ad hoc* generalization of these formulae. To obtain it, we shall first rewrite the above conditions as follows:

Let $\rho_{SW}(\mathbf{r})$ be a solution of (8.2) for each region in turn, having the necessary properties of symmetry and regularity; it differs from $\rho_{\text{diff}}(\mathbf{r})$ only in that the constants of integration are determined by different conditions. Let $\mathbf{j}_{SW}(\mathbf{r})$ be related to $\rho_{SW}(\mathbf{r})$ in the same way that $\mathbf{j}_{\text{diff}}(\mathbf{r})$ is related to $\rho_{\text{diff}}(\mathbf{r})$ (see (8.14)). Let $\rho_{SW,j}(\mathbf{r})$ coincide with $\rho_{SW}(\mathbf{r})$ when $\mathbf{r}$ is in the jth region, and otherwise let it be defined by analytic continuation. Thus, if $\rho_{SW,j}(\mathbf{r})$ is regular in all space, it can be interpreted as the neutron flux that would exist if the jth medium occupied

all space. Let $\psi_{SW}(\mathbf{r},\Omega)$ be, for $\mathbf{r}$ in the jth region, the angular distribution which would arise from a flux $\rho_{SW,j}(\mathbf{r})$ throughout space, with l and c having everywhere their values in the jth region. According to (4.16),

$$\psi_{SW}(\mathbf{r},\Omega) = \frac{c_j}{4\pi l_j}\int_0^\infty \rho_{SW,j}(\mathbf{r}-R\Omega)e^{-R/l_j}\,dR, \tag{9.6}$$

if $\mathbf{r}$ is in the jth medium.

In our application to spherically symmetrical solutions, we shall be concerned only with the radial component $j_{SW}(r)$ of $\mathbf{j}_{SW}(\mathbf{r})$, while $\psi_{SW}(\mathbf{r},\Omega)$ will depend only on r and on the radial component of Ω; the latter we call μ and accordingly write $\psi_{SW}(\mathbf{r},\Omega)$ as $\psi_{SW}(r,\mu)$.

We now express the conditions (9.5) and (9.1) in this notation. It has already been remarked that (6.34) is applicable to the diffusion approximation as well as to the exact solution. If (8.2) is assumed, either of (6.34) and (8.14) can be derived from the other. Since (8.2) and (8.14) hold for $\rho_{SW}(r)$ and $j_{SW}(r)$, so does (6.34), and so, expressing the condition (9.5) in terms of $j_{SW}(r)$, we obtain

$$4\pi a_1^2 j_{SW}(a_1-)+[4\pi a_2^2 j_{SW}(a_2-)-4\pi a_1^2 j_{SW}(a_1+)]+\ldots+$$
$$+[4\pi a_n^2 j_{SW}(a_n-)-4\pi a_{n-1}^2 j_{SW}(a_{n-1}+)] = 4\pi a_n^2 j_{SW}(a_n+), \tag{9.7}$$

where a_1, a_2, etc., are the radii of the interfaces between the successive regions, and a_n is the outer radius of the outermost region. If we approximate $\rho(r)$ by $\rho_{SW}(r)$, the first term in (9.7) represents the excess number (positive or negative) of neutrons created per unit time in the inmost region; the second term, that in the first spherical shell, and so on. This is the required form of (9.5).

For the centre of the system, again approximating $\rho(r)$ by $\rho_{SW}(r)$, and using polar coordinates and (9.6), (4.15′) becomes

$$\rho_{SW}(0) = 4\pi\{\psi_{SW}(0,-1)-\psi_{SW}(a_1-,-1)\}+$$
$$+4\pi\{\psi_{SW}(a_1+,-1)-\psi_{SW}(a_2-,-1)\}+\ldots. \tag{9.8}$$

For the inmost region, the solution of (8.2) will be regular at $r=0$, and is therefore of the form (9.2); its analytical continuation is thus regular in all space. Hence it is an exact solution of the corresponding infinite medium problem (see § 5.1), and therefore $\rho_{SW}(0) = 4\pi\psi_{SW}(0,-1)$. With (9.8) this gives

$$[\psi_{SW}(a_1-,-1)-\psi_{SW}(a_1+,-1)]+$$
$$+[\psi_{SW}(a_2-,-1)-\psi_{SW}(a_2+,-1)]+\ldots+\psi_{SW}(a_n-,-1) = 0. \tag{9.9}$$

This is the required form of (9.1).

When the conditions (9.1) and (9.5) have been put in the forms (9.9) and (9.7), it is seen that, if twice as many conditions are required as there are interfaces, together with one for a free surface, this is most naturally achieved by stipulating

$$j_{SW}(r) \text{ is continuous} \tag{9.10}$$

and $$\psi_{SW}(r, -1) \text{ is continuous and } \psi_{SW}(a_n, -1) = 0. \tag{9.11}$$

The discussion given above has assumed the absence of gaps in the system. If these are present, the expressions (9.7) and (9.10) must be modified as indicated by (8.26).

The condition (9.10) is identical with the condition (8.13) used in the diffusion approximation; the latter, as we have seen, was more justifiable than (8.12), which is now replaced by (9.11).

The conditions (9.10) and (9.11) were proposed by Wilson also, but on different grounds. These were that $j(r)$ and $\psi(r, -1)$ were less likely to be distorted, in approximating $\rho(r)$ by a solution of (8.2), than any other functions that can be as easily constructed from $\rho(r)$.

9.3. The explicit form of the Serber-Wilson conditions

We shall now investigate the nature of the calculations involved in applying the Serber-Wilson method. The condition (9.10) introduces only elementary functions, and so we shall concentrate on (9.11). Supposing first that $c < 1$, and substituting into (9.6) the general expression for $\rho_{SW}(r)$ in such a region

$$\rho_{SW,j}(r) = (A_j^-/r)e^{-r/L_j} + (A_j^+/r)e^{r/L_j}, \tag{9.12}$$

we find, suppressing the subscript j,

$$\psi_{SW}(r, -1) = \frac{c}{4\pi l}\int_0^\infty \frac{dR}{r+R}\left[A^- e^{-(R/l)-[(R+r)/L]} + A^+ e^{-(R/l)+[(R+r)/L]}\right]$$

$$= \frac{c}{4\pi l} e^{r/l}\left[A^- E_1\left(\frac{r}{l}+\frac{r}{L}\right) + A^+ E_1\left(\frac{r}{l}-\frac{r}{L}\right)\right]. \tag{9.13}$$

Similarly, if $c > 1$, limiting ourselves for simplicity to the inmost region, where $\rho_{SW}(r)$ is given by (9.2), we obtain

$$\psi_{SW}(r, -1) = -\frac{c\rho(0)}{4\pi l\kappa} e^{r/l} \operatorname{im} E_1\left(\frac{r}{l} + i\kappa r\right). \tag{9.14}$$

The exponential integral function E_1 has been tabulated (62) for complex arguments, so that no numerical integration is needed in applying the Serber-Wilson method to spherically symmetrical problems.

If the critical radius is required for a number of situations, e.g. for various compositions of the regions of the system, it may be rapidly obtained by means of a graphical method described by Melvin (39).

9.4. The accuracy and limits of applicability of the method

We finally attempt a qualitative estimate of the accuracy obtainable by the use of the Serber–Wilson method. Considering first the case of a bare homogeneous sphere, we have seen in Chapter VI that $\rho_{tr}(r)$ is negative for an infinite half-space, and the same should be true for a bare sphere. The solution of (8.2) which agrees with the correct $\rho(r)$ at $r = 0$ will therefore be greater than $\rho(r)$ everywhere else, and from

$$\int_0^{a_{SW}} \rho_{SW}(r)e^{-r/l}\,dr = \int_0^{a} \rho(r)e^{-r/l}\,dr$$

(see (4.15) and (9.1)), it follows that $a_{SW} < a$, that is, the Serber–Wilson method necessarily underestimates the critical radius. Wilson arrived at the same conclusion in a different manner, as follows. If

$$\rho(r) = [\rho(0)/\kappa r]\sin \kappa r$$

everywhere (and this, as we know, is a permissible solution of the exact integral equation in the infinite medium case), $\psi(r, \mu)$ is the corresponding angular distribution (which can be constructed by using (4.16)), and $r = a$ is the smallest solution of $\psi(r, -1) = 0$, then it follows from (4.16) that for this r and any $\mu \neq -1$, $\psi(a, \mu)$ is positive. Hence $\rho_{SW}(r)$ is the exact solution for the problem where a certain number of neutrons fall obliquely on the sphere from outside, though none fall normally. Wilson has calculated the ratio $\int_{-1}^{0} |\mu|\psi(a, \mu)\,d\mu \Big/ \int_0^1 \mu\psi(a, \mu)\,d\mu$ for a few sample values of κl, and has found it to be very small, i.e. the number of neutrons entering the sphere, introduced in applying the Serber–Wilson method, is very small compared with the number which escape from the system. The underestimation of the critical radius is therefore by only a very small amount.

The arguments used thus far have been based to some extent on the division of $\rho(r)$ into $\rho_{diff}(r)$ and $\rho_{tr}(r)$. As $c-1$, and consequently κl, increase, this procedure loses its significance, but on the other hand the relative spread of the values of $\rho(r)$ decreases, and the satisfying of the integral equation at (at least) one point becomes more important. The net effect would be expected to ensure the accuracy of the Serber–

Wilson method for larger values of κl than those for which the diffusion approximation is applicable. This has been shown to be the case by comparison with more accurate methods to be discussed later (two sample results are given in Table I), and it would seem that the Serber–Wilson method can be applied for any value of c, unless very high accuracy is required.

TABLE I

Comparison of critical radii as given by various approximate methods

c_{core}	a_1	a_2	a_3
1·30	1·79	1·77	1·77
1·80	0·90	0·92	0·95

a_1 = critical radius as given by the diffusion approximation.
a_2 = critical radius as given by the Serber–Wilson approximation.
a_3 = critical radius as given by the P_3 spherical harmonics approximation, which can be taken as almost exact.

The values of the a_r are given in mean free paths in the core. The system consists of a sphere in an infinite reflector with $c = 0{\cdot}985$ and l nearly the same as in the core.

For more complicated systems, it has been found that if there are no gaps, and $c > 1$ in the central region and $\leqslant 1$ elsewhere, which is the normal arrangement in a reactor, the error in the critical radius is a very slowly varying function of the conditions. It is therefore often enough to determine the correction to the Serber–Wilson result for one typical case, by comparison with more accurate calculations, and then to apply the same correction over a fairly wide range of conditions.

However, if gaps are present, or if the inmost region is capturing rather than multiplying, the accuracy turns out to be no better than in the diffusion approximation. In the former case this has not been explained, or even confirmed, but the poor accuracy in the case of a capturing central region is to be ascribed to the fact that the ratio $\int_0^a \rho_{tr}(r)e^{-r/l}\,dr \Big/ \int_0^a \rho_{SW}(r)e^{-r/l}\,dr$, where a is the radius of the central region, is larger than in the cases previously considered. Roughly speaking, in this case the centre becomes a rather non-representative point, and to satisfy the integral equation there no longer guarantees a good approximation throughout the system.

When the application of the Serber–Wilson method to systems with other symmetry properties (plane or cylindrical) is considered, it is found that Serber's condition, that the exact integral equation should be satisfied at the centre of the system (i.e. in the midplane, or along the axis of the cylinder), will no longer be equivalent to Wilson's condition,

that $\psi(r, -1)$ should be continuous. If the former condition is applied, the method becomes so complex that the accuracy it gives is no longer worth the effort needed to obtain the result; if the latter condition is applied, so much accuracy is lost that the simplification of the solution confers no advantage.

X

THE SPHERICAL HARMONICS METHOD FOR PLANE GEOMETRIES

10.1. A general outline of the spherical harmonics method

THE approximate methods of solution of the neutron transport equation which were described in the last two chapters were capable of giving answers only to a limited, if quite satisfactory, degree of accuracy. We shall now turn to methods which are able to give results of arbitrarily high accuracy, provided that a sufficient amount of labour is expended on their calculation.

Of these methods, the spherical harmonics method (first applied to neutron transport problems by Wick (53) and Marshak (35), and developed in detail by Mark (32, 33)) is the most useful except in isolated cases. For the sake of simplicity, we first outline it for the case of *plane geometry*, that is, where the neutron flux is a function of one Cartesian coordinate (x, say) only, and consequently the angular distribution depends only on x and on the x-component of $\Omega = \mu$, say. In symbols,

$$\rho(\mathbf{r}) = \rho(x); \qquad \psi(\mathbf{r},\Omega) = \psi(x,\mu).$$

Unlike the methods we have previously discussed, the spherical harmonics method takes as its starting-point not the integral equation (4.15′) but the Boltzmann integro-differential equation. In the constant cross-section approximation with isotropic scattering this has the form (4.4), while in the plane case in the absence of sources it further reduces to

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\frac{\psi(x,\mu)}{l} = \frac{c}{2l}\int_{-1}^{1}\psi(x,\mu')\,d\mu'. \tag{10.1}$$

The angular distribution $\psi(x,\mu)$ is now expanded into spherical harmonics in μ, so that

$$\psi(x,\mu) = \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\psi_n(x), \tag{10.2}$$

where $P_n(\mu)$ are the Legendre polynomials, and

$$\psi_n(x) = \int\int \psi(x,\mu)P_n(\mu)\,d\Omega = 2\pi\int_{-1}^{1}\psi(x,\mu)P_n(\mu)\,d\mu. \tag{10.3}$$

The equation (10.1) is then multiplied by $(2n+1)P_n(\mu)$ and integrated over all μ, using the recurrence formula of Legendre polynomials:

$$(n+1)P_{n+1}(\mu)+nP_{n-1}(\mu) = (2n+1)\mu P_n(\mu),$$

which may be found in Whittaker and Watson (52), p. 308, and the result is

$$(n+1)\psi'_{n+1}(x)+n\psi'_{n-1}(x)+(2n+1)\frac{1-c\delta_{0n}}{l}\psi_n(x) = 0, \qquad (10.4)$$

where dashes denote differentiation with respect to x.

The quantities $\psi_n(x)$ are referred to as the *spherical harmonic moments of the angular distribution*. It is seen that the first two moments, $\psi_0(x)$ and $\psi_1(x)$, are identical with the flux $\rho(x)$ and the current $j(x)$ respectively.

The equations (10.4) form an infinite system of differential equations with an infinite number of unknowns. Though the exact solution of this system is impossible, an approximate solution can be found by assuming that $\psi'_{N+1}(x)$, say, is negligibly small. We shall later discuss the validity of this approximation, but it may be noted here that from (10.3)

$$\psi'_{N+1}(x) = 2\pi \int_{-1}^{1} P_{N+1}(\mu)\frac{\partial\psi(x,\mu)}{\partial x}\, d\mu,$$

and consequently, unless the dependence of $\partial\psi/\partial x$ on μ is of a delta-function character, $\psi'_{N+1}(x)$ should tend to zero as $N \to \infty$ while x remains constant. The uniformity of this convergence, however, cannot be thus proved.

If this approximation is made, the first $N+1$ equations in (10.4) contain only $N+1$ unknowns, $\psi_0(x),\ldots,\psi_N(x)$, so that the solution reduces to that of a system of finite order. In carrying this out, we employ the well-known procedure of seeking a solution in the form

$$\psi_n(x) = g_n e^{\nu x/l} \quad (n = 0, 1, \ldots, N), \qquad (10.5)$$

say, where the g_n are some constants. Substituting (10.5) into (10.4) we have

$$\nu[(n+1)g_{n+1}+ng_{n-1}]+[(2n+1)-c\delta_{0n}]g_n = 0, \qquad (10.6)$$

which is a system of equations to determine the g_n. If the equations (10.6) are compatible, the determinant of the coefficients must vanish, i.e.

$$\begin{vmatrix} 1-c & \nu & 0 & 0 & \ldots & 0 & 0 \\ \nu & 3 & 2\nu & 0 & \ldots & 0 & 0 \\ 0 & 2\nu & 5 & 3\nu & \ldots & 0 & 0 \\ 0 & 0 & 3\nu & 7 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & (N-1)\nu & 0 \\ 0 & 0 & 0 & 0 & \ldots & (2N-1) & N\nu \\ 0 & 0 & 0 & 0 & \ldots & N\nu & (2N+1) \end{vmatrix} = 0, \qquad (10.7)$$

and this equation determines the permissible values of ν in (10.5). The

g_n in (10.5) will, by (10.6), depend on ν; hence, if ν_j are the solutions of (10.7), the general solution of (10.4) in the present approximation will be

$$\psi_n(x) = \sum_j \tilde{A}_j g_n(\nu_j) e^{\nu_j x/l} \quad (n = 0, 1, \ldots, N), \tag{10.8}$$

where the $\tilde{A}_j$ are arbitrary constants to be determined from the boundary conditions.

The approximation in which $\psi'_{N+1}(x)$ is neglected, so that $\psi_N(x)$ is the highest spherical harmonic which appears in the equations, is called the P_N approximation. If N is odd, the approximation is called odd-order and the number of equations retained in (10.4) is even, and conversely.

10.2. The coefficients and exponents involved in the method

10.2.1. *Auxiliary functions*

It is convenient to introduce a set of auxiliary functions $G_n(\nu)$, defined by

$$G_n(\nu) = (-1)^n \left\{ P_n\left(\frac{1}{\nu}\right) - \frac{c}{\nu}\left[Q_0\left(\frac{1}{\nu}\right) P_n\left(\frac{1}{\nu}\right) - Q_n\left(\frac{1}{\nu}\right)\right]\right\}, \tag{10.9}$$

where P_n are, as before, the Legendre polynomials, and Q_n are the Legendre functions of the second kind (see Whittaker and Watson (52), pp. 316 ff.). Since the Q_n satisfy the same recurrence relations as the P_n, it follows that the G_n satisfy

$$(n+1)G_{n+1}(\nu) + \frac{2n+1}{\nu} G_n(\nu) + nG_{n-1}(\nu) = 0 \quad (n \geqslant 1), \tag{10.10}$$

while from the definition

$$G_0(\nu) = 1, \qquad G_1(\nu) = (c-1)/\nu. \tag{10.11}$$

The last two formulae can be regarded as an alternative definition of the G_n. On comparing these formulae with (10.6), we see that the coefficients g_n in (10.5) are proportional to the functions $G_n(\nu)$, so that the solution (10.8) of the equations (10.4) in the P_N approximation can be rewritten

$$\psi_n(x) = \sum_j A_j G_n(\nu_j) e^{\nu_j x/l}. \tag{10.8'}$$

Since the definition of $G_n(\nu)$ does not involve N, the recurrence relation (10.10) shows that (10.8') should hold also for $n = N+1$. Since, however, $\psi_{N+1}(x)$ is neglected in the P_N approximation, we should thus have $G_{N+1}(\nu_i) = 0$, i.e. the permissible values of ν in the P_N approximation must be the solutions of

$$G_{N+1}(\nu) = 0, \tag{10.12}$$

and therefore this should be identical with the determinantal equation (10.7). This can be directly verified by writing both equations in the form of continued fractions.

We reproduce here the tables (given by Mark (33)) of the permissible

TABLE II†

Values of ν_i for $N = 1, 3, 5, 7$

N	1	3		5			7			
i / c	1	1	2	1	2	3	1	2	3	4
1·0	0	0	1·972027	0	1·225211	3·202945	0	1·103185	1·591779	4·458086
0·9	0·547723	0·525560	2·055185	0·525430	1·240875	3·296688	0·525430	1·108937	1·615640	4·554851
0·8	0·774597	0·711876	2·145773	0·710456	1·260956	3·393117	0·710413	1·116799	1·642629	4·652966
0·7	0·948683	0·834508	2·241834	0·829085	1·286086	3·491504	0·828671	1·127655	1·672473	4·752078
0·6	1·095445	0·922637	2·341383	0·909562	1·316073	3·591190	0·907696	1·142345	1·704602	4·851871
0·5	1·224745	0·988751	2·442706	0·964672	1·349619	3·691603	0·959481	1·160900	1·738235	4·952060
0·4	1·341641	1·039801	2·544478	1·002603	1·384742	3·792265	0·992327	1·181880	1·772515	5·052401
0·3	1·449138	1·080124	2·645751	1·029086	1·419574	3·892788	1·012964	1·203058	1·806656	5·152683
0·2	1·549193	1·112590	2·745892	1·048012	1·452833	3·992862	1·026230	1·222733	1·840027	5·252727
0·1	1·643168	1·139170	2·844500	1·061917	1·483848	4·092252	1·035120	1·240155	1·872179	5·352384
0	1·732051	1·161256	2·941340	1·072421	1·512380	4·190778	1·041352	1·255230	1·902832	5·451533

† Tables II and III are extended to values of c greater than unity in a revision of Mark's report (33): Atomic Energy of Canada Limited, report CRT 338 (revised), 1957.

TABLE III

Values of $G_n(\nu_i)$ for $N = 1, 3, 5, 7$

N	1	3		5			7			
i	1	1	2	1	2	3	1	2	3	4
σ n										
1·0 0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
1	0	0	0	0	0	0	0	0	0	0
2		0	−0·500000	0	−0·500000	−0·500000	0	−0·500000	−0·500000	−0·500000
3		0	0·422577	0	0·680155	0·260177	0	0·755388	0·523528	0·186926
4				0	−0·596483	0·232846	0	−0·823284	−0·200561	0·301623
5	*			0	0·332190	−0·338997	0	0·738992	−0·192022	−0·271324
6							0	−0·542027	0·388296	−0·139774
7							0	0·279046	−0·288438	0·290791
0·9 0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
1	−0·182574	−0·190273	−0·648657	−0·190320	−0·080588	−0·030333	−0·190320	−0·090176	−0·061895	−0·021955
2		0·043058	−0·464487	0·043327	−0·402583	−0·486198	0·043328	−0·378023	−0·442535	−0·492770
3		−0·009698	0·409117	−0·010552	0·594450	0·266024	−0·010557	0·628264	0·497775	0·194946
4				0·002649	−0·536413	0·223434	0·002666	−0·707938	−0·207269	0·294678
5				−0·000633	0·302555	−0·334814	−0·000688	0·646497	−0·167299	−0·272409
6							0·000179	−0·478863	0·362566	−0·135920
7							−0·000044	0·247814	−0·273362	0·288912
0·8 0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
1	−0·259199	−0·280948	−0·093206	−0·281509	−0·158610	−0·058943	−0·281526	−0·179083	−0·121756	−0·042983
2		0·091987	−0·434844	0·094356	−0·311222	−0·473943	0·094428	−0·259469	−0·388816	−0·486143
3		−0·028064	0·399890	−0·033677	0·517229	0·272092	−0·033849	0·506610	0·475676	0·202789
4				0·012187	−0·484388	0·215126	0·012561	−0·599246	−0·215157	0·288338
5				−0·003936	0·277604	−0·331794	−0·004747	0·560546	−0·144771	−0·273775
6							0·001782	−0·420819	0·340876	−0·132410
7							−0·000591	0·219320	−0·261302	0·287513
0·7 0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
1	−0·316228	−0·359493	−0·133819	−0·361845	−0·233266	−0·085923	−0·362026	−0·266039	−0·179375	−0·063130
2		0·146177	−0·410462	0·154658	−0·227935	−0·463086	0·155312	−0·146117	−0·339123	−0·480073
3		−0·052280	0·394366	−0·069671	0·450897	0·278336	−0·071022	0·393319	0·457529	0·210460
4				0·031065	−0·442592	0·207808	0·033501	−0·500801	−0·224396	0·282551
5				−0·011707	0·258732	−0·329802	−0·015951	0·484741	−0·124517	−0·275393
6							0·007373	−0·370754	0·323490	−0·129213
7							−0·002851	0·195105	−0·252480	0·286548
0·6 0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
1	−0·365148	−0·433540	−0·170839	−0·439772	−0·303934	−0·111384	−0·440676	−0·350157	−0·234659	−0·082442
2		0·204838	−0·390552	0·225248	−0·153590	−0·458476	0·228232	−0·040213	−0·293507	−0·474512
3		−0·080996	0·391900	−0·119559	0·397128	0·284714	−0·125285	0·292108	0·443414	0·217961
4				0·061095	−0·412874	0·201365	0·070370	−0·417332	−0·235093	0·277269
5				−0·025259	0·246987	−0·328700	−0·039319	0·423905	−0·106481	−0·277233
6							0·020773	−0·332542	0·310434	−0·126302
7							−0·008799	0·177277	−0·246944	0·285973

0·5	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·408248	−0·505688	−0·204691	−0·518311	−0·370475	−0·135412	−0·521115	−0·430700	−0·287648	−0·100968
	2		0·267162	−0·374305	0·305938	−0·088245	−0·444966	0·314683	0·056509	−0·251776	−0·469416
	3		−0·113210	0·391850	−0·183030	0·355958	0·291186	−0·199211	0·206006	0·433175	0·225299
	4				0·102579	−0·395374	0·195688	0·127329	−0·352926	−0·247275	0·272444
	5				−0·044979	0·242548	−0·328365	−0·079502	0·382414	−0·090478	−0·279269
	6							0·045802	−0·309817	0·301491	−0·123647
	7							−0·020508	0·167844	−0·244562	0·285744
0·4	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·447214	−0·577033	−0·235805	−0·598442	−0·433294	−0·158217	−0·604640	−0·507666	−0·338502	−0·118755
	2		0·332419	−0·360990	0·395333	−0·030641	−0·437419	0·413972	0·144312	−0·213541	−0·464743
	3		−0·148136	0·393657	−0·258216	0·325742	0·297719	−0·292196	0·134938	0·426457	0·232478
	4				0·154205	−0·388683	0·190677	0·204818	−0·308036	−0·260884	0·268034
	5				−0·070276	0·244648	−0·328680	−0·137766	0·361187	−0·076236	−0·281474
	6							0·083842	−0·303577	0·296255	−0·121225
	7							−0·038826	0·167436	−0·245055	0·285823
0·3	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·485046	−0·648074	−0·264575	−0·680216	−0·493106	−0·179820	−0·691042	−0·581851	−0·387456	−0·135852
	2		0·400000	−0·350000	0·491486	0·021043	−0·430710	0·523297	0·225465	−0·178310	−0·460452
	3		−0·185164	0·396863	−0·342514	0·304032	0·304285	−0·400306	0·075551	0·422797	0·239504
	4				0·213844	−0·390582	0·186242	0·299097	−0·278997	−0·275806	0·263997
	5				−0·100029	0·252027	−0·329545	−0·211240	0·356991	−0·063448	−0·283826
	6							0·133070	−0·311519	0·294223	−0·119012
	7							−0·062904	0·174895	−0·248061	0·286174
0·2	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·516398	−0·719048	−0·291344	−0·763350	−0·550648	−0·200358	−0·779552	−0·654272	−0·434776	−0·152302
	2		0·469419	−0·340847	0·592570	0·068526	−0·424732	0·639440	0·302635	−0·145568	−0·456508
	3		−0·223830	0·401113	−0·433471	0·288487	0·310860	−0·518793	0·023669	0·421704	0·246382
	4				0·279395	−0·398890	0·182304	0·405101	−0·260852	−0·291895	0·260296
	5				−0·133095	0·263418	−0·330871	−0·295511	0·365069	−0·051817	−0·286304
	6							0·190337	−0·329997	0·294875	−0·116986
	7							−0·091154	0·188299	−0·253203	0·286765
0·1	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·547723	−0·790048	−0·316400	−0·847524	−0·606531	−0·219928	−0·869464	−0·725716	−0·480723	−0·168149
	2		0·540294	−0·333152	0·697162	0·113133	−0·419386	0·759946	0·377772	−0·114842	−0·452876
	3		−0·263780	0·406136	−0·529172	0·277282	0·317424	−0·643961	−0·023884	0·422718	0·253120
	4				0·349185	−0·411867	0·178798	0·518736	−0·249626	−0·309000	0·256898
	5				−0·168548	0·277795	−0·332584	−0·386876	0·381422	−0·041087	−0·288890
	6							0·252928	−0·355838	0·297735	−0·115129
	7							−0·122179	0·205938	−0·260126	0·287567
0	0	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000	1·000000
	1	−0·577350	−0·861136	−0·339981	−0·932470	−0·661209	−0·238619	−0·960290	−0·796666	−0·525532	−0·183435
	2		0·612334	−0·326619	0·804249	0·155797	−0·414591	0·883235	0·452016	−0·085724	−0·449528
	3		−0·304747	0·411728	−0·628250	0·289116	0·323962	−0·773409	−0·069066	0·425439	0·259721
	4				0·422005	−0·428246	0·175662	0·637294	−0·242723	−0·326976	0·253772
	5				−0·205712	0·294396	−0·334619	−0·482849	0·403317	−0·031046	−0·291568
	6							0·318992	−0·386798	0·302392	−0·113424
	7							−0·155019	0·226576	−0·268520	0·288555

values of ν_i and of the corresponding values of $G_n(\nu_i)$ $(n = 0, 1, ..., N)$ for various values of c and N. Only positive values of ν_i are shown, since by (10.10) and (10.11)

$$G_n(-\nu) = (-1)^n G_n(\nu). \tag{10.13}$$

This last relation shows that for $c = 1$ the equation (10.7) will have in every approximation a double root $\nu = 0$. The general theory of linear differential equations shows that the contribution of this root to (10.8′) will then be

$$\left[\exp(\nu x/l)\left\{A G_n(\nu) + B\left[\frac{x}{l} G_n(\nu) + \frac{d}{d\nu} G_n(\nu)\right]\right\}\right]_{\nu=0}.$$

It is obvious, however, that, whatever the value of N, we have for $c = 1$

$$G_n(0) = \delta_{0n}, \qquad [dG_n(\nu)/d\nu]_{\nu=0} = -\tfrac{1}{3}\delta_{1n}, \tag{10.14}$$

and the corresponding contributions to (10.8′) are easily shown to be

$$\left.\begin{aligned} A + Bx/l &\text{ to } \psi_0(x) \\ -\tfrac{1}{3}B &\text{ to } \psi_1(x) \\ 0 &\text{ to } \psi_n(x) \quad (n \geqslant 2) \end{aligned}\right\}. \tag{10.15}$$

10.2.2. *Properties of the roots of the determinantal equation*

An examination of Table II shows that for $c < 1$ the permissible values of ν_i are, for any N, real and different, and for $c = 1$ two roots are equal; for $c > 1$ not more than two roots are imaginary, all others being again real and different. These properties are valid for any N, and are proved by means of (10.10) and (10.11), as follows.

We put for brevity

$$y = \nu^2, \qquad y_{i,N} = \nu_i^2, \tag{10.16}$$

where N denotes the order of approximation to which the ν_i belong, and

$$R_n(y) = -(n+1)\nu G_{n+1}(\nu)/G_n(\nu). \tag{10.17}$$

In terms of $R_n(y)$, the relations (10.10) and (10.11) become

$$R_n(y) = (2n+1) - n^2 y / R_{n-1}(y) \tag{10.18}$$

and

$$R_0(y) = 1 - c. \tag{10.19}$$

We shall require the property, which follows from the last two equations, that, for $y \to \infty$, $R_n(y)$ is infinite for odd n and finite for even n, while the values of $R_n(\infty)$ for consecutive even n are related by

$$R_{2s}(\infty) = 4s + 1 + (2s/[2s-1])^2 R_{2s-2}(\infty),$$

and this, from (10.19), implies

$$R_{2s}(\infty) = (2s+1)^2 - \left[\frac{2}{1}\cdot\frac{4}{3}\cdots\frac{2s}{2s-1}\right]^2 c = (2s+1)^2 - \frac{c}{[P_{2s}(0)]^2}. \tag{10.20}$$

Also, unless $c = 1$, $R_n(0)$ cannot vanish for any n. It is clear that $R_n(y)$ is a rational meromorphic function of y, and for $n = N$ its zeros should coincide with the squares of the roots of the determinantal equation (10.7). Since the latter is of order not greater than $N+1$ in $\nu = \sqrt{y}$ ($N+1$ being the order of the determinant, and the elements of it being linear in ν), $R_n(y)$ has at most $\frac{1}{2}(n+1)$ zeros. Similarly it can be shown that $R_n(y)$ has at most $\frac{1}{2}n$ poles at a finite distance, and the poles of $R_n(y)$ are the zeros of $R_{n-1}(y)$.

We now consider the position of these zeros, taking first the case $c < 1$. The value of $R_1(y)$ is $3-y/(1-c)$, and this function has just one zero; for $c < 1$ it is real and positive and dR_1/dy is negative there. We prove this property to be general by induction. Let $R_{2s-1}(y)$ have, for some s, s real positive different zeros, at each of which $dR_{2s-1}(y)/dy$ is negative; let these zeros, in order of increasing y, be $y_{j,2s-1}$ ($j = 1, 2, \ldots, s$). From (10.18) we have for ϵ sufficiently small

$$\operatorname{sign} R_{2s}(y_{j,2s-1}+\epsilon) = \operatorname{sign}\epsilon;$$

also, $R_{2s}(0) > 0$. An examination of the sign changes of $R_{2s}(y)$ shows that s zeros of the latter function, which we call $y_{j,2s}$ ($j = 1, 2, \ldots, s$), are distributed as follows:

$$0 < y_{1,2s} < y_{1,2s-1} < y_{2,2s} < \cdots < y_{s,2s} < y_{s,2s-1}. \qquad (10.21)$$

Since $R_{2s}(y)$ can have not more than $\frac{1}{2}(2s+1)$ zeros, i.e. not more than s zeros, it follows that all the zeros of $R_{2s}(y)$ are shown in (10.21); the same type of argument also proves that $dR_{2s}(y)/dy < 0$ for $y = y_{j,2s}$ ($j = 1, 2, \ldots, s$).

A repetition of this argument, using the fact that $R_{2s}(\infty) > 0$ from (10.20) and so $R_{2s+1}(+\infty) = -\infty$, shows that $R_{2s+1}(y)$ has $s+1$ real positive different zeros, at each of which $dR_{2s+1}(y)/dy$ is negative. This completes the proof that for $c < 1$, $R_n(y)$ has $\frac{1}{2}n$ or $\frac{1}{2}(n+1)$ zeros (according as n is even or odd), which are real, positive, and different, and at each of which dR_n/dy is negative.

If $c > 1$, it is necessary to consider separately the cases

$$1 < c < [3P_2(0)]^2, \quad [3P_2(0)]^2 < c < [5P_4(0)]^2, \quad \text{etc.},$$

but in every case the same type of argument leads to the same result; for any $c > 1$, $R_{2s-1}(y)$ has s real different zeros, one negative and the rest positive, and $y\,dR_{2s-1}(y)/dy$ is negative at each zero, while $R_{2s}(y)$ has s real different zeros, and $dR_{2s}(y)/dy$ is negative at each zero situated at a finite distance. The sign of these zeros is more complicated, however; if $1 < c < [(2s+1)P_{2s}(0)]^2$, one zero is negative and the rest positive,

if $c = [(2s+1)P_{2s}(0)]^{-1}$ one zero is infinite and the rest positive, and if $c > [(2s+1)P_{2s}(0)]^{-1}$, all the zeros are positive.

We now apply these results to the equation (10.7). It follows from the above that in the P_{2s-1} approximation this equation always has just $2s$ roots, the same as the number of spherical harmonic moments retained. Further, $2s-2$ of these roots are always real and different, while the remaining pair are imaginary conjugates, both zero, or real and different according as $c > 1$, $c = 1$, or $c < 1$. In the P_{2s} approximation, the equation (10.7) will have $2s$ roots or, for one particular value of c, $2s-2$ roots; that is, one (or, exceptionally, three) less than the number of moments $\psi_n(x)$ retained in the approximation. Of these roots, $2s-2$ will be real and different, while the remaining two will be real and different if $c < 1$ or $c > [(2s+1)P_{2s}(0)]^{-1}$, zero and equal if $c = 1$, imaginary and different if $1 < c < [(2s+1)P_{2s}(0)]^{-1}$, and infinite (i.e. non-existent so far as (10.7) is concerned) if $c = [(2s+1)P_{2s}(0)]^{-1}$.

The fact that the number of roots equals the number of moments retained in an odd-order approximation, but not in an even-order approximation, shows that the two cases are not quite on the same footing. We shall show in § 10.3.2 that in fact the even-order approximations should be discarded entirely.

10.2.3. *Behaviour of the roots of the determinantal equation in high-order approximations*

In order to examine the behaviour of the roots of (10.7) in approximations of very high order, we first rewrite (10.18) and (10.19) as

$$R_{n-1}(\nu^2) = n^2\nu^2/\{2n+1-R_n(\nu^2)\}, \qquad c = 1-R_0(\nu^2),$$

whence
$$c = 1-\cfrac{\nu^2}{3-R_1(\nu^2)} = 1-\cfrac{\nu^2}{3-\cfrac{4\nu^2}{5-R_2(\nu^2)}} = \text{etc.}$$

The continued fraction

$$1-\cfrac{\nu^2}{3-\cfrac{4\nu^2}{5-\cfrac{9\nu^2}{7-\cdots}}} \tag{10.22}$$

converges in the complex plane cut along the real axis from $-\infty$ to -1 and from 1 to ∞. This follows, since the convergence of (10.22) depends on that of

$$(2n-1)-\cfrac{n^2\nu^2}{2n+1-\cfrac{(n+1)^2\nu^2}{2n+3-\cdots}},$$

and for n sufficiently large this can be replaced by

$$n\left[2-\cfrac{\nu^2}{2-\cfrac{\nu^2}{2-\cdots}}\right]. \tag{10.23}$$

If z_1 and z_2 are the roots of $z = 2-\nu^2/z$, it is easily verified by induction that the mth convergent of the continued fraction in (10.23) is

$$\frac{z_1^{m+2}-z_2^{m+2}}{z_1^{m+1}-z_2^{m+1}}$$

and this tends to a definite limit unless $|z_1| = |z_2|$, i.e. unless $(1-\nu^2)^{\frac{1}{2}}$ is purely imaginary. The expression (10.22) therefore converges in the cut plane, and so it converges uniformly in any closed region in the cut plane.

The expression (10.22), in fact, converges uniformly in any closed region in the cut plane to the function

$$P\left\{\frac{1}{2\nu}\log\frac{1+\nu}{1-\nu}\right\}^{-1}, \tag{10.24}$$

where P denotes that we take the principal branch of $\log[(1+\nu)/(1-\nu)]$. This is proved by expanding the latter function as a continued fraction, and showing that it is identical with (10.22) for $|\nu| < 1$. Since $(1/2\nu)\log[(1+\nu)/(1-\nu)]$ is regular and one-valued in the cut plane, this identity holds for all ν in the cut plane, by analytic continuation.

In the P_N approximation, the determinantal equation (10.7) can therefore be written

$$c = (N+1)\textit{th convergent to the continued-fraction form of } (10.24); \tag{10.25}$$

and, as the order of the approximation increases, the jth root of (10.7) will converge to the corresponding root of

$$c = P\left\{\frac{1}{2\nu}\log\frac{1+\nu}{1-\nu}\right\}^{-1}, \tag{10.26}$$

or will tend to one of the cuts. Equation (10.26), however, is identical with (5.5) and therefore has only two roots, which in the notation of Chapter V are $\nu = \pm l/L$ or $\pm i\kappa l$, according as they are real or imaginary.

Thus, if the roots of (10.7) in the P_N approximation are $\pm\nu_{1,N}$, $\pm\nu_{2,N}$, etc., where $\nu_{1,N}^2 < \nu_{2,N}^2 < \ldots$ (all the $\nu_{j,N}^2$ are real by § 10.2.2), we have

$$\lim_{N\to\infty} \nu_{1,N} = l/L \quad (\text{or } i\kappa l \text{ if } c > 1), \tag{10.27}$$

while the remaining roots satisfy the inequality

$$\nu_{j,N}^2 > 1-\epsilon_N \quad (j \geqslant 2), \tag{10.28}$$

where $\lim_{N\to\infty} \epsilon_N = 0$. If $|c-1|$ is small enough, this can in fact be replaced by

$$\nu_{j,N}^2 > 1 \quad (j \geqslant 2).$$

A slight refinement of the above analysis shows that the aggregate of the $\pm\nu_{j,N}$ for all N and all $j \geqslant 2$ fills the cuts in the ν-plane everywhere densely.

The combination of (10.27), (10.28), (10.8′), and (5.27) shows that the terms in (10.8′) corresponding to $\nu = \pm\nu_{1,N}$ tend to $\psi_{n,\mathrm{as}}(x)$ with increasing N, while the sum of the remaining terms tends to $\psi_{n,\mathrm{tr}}(x)$.

10.3. The boundary conditions

10.3.1. *Conditions at an interface between media*

In order to complete the formulation of the spherical harmonics method in the case of plane symmetry, it is necessary to give the boundary conditions on the solution of (10.4).

At an interface between two media, the boundary condition in the most general case is given by (2.19), and in a stationary problem with constant cross-sections, this becomes

$$\psi(\mathbf{r}+R\boldsymbol{\Omega},\boldsymbol{\Omega}) \text{ is a continuous function of } R;$$

in the plane case this further reduces to

$$\psi(x,\mu) \text{ is a continuous function of } x \text{ for any } \mu \text{ except (possibly) } \mu = 0. \tag{10.29}$$

On multiplying this by $P_n(\mu)$ and integrating over all μ, we see that in the exact solution $\psi_n(x)$ must be continuous for all n. In the P_N approximation, however, $\psi_{N+n'}(x)$ is not taken into account for $n' \geqslant 1$, and so the boundary conditions which may be stipulated are

$$\psi_n(x) \text{ is continuous for } n = 0, 1, \ldots, N. \tag{10.30}$$

An extra medium introduces an extra interface, and consequently the number of conditions that can be satisfied at each interface is equal to the number of arbitrary constants involved in (10.8′) for each medium, that is, to the number of roots of the determinantal equation (10.7). If N is odd, this number of roots is $N+1$ (see § 10.2.2), and so all the conditions (10.30) can be satisfied; these are therefore the boundary conditions to be imposed at an interface in an odd-order approximation. If N is even, however, the number of roots is N (we assume that N is so large that

$$c < [(N+1)P_N(0)]^2, \tag{10.31}$$

since otherwise even-order approximations are inapplicable—see § 10.3.2 below), and so only N conditions can be satisfied. These may be arrived at as follows. The quantity $\psi_0(x)$ varies rapidly near the interface, and its derivative has a logarithmic singularity there; this is seen from the fact that $\psi_0(x) = \rho(x)$, using (6.54) and $E_1(x) \sim -\log x$ for $x \to 0$. It

is therefore reasonable to permit some discontinuity of $\psi_0(x)$ in a finite-order approximation. Further, $\psi(x,\mu)$, regarded as a function of μ and $x\mu/l$, cannot be a rapidly varying function of $x\mu/l$ for any μ (see the derivation of (10.29)). Thus $\psi(x,\mu)$, regarded as a function of x and μ, can be a rapidly varying function of x only for small μ; the rapid variation of $\psi_0(x)$ therefore arises from the contribution of small μ, and this can be allowed for in a manner consistent with the spherical harmonics method by multiplying (10.29) not by $P_n(\mu)$, as in the derivation of (10.30), but by $[P_n(\mu)-P_n(0)]$. In this way the contribution of small μ is removed, and we obtain instead of (10.30)

$$[\psi_n(x)-P_n(0)\psi_0(x)] \text{ } \textit{is continuous for } n = 1,\ldots,N, \qquad (10.32)$$

i.e. exactly the number of conditions needed; these are therefore the boundary conditions to be imposed at an interface in an even-order approximation.

The conditions (10.32) can also be arrived at as follows. Taking (10.12) as an equation determining the values of $1/\nu_i$, it has always $N+1$ solutions, but, for N even, one of these is zero, so that (10.7) can be regarded as having an infinite root as well as its N finite roots. To take account of this, we add to (10.8′) a term for a large but finite $\nu = \nu^*$ and then make ν^* tend to infinity. If x_0 is the interface, the term added to (10.8′) is

$$A^* \lim_{\nu^*\to\infty} G_n(\nu^*)e^{\nu^*(x-x_0)} = \begin{cases} A^*P_n(0) & (x = x_0), \\ 0 & (x < x_0), \end{cases} \qquad (10.33)$$

which follows since $G_n(\infty) = P_n(0)$, by (10.9). Using (10.30) and adding (10.33) to (10.8′) is entirely equivalent to retaining in (10.8′) only the terms for finite ν_i, but replacing (10.30) by

$$\psi_n(x-x_0+)-\psi_n(x-x_0-) = A^*P_n(0) \quad (n = 0,1,\ldots,N), \qquad (10.34)$$

where A^* is independent of n, and eliminating A^* from (10.34) gives (10.32).

10.3.2. *Odd-order and even-order approximations*

The discussion of § 10.3.1 has shown that the boundary conditions at an interface are different according to the parity of the approximation order, and involve an additional postulate for even-order approximations. We should therefore expect that odd-order approximations would prove superior, and this is in fact the case: any P_{2s-1} approximation is invariably more accurate than the succeeding P_{2s} approximation.

The approximations of odd and even parity differ also in another respect. We have seen that, for $c > 1$, the determinantal equation (10.7) has, in approximations of high enough order, two imaginary roots,

which cause an oscillatory behaviour of $\psi(x, \mu)$ as a function of x. In odd-order approximations, these imaginary roots will be present in every approximation (see § 10.2.2), and so the oscillatory nature of $\psi(x, \mu)$ is correctly shown; however, in (e.g.) a second-order approximation with $c \geqslant 9/4$, the imaginary roots do not appear, and so the solution found for $\psi(x, \mu)$ differs greatly from the real solution. We should therefore expect that a P_2 approximation will be rather poor even if $c < 9/4$.

As a result of these considerations, it appears that odd-order approximations are much more reliable, and we shall henceforth work only with them. This difference between the odd-order and even-order approximations was not discovered for some time, and this is one of the reasons why Kourganoff (**27**, pp. 90 ff.) obtains poor results with the spherical harmonics method, since he uses only even-order approximations. Another reason is that he does not make the most favourable choice of boundary conditions at the free surface (see below).

10.3.3. *Conditions at infinity*

We now return to the boundary conditions, taking next the conditions at infinity. We discuss for definiteness the conditions at $x = +\infty$, and suppose for the moment that there is no supply of neutrons from $+\infty$, so that the neutron current vanishes there. The number of conditions to be imposed at the extreme boundary of a system should clearly equal half the number of parameters available per medium. In a physically meaningful problem with a medium occupying a half-space, we must have $c \leqslant 1$, and if $c < 1$ the roots of (10.7) are all real and half of each sign, by § 10.2.2. If the neutron current vanishes at $x = +\infty$, then the coefficients of $G_n(\nu_j)e^{\nu_j x/l}$ with $\nu_j > 0$ in (10.8′) must vanish, and this gives the required number of conditions. If $c = 1$, the equation (10.7) has two roots zero and the rest real and half of each sign. The coefficients that belong to positive roots in (10.8′) must again vanish, while the double root introduces two coefficients, of which one (called B in (10.15)) must vanish if the current at infinity is zero, while the other (A) need not. The number of conditions is therefore again correct.

If there is a source (or sink) at positive infinity, for $c < 1$ the discussion of § 2.3.3 and (10.28) show that for $j \geqslant 2$ the coefficient belonging to $+\nu_j$ must again be zero, while the coefficient belonging to the lowest positive exponent is determined by the strength of the source at infinity. Similarly, for $c = 1$ the coefficient B in (10.15) will be given by the strength of the source, while the coefficients belonging to the non-vanishing positive exponents will be zero as before.

10.3.4. *Conditions at a free surface. Mark's lemma and boundary conditions*

We next consider the conditions at a free surface. Suppose that this surface is $x = 0$ and the medium occupies the space $x > 0$. The exact boundary conditions are then

$$\psi(0, \mu) = 0 \quad \text{for } \mu > 0 \tag{10.35}$$

(see § 2.3.2). This, however, constitutes an infinite number of conditions which cannot all be exactly satisfied in an approximation of finite order. In the P_N approximation (where N is now always odd), we can satisfy only $\frac{1}{2}(N+1)$ conditions. The reduction of (10.35) to this number of conditions can be done in three ways:

(i) by choosing $\frac{1}{2}(N+1)$ positive values μ_j and satisfying (10.35) exactly at these points, i.e.

$$\psi(0, \mu_j) = 0 \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)); \tag{10.36}$$

(ii) by choosing $\frac{1}{2}(N+1)$ functions defined in $0 \leqslant \mu \leqslant 1$, which we call $\chi_j(\mu)$, say, and making $\psi(0, \mu)$ orthogonal to each function in that interval, i.e.

$$\int_0^1 \psi(0, \mu)\chi_j(\mu)\, d\mu = 0 \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)); \tag{10.37}$$

(iii) by replacing the vacuum in $x < 0$ by a black body ($c = 0$), and regarding the free surface as an interface where the boundary conditions are given by (10.30). This procedure avoids the arbitrariness involved in (i) and (ii).

It was proved by Mark (33) that (iii) is equivalent to (i) with the μ_j in (10.36) given by the positive roots of

$$P_{N+1}(\mu_j) = 0. \tag{10.38}$$

This is shown as follows. For the fictitious black medium in $x < 0$, $c = 0$, so that for this medium

$$G_n(\nu) = (-1)^n P_n(1/\nu) = P_n(-1/\nu), \tag{10.39}$$

from (10.9), and the determinantal equation (10.7), which is identical with (10.12), becomes

$$P_{N+1}(1/\nu_j) = 0. \tag{10.40}$$

Further, from § 10.3.3, only the positive roots of (10.40) will contribute to the expressions (10.8′) for $x < 0$. Thus, combining (10.2), (10.8′), and (10.39), we obtain

$$\psi(0-, \mu) = \frac{1}{4\pi} \sum_{\nu_j > 0} A_j \sum_{n=0}^{N} (2n+1) P_n(\mu) P_n(-1/\nu_j). \tag{10.41}$$

The inner sum in (10.41) can be simplified as follows. From the recurrence relation of Legendre polynomials

$$(n+1)P_{n+1}(z)+nP_{n-1}(z) = (2n+1)zP_n(z)$$

we have at once

$$\begin{aligned}(m+1)&[P_{m+1}(z_1)P_m(z_2)-P_{m+1}(z_2)P_m(z_1)]\\ &= (2m+1)(z_1-z_2)P_m(z_1)P_m(z_2)+m[P_m(z_1)P_{m-1}(z_2)-P_m(z_2)P_{m-1}(z_1)]\\ &= (z_1-z_2)\sum_{n=0}^{m}(2n+1)P_n(z_1)P_n(z_2). \qquad (10.42)\end{aligned}$$

Using this identity with $m = N, z_1 = \mu, z_2 = -1/\nu_j$ together with (10.40) and $\psi(0-,\mu) = \psi(0+,\mu)$ (which comes from (10.30)), we have

$$\psi(0+,\mu) = \frac{N+1}{4\pi}\sum_{\nu_j>0} A_j \frac{\nu_j}{\mu\nu_j+1}P_{N+1}(\mu)P_N(-1/\nu_j), \qquad (10.43)$$

which obviously vanishes for all positive roots of $P_{N+1}(\mu) = 0$. The method (iii) therefore leads to the boundary conditions (10.36) with the μ_j given by (10.38). These are known as *Mark's boundary conditions* or *black boundary conditions*.

10.3.5. *The application of Mark's boundary conditions*

The amount of numerical work involved in using Mark's boundary conditions can often be reduced by means of certain properties of the quantities

$$\sum_n (2n+1)G_n(\nu)P_n(\mu).$$

We shall illustrate this in terms of a simple, admittedly favourable, example.

Let the medium occupy the half-space $x > 0$, so that $x = 0$ is the free surface, and let $c = 1$ (no capture). It is required to find, in the P_{2s-1} approximation, the extrapolated end-point z_0 and the ratio $\psi_1(0)/\psi_0(0)$ of current to flux at the free surface.

We first determine z_0. Using (10.8′) and (10.15), we have according to § 10.3.3

$$\psi(x,\mu) = \frac{1}{4\pi}\left\{A+B\left(\frac{x}{l}-\mu\right)+\sum_{j=2}^{s}A_j e^{-\nu_j x/l}\sum_{n=0}^{2s-1}(2n+1)G_n(-\nu_j)P_n(\mu)\right\}, \qquad (10.44)$$

where the ν_j are the positive roots of $G_{2s}(\nu_j) = 0$. The definition of z_0 is that $x = -z_0$ is the point where the analytical continuation of the asymptotic flux vanishes; this asymptotic flux is clearly the first two terms in (10.44), and so

$$A = Bz_0/l. \qquad (10.45)$$

Using the recurrence relations (10.10) for $n \geqslant 1$ and (10.11) for $n = 0$, we have as in the derivation of (10.42)

$$\sum_{n=0}^{2s-1}(2n+1)G_n(-\nu_j)P_n(\mu) = \frac{c-2s\nu_j P_{2s}(\mu)G_{2s-1}(-\nu_j)}{1-\mu\nu_j}. \qquad (10.46)$$

If the μ_i are the positive roots of $P_{2s}(\mu_i) = 0$, for $c = 1$ these expressions reduce Mark's boundary conditions $\psi(0, \mu_i) = 0$ to

$$0 = B\left[\frac{z_0}{l} - \mu_i\right] + \sum_{j=2}^{s}\frac{A_j}{\nu_j}.\frac{1}{(1/\nu_j)-\mu_i} \qquad (i = 1, 2, \ldots, s). \qquad (10.47)$$

These may be regarded as a system of linear homogeneous equations for $B, A_2, A_3, \ldots, A_s$, with coefficients depending on z_0. The determinant of the coefficients must vanish, and this gives z_0. After suitable manipulation the determinant becomes

$$\begin{vmatrix} \left[\frac{z_0}{l} + \sum_{j=2}^{s}\frac{1}{\nu_j} - \mu_1\right]\mu_1^{s-1} & \mu_1^{s-2} & \ldots & \mu_1 & 1 \\ \left[\frac{z_0}{l} + \sum_{j=2}^{s}\frac{1}{\nu_j} - \mu_2\right]\mu_2^{s-1} & \mu_2^{s-2} & \ldots & \mu_2 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \left[\frac{z_0}{l} + \sum_{j=2}^{s}\frac{1}{\nu_j} - \mu_s\right]\mu_s^{s-1} & \mu_s^{s-2} & \ldots & \mu_s & 1 \end{vmatrix} = 0, \qquad (10.48)$$

and this immediately gives

$$z_0 = l\left[\sum_{i=1}^{s}\mu_i - \sum_{j=2}^{s}1/\nu_j\right]. \qquad (10.49)$$

This can be evaluated, for given s, merely by consulting the tables of zeros of the P and G functions, and substituting into (10.49). The result in the P_1 approximation is $z_0 = 0{\cdot}58l$, and in the P_3 approximation $z_0 = 0{\cdot}69l$, compared with the exact value $z_0 = 0{\cdot}71l$ (see (6.27)). The convergence is seen to be good.

We now determine $\psi_1(0)/\psi_0(0)$. To find this, it is necessary to combine in (10.44) all terms contributing to $\psi_0(0)$, and all terms contributing to $\psi_1(0)$. Thus, for $x = 0$, (10.44) should be rewritten

$$\psi(0, \mu) = \frac{1}{4\pi}\left\{\psi_0(0)\left[1 + 3\mu\frac{\psi_1(0)}{\psi_0(0)}\right] + \sum_{j=2}^{s}A_j\sum_{n=2}^{2s-1}(2n+1)G_n(-\nu_j)P_n(\mu)\right\}.$$

and consequently, since, for $c = 1$, $G_0(\nu) = P_0(\mu) = 1$ and $G_1(\nu) = 0$, the equations (10.47) are replaced by

$$\psi_0(0)\left[1+3\mu_i\frac{\psi_1(0)}{\psi_0(0)}\right]+\sum_{j=2}^{s}A_j\nu_j\frac{\mu_i}{1-\mu_i\nu_j}=0.$$

Eliminating $\psi_0(0)$ and the A_j and simplifying the resulting determinant as before, we find instead of (10.48) the equation

$$\begin{vmatrix} 1+3\dfrac{\psi_1(0)}{\psi_0(0)}\mu_1^s\displaystyle\prod_{j=2}^{s}(-\nu_j) & \mu_1^{s-1} & \mu_1^{s-2} & \cdots & \mu_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1+3\dfrac{\psi_1(0)}{\psi_0(0)}\mu_s^s\displaystyle\prod_{j=2}^{s}(-\nu_j) & \mu_s^{s-1} & \mu_s^{s-2} & \cdots & \mu_s \end{vmatrix} = 0,$$

which gives

$$3\frac{\psi_1(0)}{\psi_0(0)}\prod_{j=2}^{s}\nu_j\prod_{i=1}^{s}\mu_i=-1. \tag{10.50}$$

In this particular case, a further simplification arises, since $\prod\mu_i$ can be expressed in terms of the first and last coefficients in the power-series expansion of $P_{2s}(\mu)$. Using the recurrence relation for Legendre polynomials to relate these coefficients for P_{2s} and P_{2s-2}, we find

$$\prod_{i=1}^{s}\mu_{i,2s-1}=\frac{2s-1}{(4s-1)^{\frac{1}{2}}(4s-3)^{\frac{1}{2}}}\prod_{i=1}^{s-1}\mu_{i,2s-3},$$

where $\mu_{i,N}$ are the values of μ_i in the P_N approximation, i.e. the positive roots of $P_{N+1}(\mu) = 0$. Similarly we find

$$\prod_{j=2}^{s}\nu_{j,2s-1}=\frac{(4s-1)^{\frac{1}{2}}(4s-3)^{\frac{1}{2}}}{2s-1}\prod_{j=2}^{s-1}\nu_{j,2s-3},$$

and substituting into (10.50) we obtain the result

$$\frac{\psi_1(0)}{\psi_0(0)}=-\frac{1}{3\mu_{1,1}}=-\frac{1}{\sqrt{3}}, \tag{10.51}$$

whatever the value of s. Thus (10.51) is true in every order of approximation, and is in fact an exact relation (see (6.47)). That is to say, by using Mark's boundary conditions we obtain in this example the correct value for $\psi_1(0)/\psi_0(0)$ in all odd approximations. This will not, however, be the case for other values of c or for other geometries.

The advantages of this mode of application of Mark's conditions were accentuated in these instances because it was desired to find a particular quantity and not the whole neutron distribution, and because that

quantity could easily be isolated. The technique illustrated nevertheless retains its advantageousness even when these characteristics are not present.

10.3.6. *Conditions at a free surface (continued). Marshak's boundary conditions*

We now consider the other approach to the determination of the boundary conditions, namely (ii) of § 10.3.4. In order that the conditions (10.37) should be a satisfactory approximation to (10.35), it is necessary that the functions $\chi_j(\mu)$ should be the first $\frac{1}{2}(N+1)$ members of a complete set of functions orthogonal in the interval $0 \leqslant \mu \leqslant 1$. It is natural, in view of the important part played by the Legendre polynomials in the spherical harmonics method, to select the $\chi_j(\mu)$ from among the $P_n(\mu)$, rather than from, say, the $P_n(2\mu-1)$. In the interval $0 \leqslant \mu \leqslant 1$, the odd-order Legendre polynomials form a complete set, and the even-order ones form another. The most promising alternatives are therefore

$$\chi_j(\mu) = P_{2j-1}(\mu) \tag{10.51 a}$$

and

$$\chi_j(\mu) = P_{2j-2}(\mu). \tag{10.51 b}$$

Among conditions that can be imposed, the most important is evidently that relating to the total number of neutrons in the system, and so the most relevant corollary of (10.35) is that the total number of neutrons entering the system from outside is zero:

$$\int_0^1 \psi(0,\mu)\mu \, d\mu = 0. \tag{10.52}$$

This last condition, however, is included among (10.37) if (10.51 a) is used, while if (10.51 b) is used, (10.52) will not be satisfied. We therefore choose (10.51 a), and (10.37) become

$$\int_0^1 \psi(0,\mu)P_{2j-1}(\mu) \, d\mu = 0 \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)), \tag{10.53}$$

or, what is the same thing,

$$\int_0^1 \psi(0,\mu)\mu^{2j-1} \, d\mu = 0 \qquad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)). \tag{10.53'}$$

These are known as *Marshak's boundary conditions.*

The strongest argument in favour of Marshak's boundary conditions is that they automatically include (10.52). The choice of the other conditions (10.53) involves a certain arbitrariness which is not present in Mark's conditions. We should therefore expect that in low-order

approximations Marshak's conditions would give better results, but that the accuracy would increase faster in higher approximations when Mark's conditions are used. This is in fact the case. It is usually found that Mark's conditions become superior in the P_5 or sometimes the P_7 approximation.

Since the integrals

$$\int_0^1 P_n(\mu)\mu^{2j-1}\,d\mu$$

are simple rational numbers, Marshak's conditions are the easier to apply, except when the technique discussed in § 10.3.5 can be used in connexion with Mark's conditions.

In some cases the choice between the two types of boundary condition is decided by the nature of the problem, irrespective of the approximation used and of the comparative ease of application. If, for instance, it is desired to find how the neutron distribution in a system will be altered by the presence of a strong (but not black) absorber on the free surface, the conditions to be applied at the free surface are clearly to be the same with or without the absorber, and in this case Mark's conditions are required. Other situations may arise in which the use of Marshak's conditions is mandatory.

10.3.7. *Conditions at a surface exposed to neutrons*

It is sometimes necessary to deal with an exposed surface, i.e. an external surface on which a known neutron radiation falls from outside. Taking as before a medium in $x > 0$ with the surface $x = 0$ exposed, the exact condition is

$$\psi(0,\mu) = F(\mu) \quad \text{for } \mu > 0, \tag{10.54}$$

where $F(\mu)$ is some known function. Two cases must be distinguished, according as $F(\mu)$ is highly singular (e.g. contains delta functions of μ) or is not highly singular (e.g. is bounded for all μ).

In the former case, the neutron angular distribution in the medium will also be highly singular, and the series (10.2) cannot be expected to converge rapidly. The best method of dealing with such a case is to divide the neutrons into those coming from outside directly and those which have undergone at least one collision in the medium. For the first class, the Boltzmann equation contains no integral term and can therefore easily be solved exactly; for the second class, there are no sources outside the medium and the surface $x = 0$ therefore becomes a free surface. If $F(\mu) = \delta(\mu-\mu_0)$, the source term for neutrons scattered at least once in the medium is simply $\exp(-x/l\mu_0)$ with a factor of propor-

tionality, and a source term of this kind will not introduce any appreciable complication. If $F(\mu)$ is less singular, the source term in the medium will be more complex, but in this case there is no reason to fear poor convergence of (10.2) and the problem may be treated in a straightforward manner. As in the case of (10.35), the question then arises of replacing the infinite set of conditions (10.54) by the most suitable $\frac{1}{2}(N+1)$ conditions. Only the first two approaches discussed in § 10.3.4 are here applicable; the third has no obvious counterpart.

Following the approach (ii) and repeating the arguments of § 10.3.6 we arrive at the conditions

$$\int_0^1 \psi(0,\mu)\mu^{2j-1}\,d\mu = \int_0^1 F(\mu)\mu^{2j-1}\,d\mu \quad (j = 1,2,\ldots,\tfrac{1}{2}(N+1)), \quad (10.55)$$

which are a generalization of Marshak's boundary conditions.

The justification of Mark's conditions lay chiefly in the approach (iii), and so there is in the present case no reason to attempt to generalize them. It can be shown that, unless $F(\mu)$ is very smooth, the convergence may be extremely slow if Mark's type of condition is used. For instance, suppose $F(\mu) = 1$ for $0 \leqslant \mu < \mu_0$ and $= 0$ for $\mu_0 < \mu \leqslant 1$. In the P_N approximation, the generalization of Mark's conditions is

$$\psi(0,\mu_j) = F(\mu_j) \quad \text{for} \quad P_{N+1}(\mu_j) = 0, \quad \mu_j > 0.$$

As μ_0 varies, the neutron distribution in the system, as given by this approximation, will change suddenly as μ_0 passes through one of the μ_j, and then remain unchanged until the next μ_j is reached. Thus, unless N is very large, the neutron distribution given by this method will be extremely inaccurate.

10.3.8. *A thin absorbing slab at an interface between media*

We finally consider, for the case of plane symmetry, a layer of purely absorbing material placed between two scattering media. In principle such a layer could be treated like any other medium, but in practice this may be undesirable, since the rate of convergence of the spherical harmonic method is faster, the smaller the values of $|c-1|$ involved, except in the case of layers of small optical thickness. This follows since, unless the optical thickness is small, the main contribution to $\psi(x,\mu)$ inside a layer comes from the roots $\pm\nu_1$ (in the notation of § 10.2.3) of (10.7), and so the method cannot converge any faster than the quantities

$$\sum_0^N (2n+1)P_n(\mu)\{A_{+1}\,G_n(\nu_1)e^{\nu_1 x/l} + A_{-1}\,G_n(-\nu_1)e^{-\nu_1 x/l}\} \quad (10.56)$$

for the various media converge to their ultimate values. This is governed for each medium by the rate of convergence of (10.22) for $\nu = l/L$, and this is the faster, the smaller $|l/L|$, i.e. the smaller $|c-1|$ for the layer in question.

If, therefore, the absorbing layer is not thin optically, while for the other media in the system c is not very different from unity, then to treat the absorber the same way as the other media would mean going to a higher degree of approximation than would otherwise be necessary. It is for this reason that a special boundary condition is devised for the case in question.

Let the absorbing layer, of negligible geometrical thickness, be situated at $x = 0$, so that $x > 0$ and $x < 0$ are the scattering media on each side of it, and let the optical thickness of the layer be h. The exact boundary conditions are

$$\begin{aligned} \psi(0+,\mu) &= e^{-h/\mu}\psi(0-,\mu) \quad (\mu > 0); \\ \psi(0-,\mu) &= e^{-h/|\mu|}\psi(0+,\mu) \quad (\mu < 0). \end{aligned} \tag{10.57}$$

To reduce these to the finite number of conditions needed, we again apply the argument of § 10.3.6, obtaining

$$\left.\begin{aligned} \int_0^1 \psi(0+,\mu)\mu^{2j-1}\,d\mu &= \int_0^1 e^{-h/\mu}\psi(0-,\mu)\mu^{2j-1}\,d\mu \\ \int_{-1}^0 \psi(0-,\mu)\mu^{2j-1}\,d\mu &= \int_{-1}^0 e^{h/\mu}\psi(0+,\mu)\mu^{2j-1}\,d\mu \end{aligned}\right\} \quad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)). \tag{10.58}$$

In order to apply these conditions, a knowledge is required of the functions

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n}\,dt, \tag{10.59}$$

which are a generalization of those introduced in (6.2) and (6.32). These functions have, however, been tabulated by Placzek (40). The conditions (10.58) are the generalization of Marshak's boundary conditions to the present case. The corresponding generalization of Mark's conditions is

$$\psi(0+,\mu_j) = e^{-h/\mu_j}\psi(0-,\mu_j), \qquad P_{N+1}(\mu_j) = 0, \qquad \mu_j \gtrless 0, \tag{10.60}$$

but it can be shown by the argument of § 10.3.4 that this is also the result obtained by treating the absorbing layer like the others and then eliminating its constants to relate the angular distributions in the two scattering media. The boundary conditions (10.60) will therefore, by

the reasoning at the beginning of the present sub-section, lead to less accurate results than (10.58).

We here conclude our analysis of various types of boundary condition for plane symmetry. In general it is necessary also to discuss the boundary conditions at the surfaces of gaps, but we have seen in § 8.6.1 that in plane symmetry a gap can have no effect, and the two media adjacent to it can be regarded as contiguous.

10.4. Multi-layer problems

The boundary conditions discussed in § 10.3 allow us to determine the A_j in (10.8′), or its analogue in the presence of sources, for the various media in the system. If there are s such media, and the P_N approximation is used (N being odd), there are $s(N+1)$ such conditions with $s(N+1)$ unknowns, and it is desirable to systematize the process of their solution.

Let us first consider one particular medium, and suppose for simplicity that $c \neq 1$ and that this medium contains no sources. Let $\boldsymbol{\psi}(x)$ and $\boldsymbol{\varphi}(x)$ be vectors in a space of $N+1$ dimensions, with components

$$\psi_0(x), \psi_1(x),..., \psi_N(x)$$

and

$$A_1 \exp(\nu_1 x/l), A_{-1} \exp(-\nu_1 x/l), A_2 \exp(\nu_2 x/l),...,$$

$$A_{\frac{1}{2}(N+1)} \exp(\nu_{\frac{1}{2}(N+1)} x/l), A_{-\frac{1}{2}(N+1)} \exp(-\nu_{\frac{1}{2}(N+1)} x/l), \text{ respectively.}$$

We define the matrix

$$\mathfrak{N}(c) = \begin{bmatrix} G_0(\nu_1) & G_0(-\nu_1) & . & G_0(-\nu_{\frac{1}{2}(N+1)}) \\ G_1(\nu_1) & G_1(-\nu_1) & . & G_1(-\nu_{\frac{1}{2}(N+1)}) \\ . & . & . & . \\ G_N(\nu_1) & G_N(-\nu_1) & . & G_N(-\nu_{\frac{1}{2}(N+1)}) \end{bmatrix}, \tag{10.61}$$

its inverse matrix $\mathfrak{N}^{-1}(c)$, and the diagonal matrix $\mathfrak{F}(\boldsymbol{\nu}\tau)$, whose diagonal elements are $\exp(\nu_1 \tau)$, $\exp(-\nu_1 \tau)$,..., $\exp(\nu_{\frac{1}{2}(N+1)}\tau)$, $\exp(-\nu_{\frac{1}{2}(N+1)}\tau)$. In this notation, formula (10.8′) can be rewritten

$$\boldsymbol{\psi}(x) = \mathfrak{N}(c)\boldsymbol{\varphi}(x), \tag{10.62}$$

and hence we have also

$$\boldsymbol{\varphi}(x) = \mathfrak{N}^{-1}(c)\boldsymbol{\psi}(x), \tag{10.62′}$$

whilst the definition of $\varphi(x)$ implies that

$$\varphi(x) = \mathfrak{F}\left(\nu\frac{x-x'}{l}\right)\varphi(x'), \tag{10.63}$$

provided that x and x' both lie in the medium under consideration.

The matrix $\mathfrak{N}^{-1}(c)$ is easily obtained. The recurrence relations (10.10) for $G_n(\nu)$ $(n \geqslant 1)$ are the same as for $P_n(\mu)$. Using these together with (10.11), we therefore find as in the proof of (10.42)

$$(m+1)[G_{m+1}(\nu')G_m(\nu) - G_{m+1}(\nu)G_m(\nu')]$$
$$= \left(\frac{1}{\nu}-\frac{1}{\nu'}\right)\sum_{n=0}^{m}(2n+1-c\delta_{0n})G_n(\nu)G_n(\nu'). \tag{10.64}$$

Putting here $m = N$, $\nu' = \nu_{j'}$, and passing to the limit as $\nu \to \nu_j$, where ν_j and $\nu_{j'}$ are any two solutions of (10.12) $(\nu_{-j} = -\nu_j,\ \nu_{-j'} = -\nu_{j'})$, we obtain the following orthogonality relation between the $G_n(\nu_j)$:

$$\sum_{n=0}^{N}(2n+1-c\delta_{0n})G_n(\nu_j)G_n(\nu_{j'}) = \delta_{jj'}h(\nu_j),$$

where
$$h(\nu) = (N+1)\nu^2 G_N(\nu)\,dG_{N+1}(\nu)/d\nu. \tag{10.65}$$

Using this orthogonality relation, we see that the matrix

$$\begin{bmatrix} (1-c)G_0(\nu_1)/h(\nu_1) & 3G_1(\nu_1)/h(\nu_1) & \cdots & (2N+1)G_N(\nu_1)/h(\nu_1) \\ (1-c)G_0(-\nu_1)/h(-\nu_1) & 3G_1(-\nu_1)/h(-\nu_1) & \cdots & (2N+1)G_N(-\nu_1)/h(-\nu_1) \\ \cdots & \cdots & \cdots & \cdots \\ (1-c)G_0(-\nu_{\frac{1}{2}(N+1)})/h(-\nu_{\frac{1}{2}(N+1)}) & \cdots & \cdots & (2N+1)G_N(-\nu_{\frac{1}{2}(N+1)})/h(-\nu_{\frac{1}{2}(N+1)}) \end{bmatrix} \tag{10.66}$$

multiplied on the right by (10.61), gives the unit matrix. Thus the matrix (10.66) is simply $\mathfrak{N}^{-1}(c)$, so that all the matrices appearing in formulae (10.62), (10.62′), and (10.63) are known, and no numerical inversion of matrices is necessary. Combining the above results we obtain

$$\psi(x) = \mathfrak{N}(c)\mathfrak{F}\left(\nu\frac{x-x'}{l}\right)\mathfrak{N}^{-1}(c)\psi(x'), \tag{10.67}$$

provided that x and x' both lie in the medium under consideration.

On proceeding to consider the complete system of media, we must define the above matrices separately for each medium. If the ith medium

lies between x_{i-1} and x_i, and if c_i, l_i, and ν_i are the values of c, l, and ν in this medium, then formula (10.67) gives

$$\psi(x_i) = \mathfrak{N}(c_i)\mathfrak{F}\left(\nu_i \frac{x_i - x_{i-1}}{l_i}\right)\mathfrak{N}^{-1}(c_i)\psi(x_{i-1}). \qquad (10.67')$$

By applying this formula successively to the various media, we can eliminate all the media except the two extreme ones and reduce the system of $s(N+1)$ equations to one of $N+1$ equations only. As we have already remarked, this procedure does not involve any numerical inversion of matrices.

It has been assumed above that there are no sources in the ith medium, and that it is contiguous with the $(i-1)$th and $(i+1)$th media. The changes required in (10.67') if these assumptions are invalid are easily found.

If $c = 1$ in one of the media, the procedure is as follows. The product $\mathfrak{N}(c)\mathfrak{F}(\nu[x-x']/l)\mathfrak{N}^{-1}(c)$ is calculated; where c occurs explicitly, it is regarded as a variable parameter, as are $\nu_{\pm 1}$, whilst for the other ν_j the values corresponding to $c = 1$ are used throughout. We then pass to the limit as $c \to 1, \nu_{\pm 1} \to 0$, and $c - 1 = O(\nu_{\pm 1}^2)$. That is, formula (10.67') in such a case would be taken as

$$\psi(x) = \left\{\lim_{c \to 1}\left[\mathfrak{N}(c)\mathfrak{F}\left(\nu \frac{x-x'}{l}\right)\mathfrak{N}^{-1}(c)\right]\right\}\psi(x').$$

A similar procedure is necessary when $|c-1|$ is very small, although not zero.

10.5. The P_1 approximation

The lowest possible spherical harmonics approximation is the P_1 approximation. It resembles the diffusion equation, and differs from all higher spherical harmonics approximations, in that no account is taken of the transient terms, since equation (10.7) has in this approximation only one pair of roots.

The equations (10.4) in the P_1 approximation are

$$\left.\begin{aligned} \psi_1'(x) + \frac{1-c}{l}\psi_0(x) &= 0 \\ \psi_0'(x) + \frac{3}{l}\psi_1(x) &= 0 \end{aligned}\right\}. \qquad (10.68)$$

On elimination of $\psi_1(x)$ they give

$$\psi_0''(x) = \frac{3(1-c)}{l^2}\psi_0(x), \tag{10.69}$$

and this is the form taken by the diffusion approximation equation (8.2) if, instead of the correct expression for L,

$$\frac{1}{c} = \frac{L}{2l}\log\frac{L+l}{L-l}, \tag{5.5}$$

we use the approximate result

$$1-c = \tfrac{1}{3}l^2/L^2.$$

At the interface between two media, the conditions (10.30) are that $\psi_0(x)$, the flux, and $\psi_1(x)$, the current, are continuous. We have seen that in the diffusion approximation these are not the best possible conditions, (8.12) and (8.13) being superior. However, the relation (6.37) between the current and the flux is now approximately replaced by the second of (10.68), and hence, expressing the current in terms of the gradient of the flux and using the ratios of the quantities involved, we have

$$l\psi_0'(x)/\psi_0(x) \quad \textit{continuous},$$

which is the same as (8.15). In the absence of sources, therefore, the discrepancy between the interface boundary conditions in the diffusion and P_1 approximations is compensated at the expense of an error in the diffusion length.

In order to test the accuracy of the free-surface boundary conditions in the P_1 approximation, we shall calculate the value of the extrapolated end-point z_0, defined by (6.23); to illustrate the procedure in higher approximations, however, we shall use not (10.69), but the functions $G_n(\nu)$. From (10.10) and (10.11)

$$G_2(\nu) = -\tfrac{1}{2}\left[G_0(\nu)+\frac{3G_1(\nu)}{\nu}\right] = -\tfrac{1}{2}\left[1-\frac{3(1-c)}{\nu^2}\right],$$

and therefore from (10.12)

$$\nu = \pm\sqrt{3(1-c)}^{\frac{1}{2}},$$

so that

$$G_1(\nu) = \frac{c-1}{\nu} = -\left(\frac{1-c}{3}\right)^{\frac{1}{2}}\operatorname{sign}\nu,$$

and combining (10.2) with (10.8′)

$$\psi(x,\mu) = \frac{1}{4\pi}\{A^+e^{\nu x/l}[1-\mu\sqrt{3}(1-c)^{\frac{1}{2}}]+A^-e^{-\nu x/l}[1+\mu\sqrt{3}(1-c)^{\frac{1}{2}}]\}. \tag{10.70}$$

Let the medium occupy the half-space $x > 0$ with the free surface at $x = 0$. According to Mark's boundary conditions, $\psi(0,\mu)$ as given by (10.70) should vanish at the positive root of $P_2(\mu) \equiv \frac{1}{2}(3\mu^2-1) = 0$, i.e. at $\mu = 1/\sqrt{3}$, and this gives

$$A^+[1-(1-c)^{\frac{1}{2}}]+A^-[1+(1-c)^{\frac{1}{2}}] = 0. \tag{10.71}$$

Thus (10.70) gives

$$\psi_0(x) = \text{constant} \times \left[\sinh\frac{\nu x}{l}+(1-c)^{\frac{1}{2}}\cosh\frac{\nu x}{l}\right]. \tag{10.72}$$

The definition of z_0, however, is $\psi_0(-z_0) = 0$, and (10.72) therefore gives

$$\sinh(\nu z_0/l) = (1-c)^{\frac{1}{2}}\cosh(\nu z_0/l),$$

or, on substituting for ν,

$$(z_0)_{P_1,\,\text{Mark}} = \frac{l}{\sqrt{3}(1-c)^{\frac{1}{2}}}\tanh^{-1}(1-c)^{\frac{1}{2}} = \frac{l}{\sqrt{3}}\left[1+\frac{1-c}{3}+\frac{(1-c)^2}{5}+\ldots\right]$$

$$\cong 0{\cdot}58l[1+\tfrac{1}{3}(1-c)+\ldots]. \tag{10.73}$$

According to Marshak's boundary conditions, we should have

$$\int_0^1 \psi(0,\mu)\mu\,d\mu = 0,$$

and on applying this to (10.70) we obtain, instead of (10.71),

$$A^+\left[\frac{1}{2}-\left(\frac{1-c}{3}\right)^{\frac{1}{2}}\right]+A^-\left[\frac{1}{2}+\left(\frac{1-c}{3}\right)^{\frac{1}{2}}\right] = 0,$$

and hence, instead of (10.73),

$$(z_0)_{P_1,\,\text{Marshak}} = \frac{l}{\sqrt{3}(1-c)^{\frac{1}{2}}}\tanh^{-1}\left[2\left(\frac{1-c}{3}\right)^{\frac{1}{2}}\right]$$

$$= \frac{2l}{3}\left[1+\frac{4}{9}(1-c)+\frac{16}{45}(1-c)^2+\ldots\right]$$

$$\cong 0{\cdot}67l\left[1+\frac{4}{9}(1-c)+\ldots\right]. \tag{10.74}$$

Comparing these with the exact result (6.29), we see that, if $|c-1|$ is small, either boundary condition gives a fair approximation to z_0 even in the P_1 form, while the error using Marshak's boundary condition is considerably smaller than that using Mark's, as was anticipated in § 10.3.6.

10.6. A remark on orthogonality

To conclude the present chapter, we shall discuss one further question. It is known that, for a bare homogeneous source-free body, if c is treated as a variable parameter, then the exact eigenfunctions $\rho(\mathbf{r})$ (i.e. $\psi_0(\mathbf{r})$) are mutually orthogonal (see § 4.7, formula (4.30)). For a source-free system of several media in a fixed geometrical arrangement, where the value of c in the ith medium is $c_{i,0}+\gamma c_{i,1}$, say, with γ regarded as a parameter, the corresponding eigenfunctions satisfy the orthogonality relation

$$(\gamma_i-\gamma_j)\iiint \{c_1(\mathbf{r})/l(\mathbf{r})\}\psi_{0,i}(\mathbf{r})\psi_{0,j}(\mathbf{r})\,dV = 0, \qquad (10.75)$$

where γ_i and γ_j are two eigenvalues and $\psi_{0,i}(\mathbf{r})$ and $\psi_{0,j}(\mathbf{r})$ are the corresponding eigenfunctions for the neutron flux $\psi_0(\mathbf{r})$, while $c_1(\mathbf{r})$ and $l(\mathbf{r})$ are functions equal to $c_{i,1}$ and l_i when $\mathbf{r}$ is in the ith medium.

It is also known that approximate solutions do not always have such properties, as, for instance, the diffusion approximation (see § 8.8). The question arises whether the solutions obtained by the spherical harmonics approximation have the property of orthogonality. We shall show that they do, at least in the odd-order approximations.

We first take the case of plane symmetry, with Mark's boundary conditions and all layers treated in the same manner. The vacua outside the system can be replaced by black half-spaces, and the free surfaces are thus replaced by interfaces. In an odd-order approximation, the spherical harmonic moments $\psi_{n,i}(x)$ and $\psi_{n,j}(x)$ corresponding to the eigenvalues γ_i and γ_j are continuous, so that, using also the conditions at infinity,

$$\int_{-\infty}^{\infty}\left[\psi_{n,i}\frac{d\psi_{m,j}(x)}{dx}+\psi_{m,j}\frac{d\psi_{n,i}(x)}{dx}\right]dx$$

$$= \psi_{n,i}(\infty)\psi_{m,j}(\infty)-\psi_{n,i}(-\infty)\psi_{m,j}(-\infty) = 0. \qquad (10.76)$$

The first equation (10.4) for the $\psi_{n,i}$ is multiplied by $\psi_{0,j}$ and integrated. The quantity $d\psi_{1,i}/dx$ is eliminated by integration by parts, using (10.76), and $d\psi_{0,j}/dx$ is then eliminated by the second equation (10.4) for

the $\psi_{n,j}$, and so on. We thus obtain

$$\int_{-\infty}^{\infty} \frac{c_0(x)+\gamma_i c_1(x)}{l(x)}\,\psi_{0,i}(x)\psi_{0,j}(x)\,dx$$

$$= \int_{-\infty}^{\infty} \psi_{0,i}(x)\psi_{0,j}(x)\,\frac{dx}{l(x)} + \int_{-\infty}^{\infty} \psi_{0,j}(x)\,\frac{d\psi_{1,i}(x)}{dx}\,dx$$

$$= \int_{-\infty}^{\infty} [\psi_{0,i}(x)\psi_{0,j}(x)+3\psi_{1,i}(x)\psi_{1,j}(x)]\,\frac{dx}{l(x)} + 2\int_{-\infty}^{\infty} \psi_{1,i}(x)\,\frac{d\psi_{2,j}(x)}{dx}\,dx = \ldots$$

$$= \int_{-\infty}^{\infty} \sum_{n=0}^{m} (2n+1)\psi_{n,i}(x)\psi_{n,j}(x)\,\frac{dx}{l(x)} +$$

$$+ \begin{cases} (m+1)\displaystyle\int_{-\infty}^{\infty} \psi_{m,i}(x)\,\frac{d\psi_{m+1,j}(x)}{dx}\,dx & (m \text{ odd}), \\ (m+1)\displaystyle\int_{-\infty}^{\infty} \psi_{m,j}(x)\,\frac{d\psi_{m+1,i}(x)}{dx}\,dx & (m \text{ even}). \end{cases} \tag{10.77}$$

Applying this result with $m = N$, we find, since in the P_N approximation $d\psi_{N+1,j}/dx$ is neglected, that the right side of (10.77) is unaltered if on the left side γ_i is replaced by γ_j. Hence

$$(\gamma_i-\gamma_j)\int_{-\infty}^{\infty} \frac{c_1(x)}{l(x)}\,\psi_{0,i}(x)\psi_{0,j}(x)\,dx = 0. \tag{10.78}$$

For the black half-spaces which have replaced the vacua outside the system, $c_1(x)$ is zero; (10.78) is therefore identical with (10.75), and this proves the result when Mark's boundary conditions are used.

Two results have been proved above: firstly, that irrespective of the boundary conditions used at the free surfaces ($x = a$ and $x = b$, say),

$$\int_a^b \frac{c_0(x)+\gamma_i c_1(x)}{l(x)}\,\psi_{0,i}(x)\psi_{0,j}(x)\,dx$$

$$= \sum_{n=0}^{N} (2n+1)\int_a^b \psi_{n,i}(x)\psi_{n,j}(x)\,\frac{dx}{l(x)} + \Phi(b)-\Phi(a), \tag{10.79}$$

where

$$\Phi(x) = \psi_{1,i}(x)\psi_{0,j}(x) + 2\psi_{2,j}(x)\psi_{1,i}(x)+\ldots+N\psi_{N,i}(x)\psi_{N-1,j}(x); \tag{10.80}$$

and secondly, that if Mark's conditions are used at the free surfaces, then (supposing $b > a$)

$$\Phi(a) = -\sum_{n=0}^{N}(2n+1)\int_{-\infty}^{a}\psi_{n,i}(x)\psi_{n,j}(x)\,\frac{dx}{l(x)}, \tag{10.81}$$

and similarly for $\Phi(b)$, and these are unaltered by interchanging i and j.

In order to prove (10.75) when Marshak's boundary conditions are used, it is therefore sufficient to prove that in the latter case, though $\Phi(a)$ and $\Phi(b)$ are no longer given by (10.81), they are still unchanged by the interchange of i and j. We show this as follows. For $\psi_i(x,\mu)$, say, Marshak's boundary conditions give

$$\int_0^1 P_{2k-1}(\mu)\psi_i(a,\mu)\,d\mu = 0. \tag{10.82}$$

Substituting from (10.2), we find, since the contribution of odd moments other than $\psi_{2k-1,i}(a)$ vanishes by the orthogonality of Legendre polynomials,

$$\psi_{2k-1,i}(a) = -\int_0^1 P_{2k-1}(\mu)\sum_{s=0}^{\frac{1}{2}(N-1)}(4s+1)P_{2s}(\mu)\psi_{2s,i}(a)\,d\mu. \tag{10.83}$$

The recurrence relations between the Legendre polynomials give

$$P_1(\mu)\psi_{0,j}(a)+[2P_1(\mu)+3P_3(\mu)]\psi_{2,j}(a)+\ldots+$$
$$+[(N-1)P_{N-2}(\mu)+NP_N(\mu)]\psi_{N-1,j}(a) = \mu\sum_{s=0}^{\frac{1}{2}(N-1)}(4s+1)P_{2s}(\mu)\psi_{2s,j}(a),$$

and so, substituting (10.83) into (10.80), we have

$$\Phi(a) = -\int_0^1 \mu\Big[\sum_{s'}(4s'+1)P_{2s'}(\mu)\psi_{2s',j}(a)\Big]\Big[\sum_{s}(4s+1)P_{2s}(\mu)\psi_{2s,i}(a)\Big]\,d\mu,$$

and this is unaltered when i and j are interchanged. This proves (10.75) when Marshak's boundary conditions are used.

The assumption that all media are treated in the same manner amounts to supposing that there are no interfaces at which (10.58) should be satisfied rather than (10.30). It is not yet known whether the presence of such interfaces will cause (10.75) to be no longer satisfied; it can be shown by direct investigation that in the P_1 approximation (10.75) holds even when such boundaries are present.

So far only the plane case has been discussed. In order to avoid having to return to the topic of orthogonality, we indicate here how these results may be extended to other geometries, but we shall discuss only the cases where the internal boundaries are interfaces with the media in contact

everywhere, and Mark's conditions are used at the free surfaces, so that they also reduce to this kind of interface.

The equation determining $\psi_i(\mathbf{r},\Omega)$ is

$$\Omega\,.\,\mathbf{grad}\,\psi_i(\mathbf{r},\Omega)+\frac{1}{l(\mathbf{r})}\,\psi_i(\mathbf{r},\Omega)=\frac{c_0(\mathbf{r})+\gamma_i c_1(\mathbf{r})}{4\pi l(\mathbf{r})}\,\psi_{0,i}(\mathbf{r}).$$

This is multiplied by $[\psi_j(\mathbf{r},\Omega)+\psi_j(\mathbf{r},-\Omega)]$ (where $-\Omega$ is the direction opposite to Ω), and added to the equation determining $\psi_j(\mathbf{r},\Omega)$ multiplied by $[\psi_i(\mathbf{r},\Omega)-\psi_i(\mathbf{r},-\Omega)]$. The result is integrated over all Ω, and at the same time Ω is replaced by $-\Omega$ in the term

$$-\int\!\!\int \psi_i(\mathbf{r},-\Omega)\Omega\,.\,\mathbf{grad}\,\psi_j(\mathbf{r},\Omega)\,d\Omega.$$

This gives

$$\int\!\!\int\{\Omega\,.\,\mathbf{grad}\,[\psi_i(\mathbf{r},\Omega)\psi_j(\mathbf{r},\Omega)]+\Omega\,.\,\mathbf{grad}\,[\psi_i(\mathbf{r},\Omega)\psi_j(\mathbf{r},-\Omega)]\}\,d\Omega+$$

$$+2\int\!\!\int\psi_i(\mathbf{r},\Omega)\psi_j(\mathbf{r},\Omega)\,\frac{d\Omega}{l(\mathbf{r})}=2\,\frac{c_0(\mathbf{r})+\gamma_i c_1(\mathbf{r})}{l(\mathbf{r})}\,\psi_{0,i}(\mathbf{r})\psi_{0,j}(\mathbf{r}). \quad (10.84)$$

This relation has been derived on the assumption that $\psi_i(\mathbf{r},\Omega)$ and $\psi_j(\mathbf{r},\Omega)$ are the exact angular distributions corresponding to γ_i and γ_j. It is evident, however, that the same arguments will apply if $\psi_i(\mathbf{r},\Omega)$ and $\psi_j(\mathbf{r},\Omega)$ are interpreted as the angular distributions in the P_N approximation, provided that the corresponding $\psi_{n,i}(\mathbf{r})$ and $\psi_{n,j}(\mathbf{r})$ are continuous; this will be so, in odd-order approximations, in most of the cases met with in practice. The equation (10.75) now follows by integrating over all $\mathbf{r}$ and subtracting the same expression with i and j interchanged.

XI

THE SPHERICAL HARMONICS METHOD FOR SPHERICAL GEOMETRIES

11.1. The differential equations and their solution

We now turn to spherically symmetrical systems, where the neutron flux depends only on a radial coordinate. The angular distribution $\psi(\mathbf{r},\Omega)$ can then depend only on r and on the angle between Ω and the direction in which r is measured. The cosine of this angle is clearly

$$\mu = (\mathbf{r}.\Omega)/r \tag{11.1}$$

if the centre of the system is taken as origin so that $|\mathbf{r}| = r$, and so

$$\psi(\mathbf{r},\Omega) = \psi(r,\mu) = \psi(r,[\mathbf{r}.\Omega]/r). \tag{11.2}$$

In order to transform the Boltzmann equation (4.3) to the variables r and μ, we first calculate the expression for $\Omega.\,\mathbf{grad}\,\psi(\mathbf{r},\Omega)$ in these variables. Since μ depends on $\mathbf{r}$ as well as on Ω, it follows that μ changes when $\mathbf{r}$ varies, even if Ω is constant. We thus get

$$\begin{aligned}\Omega.\,\mathbf{grad}\,\psi(\mathbf{r},\Omega) &= \Omega.\,\mathbf{grad}\,\psi(r,[\mathbf{r}.\Omega]/r)\\ &= \mu(\partial\psi/\partial r)+(\partial\psi/\partial\mu)[\Omega.\,\mathbf{grad}(\mathbf{r}.\Omega)/r]. \end{aligned}\tag{11.3}$$

But $\quad \Omega.\,\mathbf{grad}[(\mathbf{r}.\Omega)/r] = (\Omega.\Omega)/r-(\mathbf{r}.\Omega)(\Omega.\,\mathbf{grad}\,r)/r^2,$

$$\mathbf{grad}\,r = \mathbf{r}/r, \qquad \text{and} \qquad \Omega.\Omega = 1,$$

since Ω is a unit vector. The expression (11.3) therefore becomes

$$\mu\frac{\partial\psi}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial\psi}{\partial\mu},$$

and the Boltzmann equation (4.3) becomes (since the scattering is assumed isotropic)

$$\mu\frac{\partial\psi(r,\mu)}{\partial r}+\frac{1-\mu^2}{r}\frac{\partial\psi(r,\mu)}{\partial\mu}+\frac{\psi(r,\mu)}{l} = \frac{c}{4\pi l}\int\int\psi(r,\mu')\,d\Omega'. \tag{11.4}$$

We can now proceed as in the plane case, multiplying (11.4) by $(2n+1)P_n(\mu)$, integrating over all μ and eliminating $\partial\psi/\partial\mu$ by integration by parts. Thus, if the $\psi_n(r)$ are defined by (10.3) as before (which implies

that $\psi(r,\mu)$ is again given in terms of the $\psi_n(r)$ by (10.2)), the equation (11.4) gives

$$(n+1)\left[\frac{d}{dr}+\frac{n+2}{r}\right]\psi_{n+1}(r)+n\left[\frac{d}{dr}-\frac{n-1}{r}\right]\psi_{n-1}(r)+$$
$$+(2n+1)\frac{1-c\delta_{0n}}{l}\psi_n(r)=0. \quad (11.5)$$

In the P_N approximation, this infinite system of equations is converted into a finite one by neglecting $\psi_{N+1}(r)$ and retaining only the first $N+1$ equations of (11.5).

It is easily verified that the resulting system of equations is satisfied if

$$\psi_n(r)=G_n(\nu_j)\frac{1}{\sqrt{r}}K_{n+\frac{1}{2}}(-\nu_j r/l), \quad (11.6)$$

where $K_s(z)$ is the modified Bessel function of the second kind, the $G_n(\nu)$ are defined and discussed in § 10.2, and ν_j satisfies (10.12), i.e. $G_{N+1}(\nu_j)=0$. This may be shown by using the recurrence relations of Bessel functions, namely

$$\frac{d}{dz}[z^sK_s(z)]=-z^sK_{s-1}(z) \quad \text{and} \quad \frac{d}{dz}[z^{-s}K_s(z)]=-z^{-s}K_{s+1}(z);$$

see Whittaker and Watson (52), pp. 373f. We use throughout Macdonald's definition of $K_s(z)$, i.e. $K_s(z)=\frac{1}{2}\pi[I_{-s}(z)-I_s(z)]\operatorname{cosec} s\pi$.

The condition $G_{N+1}(\nu_j)=0$ arises since in the P_N approximation $\psi_{N+1}(r)$ must vanish identically.

The general solution of (11.5) in the P_N approximation can then be written

$$\psi_n(r)=\sum_j A_j\nu_j G_n(\nu_j)\left(\frac{2l}{\pi\nu_j r}\right)^{\frac{1}{2}}K_{n+\frac{1}{2}}(-\nu_j r/l), \quad (11.7)$$

where the A_j are constants of integration, and the numerical factor is introduced for convenience. Substituting (11.7) into (10.2) we obtain for the angular distribution in the P_N approximation

$$\psi(r,\mu)=\frac{1}{4\pi}\sum_j A_j\nu_j\sum_{n=0}^{N}(2n+1)\left(\frac{2l}{\pi\nu_j r}\right)^{\frac{1}{2}}K_{n+\frac{1}{2}}(-\nu_j r/l)G_n(\nu_j)P_n(\mu). \quad (11.8)$$

Since the ν_j occur in pairs, which in the notation of § 10.2.3 are ν_j and $\nu_{-j}=-\nu_j$, the relations between $K_s(z)$, $K_s(-z)$ and $I_s(z)$ give

$$\psi_n(r)=\sum_{j>0}G_n(-\nu_j)\left(\frac{2l}{\pi\nu_j r}\right)^{\frac{1}{2}}[\tilde{A}_j\nu_j K_{n+\frac{1}{2}}(\nu_j r/l)+B_j I_{n+\frac{1}{2}}(-\nu_j r/l)]. \quad (11.7')$$

If $c = 1$, the equation (10.7) has a double root zero, and the contribution of this to $\psi_n(r)$ is easily found by direct substitution into (11.5) to be

$$\frac{n!}{2n+1} A \left(\frac{l}{r}\right)^{n+1} + B\delta_{0n}, \tag{11.9}$$

where, of course, n does not exceed N in the P_N approximation.

The relation (11.9) can be deduced from the properties of the $G_n(\nu)$ as follows. The $K_{n+\frac{1}{2}}$ in (11.7′) are elementary functions:

$$(2l/\pi\nu r)^{\frac{1}{2}} K_{n+\frac{1}{2}}(-\nu r/l) = e^{\nu r/l} \times \text{a polynomial of order } (n+1) \text{ in } l/\nu r. \tag{11.10}$$

If ν is the root of (10.7) which tends to zero as $c \to 1$, then $G_{N+1}(\nu) = 0$ gives for $n \leqslant N$, $G_n = O(\nu G_{n-1}) = O(\nu^n)$. Thus, for $\nu \to 0$, the quantity

$$\nu G_n(-\nu)(2l/\pi\nu r)^{\frac{1}{2}} K_{n+\frac{1}{2}}(\nu r/l)$$

tends to a numerical multiple of $(l/r)^{n+1}$. On calculating the numerical factor, we find the first term of (11.9). Similarly the second term of (11.7′) gives the second term of (11.9).

11.2. Discussion of the solution

In the present section we consider two questions, the reason why the solution in § 11.1 involves the same G_n functions as the solution for the plane case, and the reason why, on substituting (11.9) into (10.2), we obtain the first $N+1$ terms of an everywhere divergent series.

11.2.1. *The reappearance of the functions $G_n(\nu)$*

The relation (11.10) shows that the expressions (11.6) differ from the corresponding ones for the plane case only by a polynomial in $1/r$. If r increases indefinitely, the solution for the spherical case should become that for the plane case, and since the ν_j are independent of position, this can only be so if they are the same as for the plane case. This suggests also that the same values of ν_j should appear for other geometries as well, and we shall show in § 12.2 that this is true.

We now indicate why the functions $G_n(\nu)$ should also figure in the solutions for both the plane and the spherical case. Consider an infinite medium and a spherically symmetrical solution of the equations, regular at the origin. This solution can be regarded as a superposition of plane waves, and we should therefore have

$$\psi(r,\mu) = \iint \psi_{\text{pl}}(\mathbf{r}.\boldsymbol{\Omega}_0, \boldsymbol{\Omega}.\boldsymbol{\Omega}_0)\, d\Omega_0, \tag{11.11}$$

where $\psi_{\text{pl}}(x,\mu)$ is the appropriate solution for the plane case. If this latter has been determined in the P_N approximation, and if we assume

for simplicity that it involves only one ν_j and that $A_j = 1$, distances being measured in units of l, then

$$\psi_{pl}(x,\mu) = \frac{1}{4\pi} e^{\nu_j x} \sum_{n'=0}^{N} (2n'+1) G_{n'}(\nu_j) P_{n'}(\mu).$$

Substituting into (11.11) and using (11.1), we find for the nth moment $\psi_n(r)$

$$\psi_n(r) = \frac{1}{4\pi} \int\int d\Omega \int\int d\Omega_0 \, e^{\nu_j \mathbf{r}.\Omega_0} P_n\left(\frac{\mathbf{r}.\Omega}{r}\right) \sum_{n'=0}^{N} (2n'+1) G_{n'}(\nu_j) P_{n'}(\Omega.\Omega_0),$$

which, by the properties of Legendre polynomials, reduces to

$$\psi_n(r) = \frac{2n+1}{4\pi} G_n(\nu_j) \int\int d\Omega \int\int d\Omega_0 \, e^{\nu_j \mathbf{r}.\Omega_0} P_n(\mathbf{r}.\Omega/r) P_n(\Omega.\Omega_0).$$

The appearance of the G_n functions in the solutions (11.6) is therefore not a coincidence, but a consequence of their appearance in the plane case.

11.2.2. *The asymptotic angular distribution*

The equations (11.5) can also be regarded as a means of determining the higher moments of the angular distribution in a spherically symmetric problem, when the neutron flux is known. To do this, however, requires some caution, as we shall now see.

From (10.27) and (10.28), the function $K_i(\pm\nu_1 r/l)$ will vary with r much more slowly than the other $K_i(\pm\nu_j r/l)$, and so, if we consider a point far enough from the boundaries, the value of $\psi_0(r)$ there will be almost entirely due to the contribution from the roots $\pm\nu_1$ of (10.7) (the *asymptotic term*), while the contributions of the remaining terms in (11.7) will be negligible (the *transient term*). This, of course, is no more than we have said already in Chapter V.

The quantities $G_n(\nu)$ have been shown to decrease (on the average) as n increases, and the smaller ν, the faster this decrease takes place. The ratio $G_n(\pm\nu_j)/G_n(\pm\nu_1)$ for $j \geqslant 2$ will therefore increase with n. For a given system, the distance from the boundary at which the transient terms can safely be neglected will thus increase with the order of the moment considered; conversely, at a given point the order of the moment cannot be increased indefinitely if the transient terms are to remain negligible.

The convergence of the series $\sum (2n+1)\psi_n(r)P_n(\mu)$ therefore gives no information about that of the series

$$\sum (2n+1)\psi_{n,as}(r)P_n(\mu), \tag{11.12}$$

since $\psi_n(r)$ may be the small difference of two large quantities $\psi_{n,as}(r)$ and $-\psi_{n,tr}(r)$. In fact, as we have mentioned, the substitution of (11.9) into (10.2) gives only an asymptotic series.

We now derive, for any geometry, the sufficient condition for (11.12) to be convergent and not merely asymptotic. Let a volume V be filled with some material, so that $\psi_{0,as}(\mathbf{r})$ is a particular analytical function, the same throughout V. Let $\psi_{0,as}(\mathbf{r}, V)$ coincide with $\psi_{0,as}(\mathbf{r})$ in V, and be given by the same analytical function for $\mathbf{r}$ outside V. Then if $\psi_{0,as}(\mathbf{r}, V)$ is regular in all space, the series

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} C_{n,m} \psi_{n,m,as}(\mathbf{r}) P_{n,m}(\Omega) \tag{11.12'}$$

(an obvious generalization of (11.12), with $C_{n,m}$ normalization coefficients) will converge for all $\mathbf{r}$ in V.

The proof is as follows. By definition, $\psi_{0,as}(\mathbf{r}, V)$ is a solution of

$$(\nabla^2 - 1/L_V^2)\psi_{0,as}(\mathbf{r}, V) = 0, \tag{11.13}$$

where L_V is the diffusion length for the material in V; and we have seen in Chapter V that, if the solution of (11.13) is regular in all space, then it is also the exact solution of the integral equation for the infinite medium. For the infinite medium, the angular distribution can be found from (4.16). Let this distribution be $\psi_\infty(\mathbf{r}, \Omega, V)$; it is a well-defined continuous function, and therefore its expansion in spherical harmonics, of the same form as (11.12'), converges for all $\mathbf{r}$. We now compare the moments $\psi_{n,m,\infty}(\mathbf{r}, V)$ which appear in this last series with the corresponding ones $\psi_{n,m,as}(\mathbf{r})$ which appear in (11.12'), for $\mathbf{r}$ in V. For the plane and spherical cases, these moments are uniquely defined when the zeroth moment is given, and we shall see in § 12.1 that this is true for any geometry, at least as regards the asymptotic part. Since, however, $\psi_{0,\infty}(\mathbf{r}, V)$ was constructed so as to coincide with $\psi_{0,as}(\mathbf{r})$ for $\mathbf{r}$ in V, it follows that for $\mathbf{r}$ in V each term of (11.12') is identical with the corresponding term in a series that is known to converge, and so the result is proved.

In the plane case, any solution of (11.13) which depends on only one Cartesian coordinate (x, say) is necessarily regular in all space, and therefore the series corresponding to (11.12) will always converge in the case of plane symmetry.

In the case of spherical symmetry, a solution of (11.13) may have a singularity at the origin, but is always regular elsewhere. Thus if $\psi_{0,as,i}(r) \equiv \psi_{0,as}(r)$ in the ith medium and is elsewhere defined by analytic continuation, then the series (11.12) will converge for r in the ith region if $\psi_{0,as,i}(0)$ is finite, but not otherwise. This statement involves proving

that the above condition ($\psi_{0,as}(\mathbf{r}, V)$ regular in all space) is necessary as well as sufficient for (11.12′) to converge for $\mathbf{r}$ in V, but this follows, at least in the spherical case, because $c_i = 1$ and $\psi_{0,as,i}(0) = \infty$ makes (11.12) diverge for all r in the ith region, and for $c_i \neq 1$ the same can be proved from the asymptotic behaviour of $K_{n+\frac{1}{2}}(-\nu r/l)$ and $G_n(\nu)$ for large n.

For instance, if there is a spherical core in an infinite slightly capturing ($c < 1$) reflector, then for the core $\psi_{0,as}(r)$ is regular for $r = 0$ and consequently (11.12) converges everywhere in the core; for the reflector, however, $\psi_{0,as}(r)$ must vanish at $r = \infty$, and it is a solution of the diffusion equation. Its analytical continuation cannot therefore be regular at $r = 0$, and so (11.12) cannot converge for any point in the reflector.

11.3. The boundary conditions

11.3.1. *Conditions at the origin*

We now determine the boundary conditions for the solution of (11.5). In the absence of sources, the exact solution of the integral equation (4.15′) is regular in any region, and so the solution of (11.5) should also be regular. In the plane case, the corresponding condition was automatically satisfied for all x, but in the spherical case it is satisfied only for $r \neq 0$ (see (11.7), for instance). Thus, apart from the boundary conditions, we must also impose a condition at the origin

$$\psi_n(0) \text{ is finite} \quad (n = 0, 1, \ldots, N). \tag{11.14}$$

The formula (11.7′) shows that (11.14) actually represents $\frac{1}{2}(N+1)$ conditions, i.e. exactly the number required.

If there are distributed sources, but the source term is regular at $r = 0$, the same argument as before leads to the conditions (11.14); if the source term is singular at $r = 0$, a slightly more general condition may be derived as follows. In physical applications $\psi(r, \mu)$ should be positive, while the Legendre polynomials satisfy the inequality $|P_n(\mu)| \leqslant P_0(\mu)$ for $|\mu| \leqslant 1$. Thus (10.3) gives

$$|\psi_n(r)| \leqslant \psi_0(r). \tag{11.15}$$

Using (11.7) and (11.5), it is easily shown that, in the absence of sources, (11.15) implies (11.14). The former therefore provides the necessary $\frac{1}{2}(N+1)$ conditions in a form which can be used even if the source term is singular at the origin.

If there are concentrated sources, however, the procedure is that given in § 10.3.7; that is, the neutrons are divided into those that come directly from the source and those which have had at least one collision

in the medium. The approximate method is applied only to the latter group.

11.3.2. *Boundaries already encountered in the plane case*

We now consider the conditions at the actual physical boundaries of the system. The conditions at infinity will be essentially the same as in the plane case, except that a supply of neutrons from infinity is now understood in the sense of total supply rather than supply per unit area, and is given by

$$\lim_{r\to\infty} 4\pi r^2\psi_1(r) \quad \text{rather than by} \quad \lim_{x\to\infty} \psi_1(x).$$

The conditions at the interface between two media in direct contact are purely local conditions and will not depend on the geometry. They are therefore the same as in the plane case. This also applies to Marshak's conditions at a free surface.

The situation with respect to Mark's conditions is more complicated. These were derived in the plane case by replacing the vacuum by a black medium, applying the interface conditions and eliminating the constants of integration for the black medium. The same procedure applied to the spherical case gives a set of conditions of the form

$$\sum_{n=0}^{N} \psi_n(a) U_{n,j}(l_b/a) = 0 \quad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)), \tag{11.16}$$

where a is the value of r at the free surface, l_b is the assumed mean free path in the black medium, and the $U_{n,j}$ are some functions of l_b/a, which for $l_b/a = 0$ become

$$U_{n,j}(0) = (2n+1)P_n(\mu_j),$$

so that in this case (11.16) coincide with the plane form of Mark's conditions.

If a non-re-entrant system is surrounded by a purely absorbing medium, the neutron distribution in the system cannot depend on the mean free path in the absorber. The appearance of l_b in (11.16) is due to the fact that the boundary conditions cannot be exactly satisfied in any approximation of a finite order.

Normally, l_b is taken equal to zero. Mark's boundary conditions then take the same form as in the plane case. This assumption is simple from the point of view of numerical work, and the magnitude of the error committed at the free surface in a given order of approximation is independent of the curvature of the surface, i.e. is the same as in the plane

case, and for the plane case the convergence of the spherical harmonics method is good. Further, the application of (11.16) with $l_b \gg a$ in low-order approximations has led to impossible results.

However, as at the end of § 10.3.6, there may be situations in which his is not the best course. For instance, if it is required to find the effect on the neutron distribution of surrounding the system by a strong but imperfect absorber ($0 < c \ll 1$), then the calculations should clearly be carried out for situations as similar as possible, and this means that the perfect absorber referred to in the derivation of (11.16) should be considered to have the same mean free path as the imperfect absorber whose effect is being assessed.

It should also be remarked that the technique described in § 10.3.5 for the application of Mark's boundary conditions cannot be extended to the spherical case. This concludes our discussion of conditions at the interface.

The conditions at an exposed surface (§ 10.3.7) are again of Marshak's type, and so are likewise applicable irrespective of the geometry of the system.

In the case of a purely absorbing medium of negligible geometrical thickness between two other media, there were in the plane case two alternative sets of conditions: (10.58), which are more reliable, and (10.60), which are simpler. The former are of Marshak's type, and so applicable at once to the spherical case. Though the latter are of Mark's type, it follows from the nature of the layer that its radius of curvature must be very large compared with the mean free path in it. The solution within the layer will therefore be indistinguishable from that in the plane case, and so the conditions (10.60) can be taken over without modification. In this case we do not encounter the ambiguity which was present in the case of (11.16).

11.3.3. *Conditions at the surfaces of a gap*

In the case of plane symmetry, gaps have no effect, as we have seen (§ 10.3.8), but in the spherical case they must be taken into account, and this may be done in either of two ways. The gap may be treated as a medium where the mean free path is infinite, and then the interface conditions are applied; or, the surfaces of the gap may be considered directly, and then the exposed-surface conditions are applied (10.55).

We first consider the former approach. Let the inner and outer surfaces of the gap be $r = a$ and $r = b$ respectively. Since in the gap $l = \infty$, the equations (11.5) for the gap separate into two groups, one involving

only the odd-order moments,

$$\left.\begin{aligned} \left(\frac{d}{dr}+\frac{2}{r}\right)\psi_1(r) &= 0, \\ 3\left(\frac{d}{dr}+\frac{4}{r}\right)\psi_3(r)+2\left(\frac{d}{dr}-\frac{1}{r}\right)\psi_1(r) &= 0,\ldots, \\ N\left(\frac{d}{dr}+\frac{N+1}{r}\right)\psi_N(r)+(N-1)\left(\frac{d}{dr}-\frac{N-2}{r}\right)\psi_{N-2}(r) &= 0, \end{aligned}\right\} \quad (11.17)$$

and the other involving only the even-order moments,

$$\left.\begin{aligned} 2\left(\frac{d}{dr}+\frac{3}{r}\right)\psi_2(r)+\frac{d}{dr}\psi_0(r) &= 0,\ldots, \\ (N-1)\left(\frac{d}{dr}+\frac{N}{r}\right)\psi_{N-1}(r)+(N-2)\left(\frac{d}{dr}-\frac{N-3}{r}\right)\psi_{N-3}(r) &= 0, \\ N\left(\frac{d}{dr}-\frac{N-1}{r}\right)\psi_{N-1}(r) &= 0, \end{aligned}\right\} \quad (11.18)$$

where N is, as usual, assumed odd. Each of the equations (11.17) is exact, while the last of (11.18) has been distorted by the omission of the term in $\psi_{N+1}(r)$. Thus, by solving (11.17) and applying the interface conditions (10.30) (also exact), we obtain the exact relations between the odd-order moments on the inner and outer faces of the gap. These are found to be

$$a^2\psi_1(a) = b^2\psi_1(b), \qquad a^4[3\psi_3(a)-3\psi_1(a)] = b^4[3\psi_3(b)-3\psi_1(b)],$$
$$a^6[5\psi_5(a)-14\psi_3(a)+9\psi_1(a)] = b^6[5\psi_5(b)-14\psi_3(b)+9\psi_1(b)], \text{ etc.} \quad (11.19)$$

Similarly, solving equations (11.18) and using the interface boundary conditions, we obtain

$$\psi_0(a)+5\psi_2(a)+16\psi_4(a)+\ldots = \psi_0(b)+5\psi_2(b)+16\psi_4(b)+\ldots,$$
$$\frac{1}{a^2}[\psi_2(a)+6\psi_4(a)+\ldots] = \frac{1}{b^2}[\psi_2(b)+6\psi_4(b)+\ldots],$$

and in general

$$\frac{1}{a^{2s}}\left[\psi_{2s}(a)+\frac{(s+1)(4s+5)}{2s+1}\psi_{2s+2}(a)+\ldots\right]$$
$$= \frac{1}{b^{2s}}\left[\psi_{2s}(b)+\frac{(s+1)(4s+5)}{2s+1}\psi_{2s+2}(b)+\ldots\right]. \quad (11.20)$$

However, since the last of (11.18) is incorrect, the relations (11.20), with the sums terminated, in practice, after the term in ψ_{N-1}, are not exact; further, if the sums are not terminated, the resulting series may converge

very slowly, if at all. Unless one goes to a fairly high approximation, therefore, (11.20) may be quite inaccurate.

We therefore consider the second approach mentioned at the beginning of this sub-section. In this, the exposed-surface boundary conditions are applied to the surfaces of the gap, and we first take the inner surface.

If a neutron enters the gap at a point **b** of the outer surface and first meets the inner surface at **a** after travelling a distance R in the gap, and μ, μ' are the cosines of the angles between its direction of travel $\boldsymbol{\Omega}$ and **a**, **b** respectively, then in the case considered both μ and μ' are negative. From the triangle formed by **a**, **b**, and the path of the neutron in the gap, we have

$$b^2 = a^2 + R^2 - 2aR\mu,$$

where a and b are the magnitudes of the corresponding vectors, and

$$b\mu' = a\mu - R.$$

Eliminating R, we find

$$\mu' = -(b^2 - a^2 + a^2\mu^2)^{\frac{1}{2}}/b. \tag{11.21}$$

The Boltzmann equation in the gap is simply $\boldsymbol{\Omega} \cdot \mathbf{grad}\,\psi(\mathbf{r},\boldsymbol{\Omega}) = 0$, i.e. $\psi(\mathbf{r},\boldsymbol{\Omega})$ is a constant along the path of the neutron in the gap. In particular, $\psi(\mathbf{b},\boldsymbol{\Omega}) = \psi(\mathbf{a},\boldsymbol{\Omega})$, that is, $\psi(b,\mu') = \psi(a,\mu)$, which from (11.21) gives

$$\psi(a,\mu) = \psi(b, -(b^2 - a^2 + a^2\mu^2)^{\frac{1}{2}}/b). \tag{11.22}$$

Substituting this into the exposed surface boundary conditions (10.55), we obtain

$$\begin{aligned}\int_{-1}^{0} \mu^{2j-1}[\psi_0(a) + 3\mu\psi_1(a) + \tfrac{5}{2}(3\mu^2-1)\psi_2(a) + \ldots]\,d\mu \\ = \int_{-1}^{0} \mu^{2j-1}\Big[\psi_0(b) - \frac{3}{b}(b^2 - a^2 + a^2\mu^2)^{\frac{1}{2}}\psi_1(b) + \\ + \frac{5}{2}\Big(3\frac{b^2 - a^2 + a^2\mu^2}{b^2} - 1\Big)\psi_2(b) - \ldots\Big]\,d\mu \quad (j = 1, 2, \ldots, \tfrac{1}{2}(N+1)).\end{aligned} \tag{11.23}$$

The conditions for the outer surface, which we call (11.23′), can be derived similarly. In this case, it is necessary to bear in mind that a neutron meeting the outer surface of the gap may have come from either surface, and the integral on the left of (11.23) will therefore be replaced by the sum of two, one from 0 to $[1-(a/b)^2]^{\frac{1}{2}}$ and the other from $[1-(a/b)^2]^{\frac{1}{2}}$ to 1.

It can be shown that on eliminating the even moments between (11.23) and (11.23′) we obtain (11.19), so that using (11.19) and either (11.23) or

(11.23′) is the same as using the latter two together. However, if the integrations are effected, when the order of approximation increases the series in (11.23) converges much more rapidly than that in (11.20), and so the latter is much less accurate in a finite approximation than (11.23) and (11.23′). The best procedure, since (11.23) is simpler than (11.23′), is therefore to use (11.19) and (11.23).

Note added in second impression. We have seen in § 10.4 that, in the case of plane symmetry, the calculation for multi-layer problems can be appreciably simplified by using orthogonality relations between the $G_n(\nu_j)$; see the formula preceding (10.65). It has recently been shown (B. Davison, *Canadian J. Phys.* **35**, 55, 1957) that the treatment of multi-layer problems can be similarly simplified in the case of spherical symmetry: the A_j in (11.7) can be expressed in terms of the $\psi_n(r)$ as

$$A_j = \left(\frac{r}{l}\right)^2 \frac{d\nu_j}{dc}\sqrt{\left(\frac{-2l\nu_j}{\pi r}\right)} \sum_n \psi_n(r) \times$$

$$\times \left\{(n+1)G_{n+1}(\nu_j)K_{n+\frac{1}{2}}\left(\frac{\nu_j r}{l}\right) + nG_{n-1}(\nu_j)K_{n-\frac{1}{2}}\left(\frac{\nu_j r}{l}\right)\right\}. \quad (11.24)$$

Combining this with (11.7) and the condition of continuity of the $\psi_n(r)$ at the interface between two successive layers, we see that, if the constants A_j for one layer are known, those for the next layer can be obtained by simple multiplication of matrices, without resorting to numerical inversion.

XII

THE SPHERICAL HARMONICS METHOD FOR OTHER GEOMETRIES

12.1. The general formulation of the spherical harmonics method

12.1.1. *Preliminary*

We now consider the extension of the spherical harmonics method to other geometries. In the general case, the angular distribution $\psi(\mathbf{r},\Omega)$ will depend on both the angular coordinates of Ω, and consequently, instead of expanding $\psi(\mathbf{r},\Omega)$ in Legendre polynomials in $\Omega.\Omega_0$ (where Ω_0 is fixed for each $\mathbf{r}$), it is necessary to expand it in general spherical harmonics. In the most general case there will be $2n+1$ harmonics of order n. The procedure is, however, essentially the same as before.

Let $P_{n,m}(\Omega)$ $(m = -n, -n+1,\ldots, n)$ be the $2n+1$ spherical harmonics of order n, and $\psi_{n,m}(\mathbf{r}) = \iint \psi(\mathbf{r},\Omega)P_{n,m}(\Omega)\,d\Omega$ the corresponding moments. In the P_N approximation, we multiply the Boltzmann equation with isotropic scattering ((4.4)) by $P_{n,m}(\Omega)$ for all $n \leqslant N$ and all m for each n, and integrate over all Ω, neglecting moments of order $N+1$ and greater. We thus obtain as many differential equations as there are $\psi_{n,m}(\mathbf{r})$ involved, and so, by solving these equations with the appropriate boundary conditions, all the moments in question can be determined.

In the case of plane or spherical symmetry, the form of the solution of the differential equations could be guessed immediately, but in the general case this is no longer so; further, the number of moments involved is rather large—in the P_N approximation it is $1+3+\ldots+2N+1 = (N+1)^2$. It is therefore desirable to systematize the solution of the equations, and in particular to reduce them by elimination to a series of equations in one variable only.

The most elegant method of performing this elimination is by the use of spherical harmonics tensors. Since the properties of these are not widely known, however, we shall use a more elementary, quasi-vectorial, procedure. In the present section we discuss the method in general terms, in the next we deal with the solution of the resulting equations, and in § 12.3 we illustrate the results in terms of a finite cylinder.

12.1.2. *Notations and properties of spherical harmonics*

Let $\mathbf{U}$ be a vector of arbitrary magnitude U in the direction of $\boldsymbol{\Omega}$:

$$\mathbf{U} = U\boldsymbol{\Omega}. \tag{12.1}$$

In dealing with functions of the two vector variables $\mathbf{U}$ and $\mathbf{r}$, we distinguish the operators **grad**, div, and ∇^2 taken with respect to them by corresponding suffixes:

$$\left.\begin{aligned} \nabla_{\mathbf{U}}^2 &= \partial^2/\partial(U\Omega_x)^2+\partial^2/\partial(U\Omega_y)^2+\partial^2/\partial(U\Omega_z)^2, \\ \operatorname{div}_{\mathbf{U}} \mathbf{grad}_{\mathbf{r}} &= \operatorname{div}_{\mathbf{r}} \mathbf{grad}_{\mathbf{U}} \\ &= \partial^2/\partial(U\Omega_x)\partial x+\partial^2/\partial(U\Omega_y)\partial y+\partial^2/\partial(U\Omega_z)\partial z, \end{aligned}\right\} \tag{12.2}$$

and so on.

If
$$\psi(\mathbf{r},\boldsymbol{\Omega}) = \frac{1}{4\pi}\sum_n \sum_{m=-n}^{n} a_{n,m}\psi_{n,m}(\mathbf{r})P_{n,m}(\boldsymbol{\Omega}) \tag{12.3}$$

is the expansion of the angular distribution in general spherical harmonics, where $a_{n,m}$ is the appropriate normalization coefficient:

$$a_{n,m} = 4\pi\Big[\iint P_{n,m}^2(\boldsymbol{\Omega})\, d\boldsymbol{\Omega}\Big]^{-1},$$

let
$$\Psi_n = \frac{U^n}{2n+1}\sum_{m=-n}^{n} a_{n,m}\psi_{n,m}(\mathbf{r})P_{n,m}(\boldsymbol{\Omega}), \tag{12.4}$$

so that (12.3) can be rewritten

$$\psi(\mathbf{r},\boldsymbol{\Omega}) = \frac{1}{4\pi}\sum_n (2n+1)[\Psi_n]_{U=1}. \tag{12.5}$$

From the general properties of spherical harmonics, it follows that for all m, n
$$\nabla_{\mathbf{U}}^2[U^n P_{n,m}(\boldsymbol{\Omega})] = 0,$$

and conversely, any solution of this equation that is proportional to U^n is a spherical harmonic of order n; see Whittaker and Watson (52), p. 392. Hence, from (12.4),

$$\nabla_{\mathbf{U}}^2 \Psi_n = 0, \tag{12.6}$$

We now prove the following

LEMMA. *Any homogeneous polynomial $p_n(\mathbf{U})$ of order n in $U\Omega_x$, $U\Omega_y$, and $U\Omega_z$ can be represented uniquely in the form*

$$p_n(\mathbf{U}) = p_n^*(\mathbf{U})+U^2p_{n-2}^*(\mathbf{U})+U^4p_{n-4}^*(\mathbf{U})+\ldots, \tag{12.7}$$

where $p_n^*(\mathbf{U})$, $p_{n-2}^*(\mathbf{U})$, etc., are homogeneous polynomials of order n, $n-2$, etc., in $U\Omega_x$, $U\Omega_y$, $U\Omega_z$, satisfying the equation

$$\nabla_{\mathbf{U}}^2 p_{n-2j}^*(\mathbf{U}) = 0. \tag{12.8}$$

Proof. Using Euler's identity $\mathbf{U}.\mathbf{grad}_{\mathbf{U}} p_n(\mathbf{U}) = np_n(\mathbf{U})$, it is easily verified that

$$\nabla_{\mathbf{U}}^2\left[p_n(\mathbf{U}) - \frac{U^2}{2(2n-1)}\nabla_{\mathbf{U}}^2 p_n(\mathbf{U}) + \frac{U^4}{2.4(2n-1)(2n-3)}\nabla_{\mathbf{U}}^4 p_n(\mathbf{U}) - \ldots\right] = 0,$$

so that in (12.7) we may take

$$p_n^*(\mathbf{U}) = p_n(\mathbf{U}) - \frac{U^2}{2(2n-1)}\nabla_{\mathbf{U}}^2 p_n(\mathbf{U}) + \ldots. \tag{12.9}$$

Dividing $[p_n(\mathbf{U}) - p_n^*(\mathbf{U})]$ by U^2 and repeating the process, we find $p_{n-2}^*(\mathbf{U})$, and so on. The representation (12.7) is therefore always possible.

The number of coefficients in a general homogeneous polynomial of order n in three variables is equal to the total number of coefficients involved in $p_n^*(\mathbf{U})$, $p_{n-2}^*(\mathbf{U})$, etc., and the relations between the two sets of coefficients are linear. Thus, by the properties of linear algebraic equations, the representation (12.7) is unique. This proves the lemma.

The above lemma provides an easy means of expanding a general homogeneous polynomial of order n in Ω_x, Ω_y, Ω_z ($p_n(\Omega)$, say) in spherical harmonics. To do this, we multiply it by U^n, apply (12.7) with the $p_r^*(\mathbf{U})$ found from (12.9), and put $U = 1$; for $p_{n-2j}^*(\Omega)$, as we have seen above, is a spherical harmonic of order $n-2j$.

12.1.3. *The differential equations*

In view of (12.5), the Boltzmann equation can be written

$$\left[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \sum_n (2n+1)\Psi_n\right]_{U=1} + \frac{1}{l}\left[\sum_n (2n+1)\Psi_n\right]_{U=1} = \frac{c}{l}\psi_0(\mathbf{r}), \tag{12.10}$$

where Ψ_0 is, of course, identical with the neutron flux $\psi_0(\mathbf{r})$. In order to expand (12.10) in spherical harmonics, it is sufficient to thus expand each of the quantities

$$[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n]_{U=1}.$$

Using the above lemma and the results

$$\nabla_{\mathbf{U}}^2[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n] = 2\,\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}} \Psi_n,$$

$$\nabla_{\mathbf{U}}^4[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n] = \nabla_{\mathbf{U}}^6[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n] = \ldots = 0,$$

we easily obtain

$$[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n]_{U=1} = \left[\mathbf{U}.\mathbf{grad}_{\mathbf{r}} \Psi_n - \frac{U^2}{2n+1}\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}} \Psi_n\right]_{U=1} + \\ + \frac{1}{2n+1}[\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}} \Psi_n]_{U=1},$$

where the first term on the right is a spherical harmonic of order $n+1$ and the second is one of order $n-1$.

Substituting into (12.10), collecting spherical harmonics of order n, and multiplying by U^n, we find:

$$\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}\Psi_{n+1}+\frac{2n+1}{l}\Psi_n+$$

$$+[(2n-1)\mathbf{U}\cdot\mathbf{grad}_{\mathbf{r}}\Psi_{n-1}-U^2\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}\Psi_{n-1}]=\frac{c}{l}\delta_{n0}\psi_0(\mathbf{r}), \quad (12.11)$$

and hence also

$$\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}\Psi_{n+1}$$

$$=-\frac{1}{l}[(2n+1)\Psi_n+(2n-3)U^2\Psi_{n-2}+(2n-7)U^4\Psi_{n-4}+\ldots]-$$

$$-\mathbf{U}\cdot\mathbf{grad}_{\mathbf{r}}[(2n-1)\Psi_{n-1}+(2n-5)U^2\Psi_{n-3}+\ldots]+\frac{1+(-1)^n}{2}U^n\frac{c}{l}\psi_0(\mathbf{r}). \quad (12.12)$$

Since Ψ_n consists in general of $2n+1$ components, equation (12.11) gives, for each n, $2n+1$ relations between the components of Ψ_{n+1}, Ψ_n, and Ψ_{n-1}. If, however, the number of components of Ψ_n is less than $2n+1$ owing to the symmetry of the problem considered, then (12.11) correspondingly represents fewer equations.

The equation (12.12) for a particular n appears to contain more equations between the components of the $\Psi_{n'}$ ($n' \leqslant n+1$) than the corresponding equation (12.11). However, if (12.12) is satisfied for all $n < n_0$, then for $n = n_0$ it becomes identical with the corresponding equation (12.11), and so for $n = n_0$ it represents as many equations between the components of various Ψ_n as there are components of Ψ_{n_0}.

The equations (12.12) are exact. In the P_N approximation, the equations (12.12) are retained as they stand only for $n < N$. For $n = N$ the term $\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}\Psi_{N+1}$ is omitted, and for $n > N$ the equations are disregarded.

12.1.4. *The method of successive elimination*

The system of equations (12.12) is to be reduced to a system involving only one unknown in each equation. This is done as follows. The last equation (12.12) (i.e. that for $n = N$, with Ψ_{N+1} neglected) gives Ψ_N in terms of Ψ_n for $n < N$. Operating on this expression with $\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}$, and eliminating $\mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_{\mathbf{r}}\Psi_N$ by means of equation (12.12) for $n = N-1$, we obtain an inhomogeneous differential equation for Ψ_{N-1}, whose free term depends on Ψ_n for $n \leqslant N-2$. Similarly we get an inhomogeneous equation for Ψ_{N-2}, and so on. The result is a set of equations of the form:

$$f_{N,n}(\nabla_{\mathbf{r}}^2)\Psi_n = F_{N,n}[\nabla^2, \mathbf{U}\cdot\mathbf{grad}_{\mathbf{r}}, U^2, \Psi_{n-1}, \Psi_{n-2}, \ldots, \psi_0(\mathbf{r})]. \quad (12.13)$$

The fact that the operator on Ψ_n in (12.13) has the form shown can be proved as follows. If the dependence of the various $\Psi_{n'}$ on U is taken into account, each term in (12.13) will be proportional to U^n. Since Ψ_n is itself proportional to U^n, the operator on it cannot involve U. It is also evident from the process of elimination that the differentiation with respect to position can enter only as $\mathbf{U}\,.\,\mathbf{grad}_r$ or as ∇_r^2, and the former is excluded by the argument just given. The fact that the equation for $\psi_0(\mathbf{r})$ is of the form

$$f_{N,0}(\nabla_r^2)\psi_0(\mathbf{r}) = 0 \tag{12.13'}$$

can also be shown directly, since $\psi_0(\mathbf{r})$ is a scalar quantity and therefore must remain invariant when the (Cartesian) system of coordinates is rotated. The equation that contains it alone is therefore also invariant, and since the only combinations of $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ invariant under rotation are the linear combinations of ∇_r^2, ∇_r^4, etc., the equation for $\psi_0(\mathbf{r})$ must have the form (12.13').

The last equation of (12.13) is a homogeneous equation for $\psi_0(\mathbf{r})$, and this can be solved. The three components of Ψ_1 can then be found from (12.13) for $n = 1$, and so on. However, since the equations (12.13) are derived directly from (12.12) only for $n = N$ and $n = N-1$, while in other cases differential operators are used, it is necessary to use (12.12) as further conditions, and to retain only those solutions of (12.13) for $n = 1$ which satisfy (12.12) for $n = 0$, only those of (12.13) for $n = 2$ which satisfy (12.12) for $n = 1$, and so on.

We give below the complete set of equations (12.13) together with the supplementary conditions (12.12), for the P_1, P_3, and P_5 approximations.

P_1 approximation

$$\left.\begin{aligned}\left[\nabla_r^2 - \frac{3(1-c)}{l^2}\right]\psi_0(\mathbf{r}) &= 0,\\ \Psi_1 &= -\tfrac{1}{3}l\,\mathbf{U}\,.\,\mathbf{grad}_r\psi_0(\mathbf{r}).\end{aligned}\right\} \tag{12.14}$$

P_3 approximation

$$\left.\begin{aligned}&\left[9\nabla_r^4 - \frac{1}{l^2}(90-55c)\nabla_r^2 + 105\frac{1-c}{l^4}\right]\psi_0(\mathbf{r}) = 0,\\ &\left[\frac{1}{l^2} - \tfrac{3}{7}\nabla_r^2\right]\Psi_1 = -\tfrac{1}{3}\left[\frac{9-2c}{7l} - \frac{9l}{35}\nabla_r^2\right]\mathbf{U}\,.\,\mathbf{grad}_r\psi_0(\mathbf{r}),\\ &\left[\frac{1}{l^2} - \tfrac{1}{7}\nabla_r^2\right]\Psi_2 = -\frac{6}{7l}\mathbf{U}\,.\,\mathbf{grad}_r\Psi_1 -\\ &\qquad -\tfrac{1}{5}\left[\tfrac{3}{7}(\mathbf{U}\,.\,\mathbf{grad}_r)^2 + \frac{10(1-c)U^2}{7l^2} - \frac{U^2}{7}\nabla_r^2\right]\psi_0(\mathbf{r}),\\ &\Psi_3 = -\tfrac{1}{7}l\mathbf{U}\,.\,\mathbf{grad}_r[5\Psi_2 + U^2\psi_0(\mathbf{r})] - \tfrac{3}{7}U^2\Psi_1,\end{aligned}\right\} \tag{12.15}$$

together with

$$\left.\begin{aligned} &\operatorname{div}_{\mathbf{U}} \operatorname{grad}_{\mathbf{r}} \Psi_1 = -\frac{(1-c)}{l}\psi_0(\mathbf{r}) \\ \text{and}\quad &\operatorname{div}_{\mathbf{U}} \operatorname{grad}_{\mathbf{r}} \Psi_2 = -\frac{3}{l}\Psi_1 - \mathbf{U}.\operatorname{grad}_{\mathbf{r}}\psi_0(\mathbf{r}). \end{aligned}\right\} \tag{12.16}$$

P_5 approximation

$$\left.\begin{aligned}
&\left[25\nabla_{\mathbf{r}}^6 - \frac{1}{l^2}(525-231c)\nabla_{\mathbf{r}}^4 + \right. \\
&\qquad \left. + \frac{1}{l^4}(1575-1190c)\nabla_{\mathbf{r}}^2 - 1155\frac{1-c}{l^6}\right]\psi_0(\mathbf{r}) = 0, \\
&\left[\frac{1}{l^4} - \frac{10}{11l^2}\nabla_{\mathbf{r}}^2 + \frac{5}{33}\nabla_{\mathbf{r}}^4\right]\Psi_1 \\
&\qquad = -\frac{1}{3}\left[\frac{15-4c}{11l^3} - \frac{50-8c}{55l}\nabla_{\mathbf{r}}^2 + \frac{5l}{77}\nabla_{\mathbf{r}}^4\right]\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\psi_0(\mathbf{r}), \\
&\left[\frac{1}{l^4} - \frac{6}{11l^2}\nabla_{\mathbf{r}}^2 + \frac{1}{33}\nabla_{\mathbf{r}}^4\right]\Psi_2 = -\frac{12}{11}\left[\frac{1}{l^3} - \frac{1}{3l}\nabla_{\mathbf{r}}^2\right]\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\Psi_1 + \\
&\qquad + \frac{1}{55}\left[\left(\frac{c-10}{l^2} + \frac{10}{7}\nabla_{\mathbf{r}}^2\right)(\mathbf{U}.\operatorname{grad}_{\mathbf{r}})^2 + \right. \\
&\qquad\qquad \left. + \left(20\frac{c-1}{l^4} + \frac{10-7c}{l^2}\nabla_{\mathbf{r}}^2 - \frac{10}{21}\nabla_{\mathbf{r}}^4\right)U^2\right]\psi_0(\mathbf{r}), \\
&\left[\frac{1}{l^3} - \frac{3}{11}\nabla_{\mathbf{r}}^2\right]\Psi_3 = -\frac{15}{11}\left[\frac{1}{l} - \frac{l}{9}\nabla_{\mathbf{r}}^2\right]\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\Psi_2 - \\
&- \left[\frac{5}{11}(\mathbf{U}.\operatorname{grad}_{\mathbf{r}})^2 + \left(\frac{9}{11l^2} - \frac{2}{11}\nabla_{\mathbf{r}}^2\right)U^2\right]\Psi_1 - \left[\frac{5l}{231}(\mathbf{U}.\operatorname{grad}_{\mathbf{r}})^3 + \right. \\
&\qquad \left. + \left(\frac{5-2c}{11l} - \frac{10l}{231}\nabla_{\mathbf{r}}^2\right)U^2\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\right]\psi_0(\mathbf{r}), \\
&\left[\frac{1}{l^3} - \frac{1}{11}\nabla_{\mathbf{r}}^2\right]\Psi_4 = -\frac{14}{11l}\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\Psi_3 - \frac{7}{99}\left[7(\mathbf{U}.\operatorname{grad}_{\mathbf{r}})^2 + \right. \\
&\qquad + \left.\left(\frac{18}{l^2} - \nabla_{\mathbf{r}}^2\right)U^2\right]\Psi_2 - \frac{28}{33l}U^2\mathbf{U}.\operatorname{grad}_{\mathbf{r}}\Psi_1 - \\
&\qquad - \frac{U^2}{99}\left[10(\mathbf{U}.\operatorname{grad}_{\mathbf{r}})^2 + \left(21\frac{1-c}{l^2} - \nabla_{\mathbf{r}}^2\right)U^2\right]\psi_0(\mathbf{r}), \\
&\Psi_5 = -\tfrac{1}{11}l\mathbf{U}.\operatorname{grad}_{\mathbf{r}}[9\Psi_4 + 5U^2\Psi_2 + U^4\psi_0(\mathbf{r})] - \\
&\qquad\qquad - \tfrac{7}{11}U^2\Psi_3 - \tfrac{3}{11}U^4\Psi_1,
\end{aligned}\right\} \tag{12.17}$$

together with (12.16) and

$$\left.\begin{aligned}\operatorname{div}_{\mathbf{U}} \operatorname{grad}_{\mathbf{r}} \Psi_3 &= -\frac{5}{l}\Psi_2 - 3\mathbf{U}\cdot\operatorname{grad}_{\mathbf{r}}\Psi_1 - \frac{1-c}{l}U^2\psi_0(\mathbf{r}),\\ \operatorname{div}_{\mathbf{U}} \operatorname{grad}_{\mathbf{r}} \Psi_4 &= -\frac{1}{l}(7\Psi_3 + 3U^2\Psi_1) - \\ &\quad -\mathbf{U}\cdot\operatorname{grad}_{\mathbf{r}}[5\Psi_2 + U^2\psi_0(\mathbf{r})].\end{aligned}\right\} \tag{12.18}$$

12.2. The reduction of the differential equations

12.2.1. *The determination of the neutron flux*

We now consider the solution of the equations derived in § 12.1. We shall not attempt to solve them completely, but shall reduce them to simpler equations and relate their solutions to those of the simpler equations.

As we already know, the equation for the neutron flux $\psi_0(\mathbf{r})$ has the form

$$f_{N,0}(\nabla_{\mathbf{r}}^2)\psi_0(\mathbf{r}) = 0, \tag{12.13'}$$

where the operator $f_{N,0}(\nabla_{\mathbf{r}}^2)$ is a polynomial in $\nabla_{\mathbf{r}}^2$ with constant coefficients; see the discussion following (12.13). On factorizing this polynomial, (12.13') can be rewritten

$$\prod_i (\nabla_{\mathbf{r}}^2 - \chi_i^2/l^2)\psi_0(\mathbf{r}) = 0, \tag{12.19}$$

where the χ_i are some real or complex constants depending only on c and N, and not on the geometry. For the case of plane symmetry ($\psi_0(\mathbf{r})$ depending on x only), the solution of (12.19) is

$$\psi_0(x) = \sum_i [B_i^+ e^{\chi_i x/l} + B_i^- e^{-\chi_i x/l}].$$

On comparing this with the result (10.8) obtained for plane symmetry, we see that the $\pm\chi_i$ are identical with the $\pm\nu_j$ in (10.8), and the equation (12.19) therefore has the form

$$\prod_{j=1}^{\frac{1}{2}(N+1)} (\nabla_{\mathbf{r}}^2 - \nu_j^2/l^2)\psi_0(\mathbf{r}) = 0, \tag{12.20}$$

where the ν_j are the roots of the determinantal equation (10.7), it being assumed that N is odd. Further, it is known (see § 10.2.2) that all the ν_j are different ($j = 1, 2, \ldots, \frac{1}{2}(N+1)$), and so the most general solution of (12.20) is

$$\psi_0(\mathbf{r}) = \sum_{j=1}^{\frac{1}{2}(N+1)} \psi_{0,j}(\mathbf{r}), \tag{12.21}$$

where $\psi_{0,j}(\mathbf{r})$ satisfies

$$(\nabla_{\mathbf{r}}^2 - \nu_j^2/l^2)\psi_{0,j}(\mathbf{r}) = 0. \tag{12.22}$$

If $c < 1$, so that all the ν_j^2 are real and positive, this result can also be written

$$\psi_0(\mathbf{r}) = \sum_{j=1}^{\frac{1}{2}(N+1)} \Phi_j(\nu_j \mathbf{r}/l), \tag{12.21'}$$

where every $\Phi_j(\mathbf{R})$ satisfies

$$(\nabla_{\mathbf{R}}^2 - 1)\Phi_j(\mathbf{R}) = 0. \tag{12.22'}$$

The extension of this to the case $c \geqslant 1$ is obvious.

12.2.2. *The neutron current and higher moments*

The neutron current is found from equation (12.13) with $n = 1$. Since the operators $\mathbf{U}.\mathbf{grad}_{\mathbf{r}}$ and $\nabla_{\mathbf{r}}^2$ commute, the contribution to the particular integral, arising from some term $\psi_{0,j}(\mathbf{r}) = \Phi_j(\nu_j \mathbf{r}/l)$ in (12.21'), will be of the form

$$\Psi_{1,j} = g_{1,j}\, \mathbf{U}.\mathbf{grad}\, \Phi_j(\nu_j \mathbf{r}/l), \tag{12.23}$$

where the gradient is with respect to the variable $\nu_j \mathbf{r}/l$ which appears in Φ_j, while $g_{1,j}$ is a numerical coefficient depending only on ν_j, c, and N. Being independent of the geometry, $g_{1,j}$ can be found from the plane case, where it equals $G_1(\nu_j)$. One can verify that the particular integral, constructed by adding the contributions obtained in this manner from all the terms of (12.21'), also satisfies (12.12) for $n = 0$.

To complete the solution of (12.13) for $n = 1$ we must add the complementary function. This will clearly have the form

$$\sum_{i=1}^{\frac{1}{2}(N-1)} U\mathbf{X}_i(\beta_i \mathbf{r}/l), \tag{12.24}$$

where the β_i are as follows:

$$\left.\begin{aligned} &\text{in the } P_3 \text{ approximation, } \beta_1 = \sqrt{(7/3)};\\ &\text{in the } P_5 \text{ approximation, } \beta_1 \text{ and } \beta_2 \text{ are the positive roots of}\\ &\qquad 1 - \tfrac{10}{11}\beta^2 + \tfrac{5}{33}\beta^4 = 0, \end{aligned}\right\} \tag{12.25}$$

etc., and the $\mathbf{X}_i$ are vectors each of whose components satisfies (12.22'). In order to satisfy the appropriate supplementary conditions (12.12), the three components of each $\mathbf{X}_i(\mathbf{R})$ must be related by

$$\mathrm{div}\, \mathbf{X}_i(\mathbf{R}) = 0. \tag{12.26}$$

The factor U is introduced in (12.24) so that $\mathbf{X}_i(\mathbf{R})$ is independent of U.

Collecting the above results, and expressing the gradient with respect to the argument of each Φ_j in terms of $\mathbf{grad}_{\mathbf{r}}$, we find

$$[\Psi_1]_{U=1} = l\boldsymbol{\Omega}.\mathbf{grad}_{\mathbf{r}}\Big[\sum_{j=1}^{\frac{1}{2}(N+1)} (1/\nu_j)\, G_1(\nu_j)\Phi_j(\nu_j \mathbf{r}/l)\Big] + \sum_{i=1}^{\frac{1}{2}(N-1)} \mathbf{X}_i(\beta_i \mathbf{r}/l). \tag{12.27}$$

Similarly,

$$[\Psi_2]_{U=1} = l^2[\tfrac{3}{2}(\Omega \cdot \mathbf{grad}_r)^2 - \tfrac{1}{2}\nabla_r^2]\left[\sum_{j=1}^{\frac{1}{2}(N+1)} \frac{1}{\nu_j^2} G_2(\nu_j)\Phi(\nu_j r/l)\right] -$$

$$-3l(\Omega \cdot \mathbf{grad}_r)\sum_{i=1}^{\frac{1}{2}(N-1)} (1/\beta_i^2)\mathbf{X}_i(\beta_i r/l) + \sum_{s=1}^{\frac{1}{2}(N-1)} \Theta_s(\gamma_s r/l), \quad (12.28)$$

where the last group of terms is a further complementary function, γ_s being given by:

in the P_3 approximation, $\gamma_1 = \sqrt{7}$;
in the P_5 approximation, γ_1 and γ_2 are the positive roots of
$$1 - \tfrac{5}{11}\gamma^2 + \tfrac{1}{33}\gamma^4 = 0, \quad (12.29)$$

etc., and $U^2\Theta_s(\mathbf{R})$ is a quadratic polynomial in $U\Omega_x$, $U\Omega_y$, $U\Omega_z$, whose coefficients satisfy (12.22′) and are related by

$$\nabla_U^2[U^2\Theta_s(\mathbf{R})] = \mathrm{div}_U\, \mathbf{grad}_r[U^2\Theta_s(\mathbf{R})] = 0. \quad (12.30)$$

It is to be noted that the entire particular integral (that is, the terms in both Φ_j and $\mathbf{X}_i$) satisfies (12.12) with $n = 1$, automatically. In general, the supplementary equation (12.12) for $n = n'-1$ restricts only the complementary function introduced in $\Psi_{n'}$, and not those introduced already in the lower moments.

The values of Ψ_3, Ψ_4 and so on can be found similarly. The results obtained show, incidentally, that in any finite order of approximation the angular distribution is always expressed in terms of the solutions of (12.22′).

12.2.3. *The complementary functions*

The appearance of complementary functions in the solutions of the equations for Ψ_1, Ψ_2, etc., shows that in general there can be different angular distributions of neutrons that have the same flux distribution, that have the same flux and current distribution, and so on. This is easily understandable; for instance, let the half-space $x > 0$ be exposed to $q(\Omega)\,d\Omega$ neutrons in the solid angle $d\Omega$ about Ω falling on $x = 0$ from outside per unit time per unit area. Let $q(\Omega) = q(\mu, \theta)$, where $\mu = \Omega_x$ and $\theta = \tan^{-1}(\Omega_z/\Omega_y)$. The neutron flux $\psi_0(x)$ in $x > 0$ is determined by the values of the integral

$$\int_0^{2\pi} q(\mu, \theta)\, d\theta$$

as a function of μ, and does not depend on the way in which θ enters $q(\mu, \theta)$; but the lateral components of the current and higher moments will not have this property.

If the scattering is isotropic (and this is the only case we have examined so far), the possible difference between the angular distributions for the same flux should decrease not slower than $\exp(-d/l)$, where d is the distance from the nearest boundary. This in fact follows from the above formulae, since the relaxation lengths of the terms appearing in the complementary functions are l/β_t, l/γ_s, etc. (see (12.27), (12.28)), and these are less than l, since all the β_t, γ_s, etc., are greater than unity.

The relation (12.26) shows that the number of independent components of each $\mathbf{X}_t$ is one less than the number consistent with the symmetry of the problem considered. That is, the number of different solutions of (12.22′) appearing in each $\mathbf{X}_t$ cannot exceed two, while, if the direction of the neutron current is given by symmetry (as in the cases of plane and spherical symmetry), the complementary function disappears altogether, and the current is then uniquely determined by the flux.

Similarly, the relation (12.30) shows that the number of independent components of each $\mathbf{\Theta}_s$ is the difference between the number of components of the second moment consistent with the symmetry of the problem, and the number of components of the current. That is, the number of independent components of each $\mathbf{\Theta}_s(\mathbf{R})$ cannot exceed two, while, if the number of components of the second moment is, by symmetry, equal to that of the current, the second moment is uniquely defined by the flux and current. Similar results hold for the higher moments.

12.2.4. *The boundary conditions*

In considering the boundary conditions under which the foregoing equations are to be solved, we shall restrict ourselves to the two most frequent types of boundary, namely, an interface between two media in contact, and a free surface; further, in the latter case we consider only boundary conditions of Marshak's type.

We first find the number of conditions that can be satisfied. Let p_n be the largest number of linearly independent spherical harmonic moments of order n that is compatible with the symmetry of the geometry considered. We have

$$1 \leqslant p_n \leqslant 2n+1 \quad \text{and} \quad p_n \leqslant p_{n+1}. \tag{12.31}$$

On examining the elimination process described in § 12.1.4, we see that Ψ_{N-1} and Ψ_{N-2} satisfy second-order differential equations, Ψ_{N-3} and Ψ_{N-4} fourth-order equations, and so on. Thus $\psi_0(\mathbf{r})$ satisfies an equation of order $N+1$, and since it is a scalar (i.e. has $p_0 = 1$ component), it will contain $p_0 \cdot \frac{1}{2}(N+1)$ solutions of (12.22′). The quantities Ψ_1 and Ψ_2

satisfy equations of order $N-1$. The complementary function introduced in Ψ_1 has p_1 components, but from § 12.2.3 only $p_1-1 = p_1-p_0$ of these can be chosen independently. The complementary function introduced in Ψ_2 has p_2 components, but again only p_2-p_1 of these can be chosen independently. Thus the complementary functions introduced in Ψ_1 and Ψ_2 contain between them only $(p_2-p_0).\frac{1}{2}(N-1)$ independent solutions of (12.22′). Similarly, those introduced in Ψ_3 and Ψ_4 contain between them only $(p_4-p_3).\frac{1}{2}(N-3)$ independent solutions of that equation, and so on. The total number of solutions of (12.22′) per medium which are at our disposal is therefore

$$p_0.\frac{N+1}{2}+(p_2-p_0)\frac{N-1}{2}+(p_4-p_2)\frac{N-3}{2}+\ldots = p_0+p_2+\ldots+p_{N-1}.$$

Every solution of (12.22′), however, corresponds to one condition satisfied all over the surface of the medium in question. Thus the number of conditions that can be satisfied is

$$\left.\begin{aligned} p_0+p_2+\ldots+p_{N-1} &\text{ at a free surface,}\\ 2(p_0+p_2+\ldots+p_{N-1}) &\text{ at an interface.}\end{aligned}\right\} \tag{12.32}$$

Considering now the boundary conditions themselves, we first take the case where the symmetry is such that $p_1 = p_0$, $p_3 = p_2$, etc., and so the total number of even-order moments that enter the solution is equal to the number of odd-order moments:

$$p_0+p_2+\ldots+p_{N-1} = p_1+p_3+\ldots+p_N. \tag{12.33}$$

In this case, the total number of moments entering the solution in each medium is equal to the number of conditions that can be satisfied at an interface. However, at an interface between two media in contact, the exact values of the moments should be continuous. At such an interface, therefore, provided that (12.33) holds, we are led to the conditions

$$\psi_{n,m}(\mathbf{r}) \textit{ is continuous for all } m \textit{ and each } n \leqslant N; \tag{12.34}$$

this is a direct generalization of (10.30).

Similarly, for a free surface, when (12.33) holds, the arguments leading to (10.53) can be repeated to give the Marshak-type boundary conditions

$$\iint \psi(\mathbf{r},\Omega)P_{n,m}(\Omega)\,d\Omega = 0 \textit{ for } \Omega \textit{ inward, } \mathbf{r} \textit{ on the free surface,}$$
$$\textit{all } m, \textit{ and } n \textit{ odd and } \leqslant N. \tag{12.35}$$

If (12.33) is not satisfied, (12.34) contains

$$p_1+p_3+\ldots+p_N-(p_0+p_2+\ldots+p_{N-1}) \tag{12.36}$$

conditions too many, and so does (12.35). We must therefore select from the $\sum p_n$ conditions (12.34) the $2(p_0+p_2+\ldots+p_{N-1})$ most relevant ones, and satisfy only these, disregarding the remaining ones; the conditions (12.35) must be similarly reduced. It might seem simpler, in the case of (12.35), to replace the odd-order harmonics in the integrand by even-order harmonics, which makes the number of conditions right; we have seen, however, in particular cases where (12.33) holds (cf. § 10.3.6), that this leads to a deterioration of the convergence, and the same will probably be true of the general case.

No detailed investigation has yet been made to find what choice of conditions from (12.34) leads to the best convergence; most applications of the spherical harmonics method have been to cases where (12.33) was satisfied. The following considerations may, however, be noted.

The inequalities (12.31) show that the difference (12.36) can never exceed p_N-1. It should therefore always be possible to satisfy the continuity of $\psi_{n,m}(\mathbf{r})$ at an interface for all m and each $n \leqslant N-1$; moreover, it seems better, on intuitive grounds, to neglect some of the continuity conditions on the Nth order moment and not those on moments of any lower order. Having satisfied (12.34) for all moments of order not exceeding $N-1$, it is still possible to satisfy it for some of order N. These are selected as follows. Let μ be the component of Ω normal to the interface, $\tan\theta$ the ratio of the two tangential components, and

$$\psi_{N,m}(\mathbf{r}) = \iint \psi(\mathbf{r},\Omega) P_N^m(\mu)[\cos m\theta + \sin m\theta]\, d\Omega, \qquad (12.37)$$

where P_N^m are the associated Legendre functions. In this notation $\psi_{N,N}(\mathbf{r})$ and $\psi_{N,-N}(\mathbf{r})$ may be described as the 'entirely tangential' moments of order N and $\psi_{N,0}(\mathbf{r})$ as the 'entirely normal' one, while in general $\psi_{N,m}(\mathbf{r})$ can be described as 'predominantly normal' if $|m|$ is small and as 'predominantly tangential' otherwise. It is seen from (12.12) that the entirely tangential moments of order N in the P_N approximation are given by direct differentiation of the lower-order moments with respect to tangential coordinates only. If the mean free paths in the two media are the same, the continuity of $\psi_{N,N}(\mathbf{r})$ and $\psi_{N,-N}(\mathbf{r})$ will therefore follow from that of the lower-order moments; if the mean free paths are different, the continuity of $\psi_{N,N}(\mathbf{r})$ and $\psi_{N,-N}(\mathbf{r})$ is incompatible with that of the lower-order moments. No condition, therefore, can be placed on the tangential moments. Further, no symmetry of the system can cause the entirely normal moment to vanish. These suggest that the predominantly normal moments should be regarded as more relevant than the predominantly tangential ones.

It is therefore possible to give the following boundary conditions for use at the interface of two media in contact:

$\psi_{n,m}(\mathbf{r})$ is continuous for each $n \leqslant N-1$ and all m belonging to them, and for $n = N$ and as many m as possible starting from the smallest $|m|$. (12.34′)

A similar selection would be made for a free surface from the conditions (12.35).

12.3. Cylindrical geometries

12.3.1. *The choice of representation*

We now illustrate the discussion of §§ 12.1 and 12.2 by applying them to the case of a finite cylinder, with symmetry about the axis of the cylinder.† Let the coordinate z be measured along the axis, r perpendicular to it and θ the azimuthal angle. The vector $\mathbf{U} = U\boldsymbol{\Omega}$ is specified in the same manner by its components U_z and U_r and its azimuthal angle θ'. In this system of coordinates

$$\left.\begin{aligned} \mathbf{U}\cdot\mathbf{grad}_r &= U_r\left[\cos(\theta'-\theta)\frac{\partial}{\partial r}+\sin(\theta'-\theta)\frac{1}{r}\frac{\partial}{\partial\theta}\right]+U_z\frac{\partial}{\partial z} \\ \text{and}\quad \mathrm{div}_{\mathbf{U}}\,\mathbf{grad}_r &= \cos(\theta'-\theta)\left[\frac{\partial^2}{\partial r\partial U_r}+\frac{1}{rU_r}\frac{\partial^2}{\partial\theta\partial\theta'}\right]+ \\ &\quad +\sin(\theta'-\theta)\left[\frac{1}{r}\frac{\partial^2}{\partial U_r\partial\theta}-\frac{1}{U_r}\frac{\partial^2}{\partial r\partial\theta'}\right]+\frac{\partial^2}{\partial z\partial U_z}, \end{aligned}\right\} \quad (12.38)$$

while ∇_r^2 and $\nabla_{\mathbf{U}}^2$ have their familiar forms (cf. (12.2)).

With the symmetry postulated above, the angular distribution can involve θ and θ' only as their difference, and must be an even function of that difference. The same must be true also of the quantities Ψ_n' introduced in (12.4). If we expand Ψ_n' as a Fourier series in $\theta'-\theta$, the series must terminate because Ψ_n' is a homogeneous polynomial of order n in $U_r\cos\theta'$, $U_r\sin\theta'$ and U_z. We therefore obtain

$$\Psi_n' = \sum_{m=0}^{n}\cos m(\theta'-\theta)A_{n,m}(r, z, U_r, U_z), \quad (12.39)$$

where $A_{n,m}$ is a homogeneous polynomial of order n in U_r and U_z. This, together with the equation $\nabla_{\mathbf{U}}^2\Psi_n' = 0$ (see (12.6)), determines the dependence of $A_{n,m}$ on U_r and U_z and it is thus known apart from a factor

† By this we mean that the system is invariant with respect to rotation about the axis and/or reflection in a plane through the axis.

which may depend on r and z. Thus, (12.39) can be rewritten

$$\Psi_n = \sum_{m=0}^{n} \cos m(\theta'-\theta) B_{n,m}(U_r, U_z)\psi_{n,m}(r,z), \tag{12.40}$$

where the $B_{n,m}$ are easily found to be

$$B_{n,m}(U_r, U_z) = U_z^{n-m} U_r^m \sum_{q=0} \frac{m!\,(n-m)!}{(m+q)!\,(n-m-2q)!}\left(-\frac{1}{4}\frac{U_r}{U_z}\right)^{2q}, \tag{12.41}$$

while the $\psi_{n,m}$ ($m = 0, 1, \ldots, n$) are, apart from a normalization factor, the spherical harmonic moments of order n.

12.3.2. *The expressions for the moments*

We now determine the expressions for the spherical harmonic moments defined by (12.40). In the problem under consideration, $\psi(\mathbf{r}, \Omega)$ can always be expanded as a Fourier series in z and each term of this expansion determined separately. That is, it is sufficient to examine the case where $\psi(\mathbf{r}, \Omega)$ is a periodic function of z with period $2\pi l/\omega$, say. The simplest way to do this is to put

$$\psi_{n,m}(r,z) = \phi_{n,m}(r)\exp(i\omega z/l), \tag{12.42}$$

and to separate the real and imaginary parts afterwards. We shall also assume for simplicity that the central core of the system is being considered, so that all solutions should be regular for $r = 0$.

The flux is obtained as follows. Substituting the expression (12.42) for $\psi_0(r,z)$ into (12.22), solving the resulting equation for $\phi_0(r)$, and using (12.21), we obtain

$$\psi_0(r,z) = \sum_{j=1}^{\frac{1}{2}(N+1)} A_j I_0((\nu_j^2+\omega^2)^{\frac{1}{2}} r/l)\exp(i\omega z/l), \tag{12.43}$$

where the A_j are constants of integration.

We next consider the current. The contribution to the complementary function for each β (given by (12.25)) is determined separately and denoted by an appropriate suffix. From (12.40), (12.41), and (12.42) we have

$$\Psi_{1,\beta} = [U_z\phi_{1,0,\beta}(r) + U_r\cos(\theta'-\theta)\phi_{1,1,\beta}(r)]\exp(i\omega z/l).$$

Substituting this into the appropriate equation (12.22), it follows that $\phi_{1,0,\beta}(r)$ and $\phi_{1,1,\beta}(r)$ satisfy

$$\left[\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{m^2}{r^2}-\frac{\beta^2+\omega^2}{l^2}\right]\phi_{1,m,\beta} = 0, \tag{12.44}$$

while (12.26) gives

$$\frac{i\omega}{l}\phi_{1,0,\beta}(r)+\left(\frac{d}{dr}+\frac{1}{r}\right)\phi_{1,1,\beta}(r) = 0.$$

These equations determine $\phi_{1,0,\beta}(r)$ and $\phi_{1,1,\beta}(r)$, since the solution is assumed regular at $r = 0$. The particular solution, corresponding to (12.43), of the equation for Ψ_1 is given directly by combining the formula (12.27) with the expression (12.38) for $\mathbf{U} \cdot \mathbf{grad}_r$ in the present system of coordinates. The result is

$$\left.\begin{aligned}\psi_{1,0}(r,z) &= ie^{i\omega z/l}\Big[\sum_{j=1}^{\frac{1}{2}(N+1)} A_j G_1(\nu_j)\frac{\omega}{\nu_j} I_0((\nu_j^2+\omega^2)^{\frac{1}{2}}r/l)+ \\ &\qquad +\sum_{j'=1}^{\frac{1}{2}(N-1)} B_{j'}(\beta_{j'}^2+\omega^2)^{\frac{1}{2}} I_0((\beta_{j'}^2+\omega^2)^{\frac{1}{2}}r/l)\Big] \\ \psi_{1,1}(r,z) &= e^{i\omega z/l}\Big[\sum_{j=1}^{\frac{1}{2}(N+1)} A_j G_1(\nu_j)\frac{(\nu_j^2+\omega^2)^{\frac{1}{2}}}{\nu_j} I_1((\nu_j^2+\omega^2)^{\frac{1}{2}}r/l)+ \\ &\qquad +\sum_{j'=1}^{\frac{1}{2}(N-1)} B_{j'}\omega I_1((\beta_{j'}^2+\omega^2)^{\frac{1}{2}}r/l)\Big]\end{aligned}\right\}, \quad (12.45)$$

where the $B_{j'}$ are further constants of integration, and $\psi_{1,0}$ and $\psi_{1,1}$ are defined according to (12.40) and (12.5), that is, so that the complete angular distribution is given by

$$\begin{aligned}\psi(\mathbf{r},\Omega) = \frac{1}{4\pi}\{&\psi_0(r,z)+3[\Omega_z\psi_{1,0}(r,z)+\Omega_r\cos(\theta'-\theta)\psi_{1,1}(r,z)]+ \\ &+5[(\Omega_z^2-\tfrac{1}{2}\Omega_r^2)\psi_{2,0}(r,z)+\Omega_z\Omega_r\cos(\theta'-\theta)\psi_{2,1}(r,z)+ \\ &+\Omega_r^2\cos 2(\theta'-\theta)\psi_{2,2}(r,z)]+\ldots\}. \quad (12.46)\end{aligned}$$

The second-order moments are determined in the same manner, and so on.

In the case of a cylinder of infinite length, the system has two further symmetry properties, in addition to those stated previously: it is invariant with respect to reflection in the plane $z = 0$ and translation parallel to the z-axis. The current therefore has no component $\psi_{1,0}$ parallel to the z-axis and, as before, no component round the axis; its direction is therefore completely prescribed. It follows from the discussion of § 12.2.3 that the complementary function introduced in Ψ_1 must vanish. This can also be seen from (12.45) by putting $\omega = 0$; if any $B_{j'}$ is not zero, the result is a solution for which $\psi_{1,0}(r,z)$ does not vanish, and which therefore does not possess the required symmetry.

12.4. Yvon's modification of the spherical harmonics method

We finally discuss a possible modification of the spherical harmonics method, suggested by Yvon (unpublished, but quoted by Kourganoff (27, p. 101)). We shall take the case of plane symmetry and use the notation of Chapter X.

Yvon's suggestion is based on the following considerations. At free surfaces and interfaces, the angular distribution as a function of the direction cosine has a discontinuity at $\mu = 0$. For points inside the media, although $\psi(x, \mu)$ is continuous, it is given by different analytical functions of μ for $\mu > 0$ and $\mu < 0$. However, for any x and either interval $-1 < \mu < 0$ or $0 < \mu < 1$, $\psi(x, \mu)$ will be a regular function of μ, at least in the open interval, and will be bounded and of bounded variation in the closed interval. These statements, of course, may not hold in the presence of highly anisotropic sources inside or outside the medium.

The polynomial approximations in an open interval to a function regular in this interval, and bounded and of bounded variation in the corresponding closed interval, will converge much better than the polynomial approximations to a discontinuous function. Accordingly, Yvon has suggested using, instead of the expansion of $\psi(x, \mu)$ in Legendre polynomials in the interval $(-1, 1)$, two different expansions: for $\mu > 0$ in terms of $P_n(2\mu - 1)$, and for $\mu < 0$ in terms of $P_n(2\mu + 1)$.

The general outline of the solution by this method (corresponding to § 10.1) is easily obtained, but the more detailed analysis (corresponding to §§ 10.2 to 10.6) has not yet been carried out.

The usefulness of Yvon's method may be estimated as follows. Though the exact $\psi(x, \mu)$ is given, for any x, by two different functions for $\mu > 0$ and $\mu < 0$, the difference between them becomes negligible at distances x from the boundaries which are large compared with the mean free path, and there is therefore a single analytical function, regular in $(-1, 1)$, whose difference from $\psi(x, \mu)$ is negligibly small for the given x. A function regular in $(-1, 1)$, however, can certainly be more accurately approximated by one polynomial of order N in the whole interval than by a polynomial of order $\frac{1}{2}(N-1)$ in each half of the interval. Thus, while Yvon's procedure should converge better near the interfaces than the usual method of spherical harmonics, the reverse is the case in the interior of the media. Whether the new method is profitable depends, therefore, on the aspect of the problem that is of interest. In some applications, for instance in astrophysics (where the emergent angular distribution is of interest, the equations of radiative transfer being the same as those of neutron transport), the values of $\psi(x, \mu)$ at the free surface are required, and the rate of convergence of the solution inside the medium is unimportant. In this case, Yvon's procedure will give good results. In neutron transport problems, however, it is usually the critical size (see § 1.4) which is required, and this will be governed

primarily by the conditions in the interior of the medium. In this case, therefore, the advantage of Yvon's procedure is lost.

Note added in proof. An account of Yvon's procedure has recently been published by Mertens (*Simon Stevin, Supplement*, **30**, 1954), who has also indicated how this procedure can be generalized to the case of anisotropic scattering. Mertens does not extend the method to geometries other than plane, however.

Note added in second impression. For the case of cylindrical symmetry, we have given in § 12.3.2 the explicit expressions for $\psi_{1,0}(r, z)$ and $\psi_{1,1}(r, z)$. In a recent paper (B. Davison, *Canadian J. Phys.* **35**, 576, 1957) the explicit expression for $\psi_{n,m}(r)$ for arbitrary n, m, and N has been derived for the particular case $\omega = 0$. In the same paper the method of dealing with multi-layer problems described, for the case of plane symmetry, in § 10.4 is extended to the case of cylindrical symmetry.

We have given in § 12.2.2 semi-explicit expressions for Ψ'_1 and Ψ'_2 for a general geometry. The corresponding expression for Ψ'_n, with n arbitrary, has also been derived recently (B. Davison, 'Spherical harmonics method for neutron-transport problems with incomplete symmetry', *Canadian J. Phys.* **36**, 1958). This latter paper considers also problems where the distribution of materials possesses spherical or cylindrical symmetry, but the distribution of sources and/or the conditions at infinity do not.

XIII

THE METHOD OF DISCRETE ORDINATES

13.1. General outline of the method

HAVING expounded the method of spherical harmonics, we now discuss other methods for the approximate solution of neutron transport problems. The first of these is Chandrasekhar's *method of discrete ordinates.*† This is not so powerful as the spherical harmonics method, which it resembles in some respects, and it has been developed so far only for the case of plane geometry.‡ A detailed exposition of the method is to be found in Chandrasekhar's book (9), and we therefore give only an outline of the method, together with some comments.

The method of discrete ordinates, like the spherical harmonics method, starts from the Boltzmann equation, which, in the case of the constant cross-section approximation, isotropic scattering, and no sources, in a system with plane symmetry, has the form

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \frac{\psi(x,\mu)}{l} = \frac{c}{2l} \int_{-1}^{1} \psi(x,\mu')\, d\mu'. \tag{10.1}$$

Instead of converting (10.1) into a system of equations in the moments of the angular distribution, however, we immediately approximate the integral on the right by a formula of numerical integration, of the type:

$$\int_{-1}^{1} f(\mu)\, d\mu = \sum_{j=1}^{N} a_j f(\mu_j), \tag{13.1}$$

where the a_j and μ_j are given by the integration formula used, and do not depend on the integrand.

The equation (10.1) need then be solved only for μ equal to each of the μ_j; that is, it is replaced by the system of differential equations

$$\mu_i \frac{d\psi(x,\mu_i)}{dx} + \frac{\psi(x,\mu_i)}{l} = \frac{c}{2l} \sum_{j=1}^{N} a_j \psi(x,\mu_j) \quad (i = 1, 2, \ldots, N). \tag{13.2}$$

The quantities $\psi(x,\mu_i)$ can be regarded as N functions of the variable x. The solution of (13.2) is clearly of the form

$$\psi(x,\mu_i) = \sum_{s} A_{s,i} \exp(\gamma_s x/l). \tag{13.3}$$

† First suggested by Wick (53), but the detailed development of the method is due to Chandrasekhar.

‡ The question of its extension to other geometries is dealt with in § 13.7.2.

The permissible values of γ_s are found by substituting a particular term of (13.3) into (13.2). This gives

$$A_{s,i} = \frac{c}{2(1+\mu_i\gamma_s)} \sum_{j=1}^{N} a_j A_{s,j}, \tag{13.4}$$

and this shows also that A_s, defined by

$$A_s = \sum_{j=1}^{N} a_j A_{s,j}, \tag{13.5}$$

cannot vanish unless every $A_{s,j}$ vanishes. Multiplying (13.4) by a_i, summing over i, and dividing by A_s, we have

$$1 = \frac{c}{2} \sum_{j=1}^{N} \frac{a_j}{1+\mu_j\gamma_s}, \tag{13.6}$$

and this defines the permissible values of γ_s. The equation (13.3) can now be written more explicitly

$$\psi(x, \mu_i) = \frac{c}{2} \sum_{s} \frac{A_s}{1+\mu_i\gamma_s} \exp(\gamma_s x/l). \tag{13.7}$$

13.2. The roots of the characteristic equation

It is seen at once that, if the order of approximation increases indefinitely, the equation (13.6) defining the permissible values of γ_s becomes

$$1 = \frac{c}{2} \int_{-1}^{1} \frac{d\mu}{1+\mu\gamma_s} = \frac{c}{2\gamma_s} \log \frac{1+\gamma_s}{1-\gamma_s}.$$

This implies, according to (5.5), that there are two solutions of (13.6), say γ_1 and γ_{-1}, which tend to $\pm l/L$ as the order of approximation increases:

$$\lim_{N\to\infty} \gamma_{\pm 1} = \pm l/L. \tag{13.8}$$

In order easily to derive further information from (13.6), we shall make the following assumptions about the summation formula (13.1):

(i) it is symmetrical about the midpoint of the interval, i.e.

$$\mu_j = -\mu_{N+1-j}, \qquad a_j = a_{N+1-j}; \tag{13.9}$$

(ii) it gives the exact value of the integral if the integrand is a polynomial of sufficiently low order, not exceeding $Z(N)$, say; this implies that, if n is a positive integer,

$$\sum_{j=1}^{N} a_j \mu_j^n = \begin{cases} 2/(n+1) & \text{for even } n \leqslant Z(N), \\ 0 & \text{for odd } n; \end{cases} \tag{13.10}$$

(iii) if the integrand is positive in $(-1, 1)$, the integral as given by (13.1) is also positive; this implies that

$$a_j > 0; \tag{13.11}$$

(iv) all the μ_j are distinct.

(The above four properties are possessed by most commonly-used summation formulae.)

(v) only approximations with an even number of μ_j are used (N is even); this implies, from (i) and (iv), that

$$\textit{no } \mu_j \textit{ vanishes.} \tag{13.12}$$

This condition (N even) may seem at first sight to be the opposite of the condition of odd N used in the spherical harmonics method, but we have seen that in the latter case the number of equations to be solved is even, and it is in this last respect that similarity should be expected.

According to (13.9), the right side of (13.6) is unaltered if γ_s is replaced by $-\gamma_s$; if one of these is a solution, the other is therefore a solution also. If the roots are suitably numbered, this can be expressed as

$$\gamma_{-s} = -\gamma_s. \tag{13.13}$$

The equation (13.6) can, in fact, be put in the more convenient form

$$1 = c \sum_{j=1+\frac{1}{2}N}^{N} \frac{a_j}{1-\mu_j^2\gamma_s^2} \tag{13.6'}$$

by combining the jth and $(N+1-j)$th terms.

Using (13.11), it can be shown, with the same argument as was given to derive (10.21), that just one root of (13.6) lies between any two consecutive $1/\mu_j$ of the same sign. This gives $N-2$ roots, and since all $|\mu_j|$ are less than unity, each of these roots must have a modulus greater than unity. If $c < 1$, it can be shown similarly that there are two more roots, one between zero and the smallest positive $1/\mu_j$, and the other between the largest negative $1/\mu_j$ and zero. For $c = 1$, these two roots become a double root at the origin, while for $c > 1$ they become two purely imaginary roots. These two roots are clearly identical with the $\gamma_{\pm 1}$ which appear in (13.8). The other roots, lying between consecutive $1/\mu_j$ of the same sign, can be called $\gamma_{\pm 2}$, $\gamma_{\pm 3}$, etc., and then

$$|\gamma_{\pm s}| > 1 \quad \text{for } s \geqslant 2. \tag{13.14}$$

The formulae (13.8) and (13.14) are closely similar to the formulae (10.27) and (10.28). This shows that the distribution of the exponents γ_s in the method of discrete ordinates is essentially the same as that of the ν_j in the spherical harmonics method. As before, the term in (13.7)

corresponding to $\gamma_{\pm 1}$ can be identified as the *asymptotic* part, and the remaining terms as the *transient* part. If $c = 1$, the asymptotic part becomes

$$A + B[x/l - \mu_i]. \tag{13.15}$$

The solution of (13.2) which we have obtained is the most general one, since we have found just N roots of (13.6).

13.3. The boundary conditions. Elimination

It is evident that the boundary conditions determining the constants A_s in (13.7) and the constants A and B in (13.15) are:

(i) at an interface of two media in contact,

$$\psi(x, \mu_j) \text{ is } \textit{continuous} \text{ for } j = 1, 2, \ldots, N; \tag{13.16}$$

(ii) at a free surface (say $x = 0$ with the medium in $x > 0$),

$$\psi(0, \mu_j) = 0 \text{ for all positive } \mu_j \text{ considered.} \tag{13.17}$$

The extension of (13.17) to the case of an exposed surface is also clear, provided that the method of discrete ordinates is still applicable. We shall discuss in §§ 13.6 and 13.7 the question of applicability.

If a purely absorbing layer is placed between two other media, it should be treated like any other medium. The values of γ_s for $c = 0$ (pure absorption) are equal to the $-1/\mu_j$ (see (13.6)).

If a medium extends from $x = x_0$ to $x = +\infty$, and there is no supply of neutrons from $x = +\infty$, then (13.7) for $x \geqslant x_0$ will contain only negative γ_s. Applying this to $x = x_0$, denoting the values of γ_s for the medium in $x_0 < x < \infty$ by $\gamma_{s,0}$, and reducing to a common denominator, we have

$$\psi(x_0, \mu_i) = \frac{\text{polynomial of order } \frac{1}{2}N - 1 \text{ in } \mu_i}{\prod_{\gamma_{s,0} < 0} (1 + \mu_i \gamma_{s,0})}. \tag{13.18}$$

If $c = 1$ for $x > x_0$, $\gamma_{1,0} = \gamma_{-1,0} = 0$, but (13.18), and (13.18′) below, are unchanged, as may be seen by proceeding to the limit $c \to 1$.

The condition (13.18) is clearly equivalent to imposing the correct boundary conditions at $x = +\infty$, and, if the distribution in $x > x_0$ is unimportant, that region need be considered no further. If there is a supply of neutrons from $x = +\infty$, (13.18) becomes

$$\psi(x_0, \mu_i) = \frac{\text{polynomial of order } \frac{1}{2}N \text{ in } \mu_i}{(1 + \mu_i \gamma_{1,0}) \prod_{\gamma_{s,0} < 0} (1 + \mu_i \gamma_{s,0})}, \tag{13.18′}$$

where $\gamma_{1,0}$ is the smallest non-negative $\gamma_{s,0}$.

If the entire system consists of a homogeneous half-space $x > 0$

with a free surface at $x = 0$, then on combining (13.18′) and (13.17) we have

$$\psi(0, \mu_i) = A \prod_{\mu_j > 0} (\mu_i - \mu_j) \Big/ (1 + \mu_i \gamma_1) \prod_{\gamma_s < 0} (1 + \mu_i \gamma_s), \qquad (13.19)$$

where A is a normalization constant, and the suffix 0 is now omitted, as there is only one medium and one set of γ_s.

The expression (13.19) contains no arbitrary constant other than A, and the shape of the angular distribution emerging from a half-space is therefore given directly in this approximation by (13.19). In the case of isotropic scattering, this distribution is known exactly in the form of a fairly simple integral (see (6.17) and (6.9)), and the result (13.19) is therefore not very important; its extension to the case of anisotropic scattering (Chapter XVII) is, however, of interest.

The simplification afforded by formulae (13.18), (13.18′), and (13.19) may be regarded as a counterpart of the procedure described in § 10.4 in connexion with the spherical harmonics method. It may be mentioned that a slight modification of that procedure could likewise be utilized to eliminate the outermost medium (if that medium extends to infinity), although in the spherical harmonics method the elimination of the intermediate media is more advantageous.

13.4. The evaluation of the spherical harmonic moments

In the method of discrete ordinates, the definite integrals in the spherical harmonic moments

$$\psi_n(x) = \int\int \psi(x, \mu) P_n(\mu)\, d\Omega = 2\pi \int_{-1}^{1} \psi(x, \mu) P_n(\mu)\, d\mu \qquad (10.3)$$

are evaluated by the approximate formula (13.1), that is, they are defined by

$$\bar{\psi}_n(x) = 2\pi \sum_{j=1}^{N} a_j \psi(x, \mu_j) P_n(\mu_j). \qquad (13.20)$$

Multiplying (13.2) by $a_i P_n(\mu_i)$, summing over i, and using the recurrence relation between the Legendre polynomials, we obtain

$$(n+1)\frac{d\bar{\psi}_{n+1}(x)}{dx} + n\frac{d\bar{\psi}_{n-1}(x)}{dx} + [2n+1-c\delta_{0n}]\frac{\bar{\psi}_n(x)}{l} = 0. \qquad (13.21)$$

The quantities $\bar{\psi}_n(x)$, defined by (13.20) with $\psi(x, \mu_j)$ the solutions of (13.2), therefore satisfy the same system of equations as the exact spherical harmonic moments (see (10.4)). (They do not, however, satisfy the undistorted system (10.4), since they do not satisfy the condition that $\sum_{n=0} (2n+1)\bar{\psi}_n(x) P_n(\mu)$ converges.)

In particular, for $c = 1$, the first of (13.21) gives

$$[\bar{\psi}_1(x)]_{c=1} = \text{constant}. \tag{13.22}$$

In the derivation of (13.21) and (13.22), no use is made of the assumption (13.10) regarding the summation formula (13.1). Chandrasekhar, however, derives (13.21) and (13.22) from (13.10) and (13.6) by means of a lengthy process.

13.5. The solution for general μ. The iterated angular distribution

We now determine $\psi(x, \mu)$ for values of μ in the interval $(-1, 1)$ which are not among the μ_j involved in (13.1). It might be thought that the formula (13.7) would be applicable for any such μ. We have seen in § 13.2, however, that each γ_s (except $\gamma_{\pm 1}$) lies between two adjacent values of $\pm 1/\mu_j$; the μ_j all lie in $(-1, 1)$, and therefore each $(1+\mu\gamma_s)^{-1}$ for $|s| \geqslant 2$ has a pole in $(-1, 1)$. Since $\psi(x, \mu)$ can have no poles in this interval, the expression

$$\frac{c}{2} \sum_s \frac{A_s}{1+\mu\gamma_s} \exp(\gamma_s x/l) \tag{13.23}$$

cannot be an acceptable approximation to $\psi(x, \mu)$ in the whole interval.

This difficulty is most consistently overcome as follows. We use (13.7) and (13.20) to determine $\bar{\psi}_0(x)$, and replace $\psi_0(x)$ on the right of (10.1) by this $\bar{\psi}_0(x)$; the resulting equation for $\psi(x, \mu)$ is then solved under the exact boundary conditions. Using (13.6), we see that $\bar{\psi}_0(x)$ defined as above is

$$\bar{\psi}_0(x) = 2\pi \sum A_s \exp(\gamma_s x/l), \tag{13.24}$$

and therefore the solution of (10.1) in this approximation is

$$\psi(x, \mu) = F(\mu)\exp(-x/\mu l) + \frac{c}{2} \sum_s \frac{A_s}{1+\mu\gamma_s} \exp(\gamma_s x/l), \tag{13.25}$$

where $F(\mu)$ is determined by the boundary conditions.

Since, in solving (13.2), the boundary conditions are satisfied for $\mu =$ each μ_j, $F(\mu)$ should vanish for $\mu =$ each μ_j, in order that (13.25) should then reduce to (13.7). For $\mu \to -1/\gamma_s$, the boundary conditions give

$$F(\mu) = -\frac{c}{2} \frac{A_s}{1+\mu\gamma_s} [1 + O(1+\mu\gamma_s)],$$

in order that the infinite parts of the poles of $F(\mu)\exp(-x/\mu l)$ should cancel those of (13.23), making the resulting expression (13.25) everywhere finite.

Since (13.25) was derived by substituting a preliminary approximation

(13.24) into the Boltzmann equation and solving it again, (13.25) may be regarded as an *iterated angular distribution*. In Chapter XVI we shall give a general discussion of the use of iterative processes in neutron transport theory.

In the spherical harmonics method it is also possible to take the expression for $\psi_0(x)$ and substitute it into (10.1), integrating the resulting equation and so obtaining the iterated angular distribution in a form similar to (13.25). However, whereas this procedure is compulsory in the method of discrete ordinates in order to find $\psi(x, \mu)$ for all μ, in the spherical harmonics method this is optional, and the simpler formula (10.2) is often adequate.

13.6. The choice of the summation formula

Hitherto we have specified the general conditions (i)–(v) (§ 13.2) on the summation formula used, but we have not discussed it more closely. This we now do. Chandrasekhar himself used the Gauss formula, on the grounds that, when the integrand is regular in the (closed) interval of integration, this formula gives the most accurate result for a given number of points μ_j. However, we have seen in § 12.4 that $\psi(x, \mu)$ is not in general regular in the closed interval $(-1, 1)$. The chief advantage of Gauss's formula is therefore lost, and other writers have suggested alternative formulae.

Sykes (45) has proposed the application of Gauss's formula to each of the intervals $(-1, 0)$ and $(0, 1)$ separately.† An argument similar to that of § 12.4, with the same restrictions (absence of exposed surfaces, etc.), shows that Sykes's formula (which he calls the *double-Gauss*) is preferable in the determination of the emergent angular distribution, while the Gauss formula should be more reliable for the determination of the critical size.

If the emergent angular distribution is required in the case of a strongly anisotropic irradiation of the free surface, so that the assumptions of § 12.4 are not valid, then there is no reason to prefer either the Gauss or the double-Gauss formula. The question of the best summation formula in these conditions has been examined by Bückner (7, pp. 111 ff.), and he advocates the use of equal-interval formulae. He also remarks that, if the mth derivative of the integrand is discontinuous, there is no advantage in using a formula which is exact for polynomials of order higher than $m-1$.

† This method is in fact equivalent to that of Yvon (§ 12.4), at least for the case of isotropic scattering (Sykes, unpublished).

13.7. Comparison of the spherical harmonics and discrete ordinates methods

13.7.1 *Plane geometries*

We shall now attempt to assess the merits of these two methods from the point of view of practical calculations. It is evident that the formal analysis, such as the derivation of the properties of the roots of the characteristic equation, is much simpler in the discrete ordinates method; this, however, is irrelevant to its convenience in practice and to its rapidity of convergence.

The close parallelism of the methods shows that for plane geometries there is not likely to be much difference in their convenience of application, though this may not be the case in certain particular problems. The available results of theory and practice show that this is true also of the relative rate of convergence and the accuracy of low-order approximations, and that the important choice is not between the spherical harmonics method and the discrete ordinates method, but between the single-interval treatment (the conventional forms of the two methods) and the double-interval treatment (Yvon's modification and the double-Gauss formula). We have seen that the decision between these must rest on the quantity of prime interest in the calculation; the former is more suitable for critical-size calculations, the latter for finding the emergent distribution.

13.7.2. *Other geometries*

The great disadvantage of the method of discrete ordinates is the difficulty of extending it to other geometries. For instance, in a spherically symmetric system, the Boltzmann equation has the form

$$\mu \frac{\partial \psi(r,\mu)}{\partial r} + \frac{1-\mu^2}{r} \frac{\partial \psi(r,\mu)}{\partial \mu} + \frac{\psi(r,\mu)}{l} = \frac{c}{2l} \int_{-1}^{1} \psi(r,\mu')\, d\mu'. \quad (11.4)$$

Since numerical differentiation is rather inaccurate even for continuous functions, it is not possible to take $\psi(r,\mu)$ for certain μ_j as the set of unknown functions. If, on the other hand, we take $\partial\psi(r,\mu)/\partial\mu$ for certain μ_j as the set of unknown functions, and determine $\psi(r,\mu_j)$ and $\int \psi(r,\mu')\, d\mu'$ by numerical integration, then for large r/l the basic functions will enter the equations with very small coefficients, and this is unlikely to give high accuracy. Further, $\psi(r,\mu_j)$ will have to be calculated by means of formulae for integrals with variable limits, and these are less accurate than those with fixed limits.

Although it may not be correct to say that the discrete ordinates method cannot be extended beyond the plane case, it seems very unlikely that any such extension can be superior to the spherical harmonics method. Chandrasekhar (9, p. 364) has, in fact, suggested an adaptation of his method for dealing with spherical systems, but he admits that this adaptation is wholly equivalent to the spherical harmonics method. We therefore need not discuss Chandrasekhar's adaptation.

XIV

THE PERTURBATION METHOD

14.1. First-order perturbations in critical-size problems

We now consider some methods of solving the transport equation which have a relatively narrow range of application, though in this range they may be extremely useful. The first of these is the perturbation method: it is concerned with the effects of small changes in the system. In other words, in applying this method two systems are considered, a simple *unperturbed* system for which the solution may be found by other methods, and the actual *perturbed* system, which differs from the former one only by a slight alteration of the conditions, or by an alteration of the conditions only in a small volume. We shall discuss the perturbation method only in relation to the critical-size problem; its extension to other problems is not difficult.

It is more convenient, in using the perturbation method, to start from the integral equation of transport, which in the constant cross-section approximation with isotropic scattering takes the form (4.15′). Omitting the source term (since we are considering a critical-size problem) and introducing a variable parameter as in (3.1), we can rewrite (4.15′) as

$$\rho(\mathbf{r}) = \frac{\gamma}{4\pi}\int\int\int \frac{c(\mathbf{r}')}{l(\mathbf{r}')}\rho(\mathbf{r}')e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{14.1}$$

where $\tau(\mathbf{r},\mathbf{r}')$ is the optical distance between $\mathbf{r}$ and $\mathbf{r}'$ (see (2.39)). If the system is critical, the lowest eigenvalue γ_0 must be unity (see § 3.1). The problem is usually stated in the following terms: if a prescribed small change is made in some part of an initially critical system, what change of a prescribed type will make the system again critical?

For instance, a piece of absorbing material may be introduced into a system; it is required to find the change in the core radius needed to keep the system critical. In a system in which only one of these changes is made, the lowest eigenvalue is no longer unity, but $1+\delta_1\gamma_0$, say, while if only the other change is made, the lowest eigenvalue is $1+\delta_2\gamma_0$, say. The condition that both together leave the system critical is then

$$\delta_1\gamma_0+\delta_2\gamma_0 = 0.$$

It is assumed that the alterations in the system are so small that the effect of each is independent of the presence or absence of the other. It is therefore possible to neglect the stipulation of criticality and the

lowest eigenvalue being unity, and simply to demand that the changes in this eigenvalue produced by the two changes in the system are equal and opposite.

14.2. The variation

The method just described is most easily applied by varying the equation (14.1). We first recall some properties of this equation, which were discussed in § 4.7. Since in (14.1) $c(\mathbf{r})$ was not divided as in (3.22), but was multiplied throughout by the variable parameter γ, the adjoint integral equation (4.28) is now simply

$$\rho^\dagger(\mathbf{r}) = \frac{\gamma}{4\pi}\frac{c(\mathbf{r})}{l(\mathbf{r})}\iiint \rho^\dagger(\mathbf{r}')e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{14.2}$$

while (4.29) reduces to

$$\rho^\dagger(\mathbf{r}) = c(\mathbf{r})\rho(\mathbf{r})/l(\mathbf{r}). \tag{14.3}$$

Varying (14.1), we have

$$\begin{aligned}\delta\rho(\mathbf{r}) = \frac{\delta\gamma}{4\pi}\iiint \frac{c(\mathbf{r}')}{l(\mathbf{r}')}\rho(\mathbf{r}')e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + \\ + \frac{\gamma}{4\pi}\iiint \frac{c(\mathbf{r}')}{l(\mathbf{r}')}\delta\rho(\mathbf{r}')e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} + \\ + \frac{\gamma}{4\pi}\iiint \rho(\mathbf{r}')\delta\left[\frac{c(\mathbf{r}')}{l(\mathbf{r}')}e^{-\tau(\mathbf{r},\mathbf{r}')}\right]\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}.\end{aligned} \tag{14.4}$$

When this equation is multiplied by $\rho^\dagger(\mathbf{r})$ and integrated over all $\mathbf{r}$, (14.2) shows that the second term on the right is equal to the left side. Simplifying the first term on the right by (14.1), we then obtain

$$\begin{aligned}&\frac{\delta\gamma}{\gamma}\iiint \rho(\mathbf{r})\rho^\dagger(\mathbf{r})\,dV \\ &\quad = -\frac{\gamma}{4\pi}\iiint \rho^\dagger(\mathbf{r})\,dV \iiint \rho(\mathbf{r}')\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\delta\left[\frac{c(\mathbf{r}')}{l(\mathbf{r}')}e^{-\tau(\mathbf{r},\mathbf{r}')}\right],\end{aligned} \tag{14.5}$$

or, using (14.3),

$$\begin{aligned}&\frac{\delta\gamma}{\gamma}\iiint \rho^2(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV \\ &\quad = -\frac{\gamma}{4\pi}\iiint \rho(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV \iiint \rho(\mathbf{r}')e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\delta\left[\frac{c(\mathbf{r}')}{l(\mathbf{r}')}\right] + \\ &\quad + \frac{\gamma}{4\pi}\iiint \rho(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV \iiint \rho(\mathbf{r}')\frac{c(\mathbf{r}')}{l(\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}e^{-\tau(\mathbf{r},\mathbf{r}')}\int_0^{|\mathbf{r}-\mathbf{r}'|} ds\,\delta\left[\frac{1}{l(\mathbf{r}'')}\right],\end{aligned} \tag{14.6}$$

where

$$\mathbf{r}'' = \mathbf{r} + s(\mathbf{r}'-\mathbf{r})/|\mathbf{r}'-\mathbf{r}|. \tag{14.7}$$

14.3. The transformation of the integrals

The first integral on the right of (14.6) is easily simplified. By interchanging $\mathbf{r}$ and $\mathbf{r}'$ and the order of integration and using (14.1), it equals

$$-\iiint \rho^2(\mathbf{r})\delta[c(\mathbf{r})/l(\mathbf{r})]\,dV. \tag{14.8}$$

The second term on the right of (14.6) is transformed as follows. Instead of specifying the positions of three collinear points $\mathbf{r}$, $\mathbf{r}''$, and $\mathbf{r}'$ by the six coordinates of $\mathbf{r}$ and $\mathbf{r}'$ and the distance between $\mathbf{r}$ and $\mathbf{r}''$, we can do so by the three coordinates of $\mathbf{r}''$, the direction of $\mathbf{r}''-\mathbf{r}$, and the distances of $\mathbf{r}$ and $\mathbf{r}'$ from $\mathbf{r}''$. In Cartesian coordinates, this amounts to the following change of variables:

$$\begin{aligned} x &= x''+s\mu & x' &= x''-s'\mu \\ y &= y''+s\sqrt{(1-\mu^2)}\cos\theta & y' &= y''-s'\sqrt{(1-\mu^2)}\cos\theta \\ z &= z''+s\sqrt{(1-\mu^2)}\sin\theta & z' &= z''-s'\sqrt{(1-\mu^2)}\sin\theta \end{aligned}$$

where μ and θ are the angular coordinates of the direction of $\mathbf{r}''-\mathbf{r}$. The Jacobian of this transformation is

$$\frac{D(x,y,z,s,x',y',z')}{D(x'',y'',z'',s,s',\mu,\theta)}$$

$$= \pm \begin{vmatrix} 1 & 0 & 0 & \mu & 0 & s & 0 \\ 0 & 1 & 0 & \sqrt{(1-\mu^2)}\cos\theta & 0 & \dfrac{-s\mu\cos\theta}{\sqrt{(1-\mu^2)}} & -(s\sin\theta)\sqrt{(1-\mu^2)} \\ 0 & 0 & 1 & \sqrt{(1-\mu^2)}\sin\theta & 0 & \dfrac{-s\mu\sin\theta}{\sqrt{(1-\mu^2)}} & (s\cos\theta)\sqrt{(1-\mu^2)} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -\mu & -s' & 0 \\ 0 & 1 & 0 & 0 & -\sqrt{(1-\mu^2)}\cos\theta & \dfrac{s'\mu\cos\theta}{\sqrt{(1-\mu^2)}} & (s'\sin\theta)\sqrt{(1-\mu^2)} \\ 0 & 0 & 1 & 0 & -\sqrt{(1-\mu^2)}\sin\theta & \dfrac{s'\mu\sin\theta}{\sqrt{(1-\mu^2)}} & -(s'\cos\theta)\sqrt{(1-\mu^2)} \end{vmatrix}$$

$$= (s+s')^2.$$

Since $s+s' = |\mathbf{r}-\mathbf{r}'|$ and $d\mu d\theta$ is the solid angle element $d\Omega$, we thus have

$$dV dV' ds = |\mathbf{r}-\mathbf{r}'|^2\, dV'' d\Omega ds ds'. \tag{14.9}$$

Although (14.9) has been derived using Cartesian coordinates, it is, of course, valid for any system of coordinates.

Since $\mathbf{r}''$ lies between $\mathbf{r}$ and $\mathbf{r}'$ on the line joining them,

$$\tau(\mathbf{r},\mathbf{r}') = \tau(\mathbf{r},\mathbf{r}'')+\tau(\mathbf{r}'',\mathbf{r}'),$$

and hence, using (14.9), we can rewrite the second term in (14.6) as

$$\frac{\gamma}{4\pi}\iiint \delta\left[\frac{1}{l(\mathbf{r}'')}\right]dV''\iint d\Omega\int_0^\infty \frac{c(\mathbf{r}''+s\Omega)}{l(\mathbf{r}''+s\Omega)}\rho(\mathbf{r}''+s\Omega)e^{-\tau(\mathbf{r}'',\mathbf{r}''+s\Omega)}\,ds\times$$

$$\times\int_0^\infty \frac{c(\mathbf{r}''-s'\Omega)}{l(\mathbf{r}''-s'\Omega)}\rho(\mathbf{r}''-s'\Omega)e^{-\tau(\mathbf{r}'',\mathbf{r}''-s'\Omega)}\,ds'. \quad (14.10)$$

The upper limits of integration over s and s' are put infinite, since $c(\mathbf{r})$ and $c(\mathbf{r}')$ vanish outside the system.

The integrals over s and s' can easily be expressed in terms of the angular distribution $\psi(\mathbf{r}'',\Omega)$. Since the unperturbed system is assumed critical, the unperturbed value of γ is unity, and so the unperturbed equation (14.1) coincides with (4.15'), and the angular distribution is then given by (4.16'). Using (4.16'), we can identify the integrals over s and s' in (14.10) as $4\pi\psi(\mathbf{r}'',-\Omega)$ and $4\pi\psi(\mathbf{r}'',\Omega)$ respectively. Substituting this into (14.10), calling $\mathbf{r}''$ now $\mathbf{r}$ and combining the result with (14.6) and (14.8), we find, since $\gamma=1$,

$$\delta\gamma\iiint \rho^2(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV=-\iiint\rho^2(\mathbf{r})\,\delta\left[\frac{c(\mathbf{r})}{l(\mathbf{r})}\right]dV+$$

$$+4\pi\iiint\delta\left[\frac{1}{l(\mathbf{r})}\right]dV\iint\psi(\mathbf{r},\Omega)\psi(\mathbf{r},-\Omega)\,d\Omega. \quad (14.11)$$

This gives the perturbation in the lowest eigenvalue when the system is initially critical.

It was only in the use of (4.16') that the above derivation made use of the assumption that the unperturbed system was critical and that the lowest eigenvalue was concerned. If a higher eigenvalue is considered, it is necessary to state the manner of introduction of this eigenvalue before the angular distribution can be found. This may be done as follows: we consider a fictitious medium for which the number of secondaries per collision at $\mathbf{r}$ is not $c(\mathbf{r})$, but $\gamma c(\mathbf{r})$. (Alternatively, we may introduce a source strength per unit volume $s(\mathbf{r})=(\gamma-1)\rho^\dagger(\mathbf{r})$, but this clearly gives the same $\psi(\mathbf{r},\Omega)$ as for a source-free medium with $\gamma c(\mathbf{r})$ secondaries per collision.) If this is done, then $c(\mathbf{r})$ in (4.16') is replaced by $\gamma c(\mathbf{r})$. Using this modified angular distribution and proceeding as before, we find

$$\delta\gamma_i\iiint\rho_i^2(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV=-\gamma_i\iiint\rho_i^2(\mathbf{r})\delta\left[\frac{c(\mathbf{r})}{l(\mathbf{r})}\right]dV+$$

$$+4\pi\iiint\delta\left[\frac{1}{l(\mathbf{r})}\right]dV\iint\psi_i(\mathbf{r},\Omega)\psi_i(\mathbf{r},-\Omega)\,d\Omega, \quad (14.11')$$

where the suffix i distinguishes the eigenvalue and eigenfunction concerned.

If we introduce for each eigenvalue γ_i of (14.1) an auxiliary function $g_i(\mathbf{r})$, defined by

$$g_i(\mathbf{r}) = \frac{4\pi \int\int \psi_i(\mathbf{r}, \Omega)\psi_i(\mathbf{r}, -\Omega)\, d\Omega}{\gamma_i \rho_i^2(\mathbf{r})}, \tag{14.12}$$

then, putting $\alpha(\mathbf{r}) = 1/l(\mathbf{r})$, $\beta(\mathbf{r}) = c(\mathbf{r})/l(\mathbf{r})$, (14.11′) becomes

$$\delta\gamma_i/\gamma_i = -\iiint \rho_i^2(\mathbf{r})[\delta\beta(\mathbf{r}) - g_i(\mathbf{r})\delta\alpha(\mathbf{r})]\, dV \Big/ \iiint \rho_i^2(\mathbf{r})\beta(\mathbf{r})\, dV. \tag{14.13}$$

14.4. The statistical weight theorem and its form in the diffusion approximation

The result derived in § 14.3 is called the *statistical weight theorem*. It can be expressed by saying that, in estimating the effects of small changes in a system, the change in the number of secondaries per unit path has a weight factor proportional to the square of the unperturbed flux, while the change in the inverse mean free path has a weight factor proportional to the square of the flux multiplied by the value of the auxiliary function $g_i(\mathbf{r})$ at the point where the change is made. The result applies to any effects, and not merely to the changes produced in the eigenvalues, since any total effect of a small change is proportional to the change which occurs in the eigenvalue. If this effect is a compensatory change such that the system remains critical, we have seen in § 14.1 that the magnitude of the change is proportional to $\delta\gamma$. If there is no compensation and the system becomes non-critical, so that the stationary solution is replaced by a time-dependent one, then the time constant of the latter is proportional to the change in α throughout the system necessary to restore criticality, and this latter change is again proportional to $\delta\gamma$.

A simple physical explanation of the appearance of the square of the unperturbed flux as a weight factor, and of the factor $g_i(\mathbf{r})$ before $\delta\alpha(\mathbf{r})$, is given in § 14.6. It may also be remarked that $g_i(\mathbf{r})$ vanishes at the free surface of a system, since for all Ω either $\psi_i(\mathbf{r}, \Omega)$ or $\psi_i(\mathbf{r}, -\Omega)$ vanishes there. This we should expect, since alterations of the mean free path at the free surface, while the number of secondaries per unit path is unaltered, will have no noticeable effect on the neutron distribution.

If the unperturbed system and the unperturbed eigenfunction have spherical symmetry, so that $\psi_i(\mathbf{r}, \Omega)$ depends only on r and μ (defined in § 11.1), then by expanding $\psi_i(r, \mu)$ in spherical harmonics, substituting

in (14.12), and using the orthogonality properties of Legendre polynomials, we find

$$g_i(r) = \frac{1}{\gamma_i}\left\{1 - 3\left[\frac{j_i(r)}{\rho_i(r)}\right]^2 + 5\left[\frac{\psi_{2,i}(r)}{\rho_i(r)}\right]^2 - \ldots\right\},$$

where $\rho_i(r) = \psi_{0,i}(r)$ and $j_i(r) = \psi_{1,i}(r)$, and thus (14.11′) can be written

$$\delta\gamma_i \iiint \rho_i^2(r)\frac{c(r)}{l(r)}\,dV = -\gamma_i \iiint \frac{\rho_i^2(r)}{l(r)}\,\delta c(r)\,dV -$$

$$-\iiint \{[\gamma_i c(r)-1]\rho_i^2(r) + 3j_i^2(r) - 5[\psi_{2,i}(r)]^2 + \ldots\}\delta[1/l(r)]\,dV. \quad (14.14)$$

We shall estimate the relative magnitudes of the terms in the braces in (14.14), assuming that we are concerned with the lowest eigenvalue of a critical system ($\gamma_i = 1$), and that the region where the mean free path has been altered is in the interior of a medium where $c \cong 1$. In this case it is possible to neglect the transient terms in the estimation of $\psi_0(r) = \rho(r)$, $\psi_1(r) = j(r)$, $\psi_2(r)$, etc., and to consider only the asymptotic parts, in the sense of Chapter V. It follows from the discussion of the functions $G_n(\nu)$ in Chapter X that the asymptotic part of $\psi_n(r)$ is, for small $(\gamma_i c - 1)$, of the order of $(\gamma_i c - 1)^{\frac{1}{2}n}$ compared with the mean value of $\psi_0(r)$. Thus the first two terms in the braces in (14.14) are of the order of $c-1$ compared with the average value of $\rho^2(r)$, the third of the order of $(c-1)^2$, the fourth of the order of $(c-1)^3$, and so on. Under these conditions, therefore, it is clearly safe to neglect all terms in the braces other than the first two, and the second term on the right of (14.14) becomes

$$-\iiint \{[c(r)-1]\rho^2(r) + 3j^2(r)\}\delta[1/l(r)]\,dV.$$

The first term in this expression can, of course, be recombined with the term in $\delta c(r)$.

This argument, though carried out for spherical systems for the sake of simplicity, can be extended to any geometry. Thus, if $\gamma_i = 1$ and the changes in the mean free path occur inside a region where $c \cong 1$, the formula (14.11) becomes

$$\delta\gamma \iiint \rho^2(\mathbf{r})\frac{c(\mathbf{r})}{l(\mathbf{r})}\,dV$$

$$\cong -\iiint \rho^2(\mathbf{r})\delta\left[\frac{c(\mathbf{r})-1}{l(\mathbf{r})}\right]dV - 3\iiint |\mathbf{j}(\mathbf{r})|^2\delta\left[\frac{1}{l(\mathbf{r})}\right]dV. \quad (14.15)$$

The quantity $[c(\mathbf{r})-1]/l(\mathbf{r}) = \beta(\mathbf{r}) - \alpha(\mathbf{r})$ is conventionally called the *capturing properties* of the medium, and $1/l(\mathbf{r}) = \alpha(\mathbf{r})$ the *scattering*

properties. With this terminology, the first term in (14.15) is called the perturbation due to the change in the capturing properties, and the second term the perturbation due to the change in the scattering properties.

Alternatively, the simplified form (14.15) of the statistical weight theorem can be derived from the differential equation (5.4) of the diffusion approximation, using the perturbation technique. This is to be expected, since (14.11) has been reduced to (14.15) on the basis of the assumptions of the diffusion approximation. Thus (14.15) can be regarded as the *statistical weight theorem in the diffusion approximation*.

14.5. Applications of the statistical weight theorem

In most applications of the statistical weight theorem, the weight factors $\rho^2(\mathbf{r})$ and $g(\mathbf{r})\rho^2(\mathbf{r})$ are known, and the theorem is used to find $\delta\gamma$. In some cases, however, the weight factors are found from a knowledge of the perturbation in the eigenvalue. We give two instances of this.

Firstly, let it be required to determine the ratio of the surface flux to the mean flux in a homogeneous sphere of critical size. As we have seen, the diffusion approximation and the Serber–Wilson method are not accurate at the free surface, and even the spherical harmonics method may need a fairly high approximation to give an accurate value of the surface flux. The critical radius, however, can be accurately determined by any of these methods, or by the variational method (Chapter XV). We might therefore expect to be able to find the derivative $d(a/l)/dc$ (where a is the critical radius) more accurately than $\rho(a)$. In this case, it seems plausible to find $\rho(a)$ by expressing it in terms of this derivative. To do this, we apply a small change in the radius and a compensating change in the value of c. If the system remains critical, $\delta\gamma = 0$, and the left side of (14.11) vanishes. The second term on the right also vanishes: the contribution from the change in radius vanishes because

$$\iint \psi(\mathbf{r}, \boldsymbol{\Omega})\psi(\mathbf{r}, -\boldsymbol{\Omega})\, d\Omega$$

vanishes at the surface, and no change has been made in l inside the medium. Thus (14.11) here reduces to

$$4\pi a^2 \frac{c}{l}\rho^2(a)\,\delta a - \frac{\delta c}{l}\iiint \rho^2(r)\, dV = 0,$$

and hence

$$\rho^2(a) = \frac{1}{ca^2 l}\frac{dc}{d(a/l)}\int_0^a r^2\rho^2(r)\, dr,$$

where all the quantities on the right can be accurately determined by comparatively simple methods.

Secondly (and this is perhaps a more interesting example), let there be a critical reactive core large compared with the mean free path in it, so that $c \cong 1$, and arranged so that

(i) a variable amount of material can be removed from a point $\mathbf{r}_1$ and replaced by other material (which we call X) having the same l and a rather larger c;

(ii) a variable amount of material can be removed from a point $\mathbf{r}_2$ which is in the core and several mean free paths from the boundary, and, if desired, replaced by other material (which we call Y) having the same l and a rather smaller c; alternatively, a cavity may be left at $\mathbf{r}_2$ if desired.

We also suppose that these are the only ways in which the system can be altered, deliberately or otherwise. In this case, equation (14.15) is applicable to any alteration in the system. Thus, if a cavity is made at $\mathbf{r}_2$ and enough X inserted at $\mathbf{r}_1$ to make the system critical again (so that $\delta\gamma = 0$) then (14.15) gives a linear relation between $\rho^2(\mathbf{r}_1)$, $\rho^2(\mathbf{r}_2)$ and $|\mathbf{j}(\mathbf{r}_2)|^2$. If, on the other hand, the material at $\mathbf{r}_2$ is replaced by Y, and X is again placed at $\mathbf{r}_1$ to restore criticality, we similarly obtain the ratio of $\rho^2(\mathbf{r}_1)$ to $\rho^2(\mathbf{r}_2)$. By elimination, we find the ratio of $|\mathbf{j}(\mathbf{r}_2)|^2$ to $\rho^2(\mathbf{r}_1)$. This shows that in some circumstances not only the density and flux but also the current can be measured experimentally, though of course the actual experimental details are more complicated than would appear from the above description.

14.6. The perturbation in the neutron flux

We have so far considered only the perturbation in the eigenvalues γ_i of (14.1). However, it is sometimes of interest to find the corresponding perturbations in the eigenfunctions, i.e. the changes in the space distribution of the neutron flux, resulting from alterations in the system. For simplicity, we discuss in detail only the case where the changes concerned are in $\beta(\mathbf{r}) = c(\mathbf{r})/l(\mathbf{r})$, while $\alpha(\mathbf{r}) = 1/l(\mathbf{r})$ remains unaltered, and shall indicate the method of extending the results to problems where the mean free path also is perturbed.

If $\rho_0(\mathbf{r})$ is the unperturbed flux, then increasing $\beta(\mathbf{r})$ by $\delta\beta(\mathbf{r})$ is equivalent to introducing a system of sources of strength

$$s(\mathbf{r}) = \rho_0(\mathbf{r})\delta\beta(\mathbf{r})/4\pi \tag{14.16}$$

per unit volume per unit solid angle. The effects of sources in the general case (i.e. without the constant cross-section approximation) have been discussed in §§ 3.4 to 3.6. We shall specialize these results to the case of

isotropic scattering and constant cross-sections more explicitly than in Chapter IV.

We assume that the various eigenfunctions $\rho_i(\mathbf{r})$ of (14.1) form a complete set, normalized so that

$$\iiint \rho_i^2(\mathbf{r})\{c(\mathbf{r})/l(\mathbf{r})\}\,dV = 1. \tag{14.17}$$

It follows from (3.20′), or by applying to (14.1) and (14.2) the argument which led to (3.20), that $\rho_i(\mathbf{r})$ and $\rho_j^\dagger(\mathbf{r})$ form a biorthogonal set. Using (14.3) and (14.17), this can be expressed as

$$\iiint \rho_i(\mathbf{r})\rho_j(\mathbf{r})\{c(\mathbf{r})/l(\mathbf{r})\}\,dV = \delta_{ij}. \tag{14.18}$$

Since the set $\rho_j^\dagger(\mathbf{r})$ is assumed complete, the source strength in (4.15′) can be represented as

$$s(\mathbf{r}) = \sum_i \left\{\frac{c(\mathbf{r})}{l(\mathbf{r})}\right\}\rho_i(\mathbf{r}) \iiint s(\mathbf{r}')\rho_i(\mathbf{r}')\,dV'. \tag{14.19}$$

Substituting this into (4.15′), the latter equation can be rewritten

$$\rho(\mathbf{r}) = \frac{1}{4\pi}\iiint \frac{c(\mathbf{r}')}{l(\mathbf{r}')}\left[\rho(\mathbf{r}')+4\pi\sum_i \rho_i(\mathbf{r}') \iiint s(\mathbf{r}'')\rho_i(\mathbf{r}'')\,dV''\right] e^{-\tau(\mathbf{r},\mathbf{r}')}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2},$$

and, from (14.1), the solution is clearly

$$\rho(\mathbf{r}) = 4\pi\sum_i \frac{\iiint s(\mathbf{r}')\rho_i(\mathbf{r}')\,dV'}{\gamma_i - 1}\rho_i(\mathbf{r}). \tag{14.20}$$

Substituting here for $s(\mathbf{r})$ from (14.16), we find, since $\rho(\mathbf{r})$ is here simply the perturbation in the lowest eigenfunction $\rho_0(\mathbf{r})$,

$$\delta\rho_0(\mathbf{r}) = \sum_i \frac{\iiint \rho_0(\mathbf{r}')\rho_i(\mathbf{r}')\delta[c(\mathbf{r}')/l(\mathbf{r}')]\,dV'}{\gamma_i - 1}\rho_i(\mathbf{r}). \tag{14.21}$$

Since the unperturbed system is assumed critical, the lowest eigenvalue γ_0 is unity, and for (14.21) to give a finite result we must have

$$\iiint \rho_0^2(\mathbf{r})\delta[c(\mathbf{r})/l(\mathbf{r})]\,dV = 0. \tag{14.22}$$

This is, of course, what we should get from (14.11) by putting

$$\delta[1/l(\mathbf{r})] = \delta\gamma = 0.$$

The derivation of (14.22) by means of (14.20) gives a simple physical explanation of why the statistical weight of $\delta\beta(\mathbf{r})$ is the square of the unperturbed neutron flux: one factor $\rho_0(\mathbf{r})$ arises because the effect of the source at $\mathbf{r}$ is proportional to $\rho_0(\mathbf{r})$, by (14.20), and the other because a

criticality', while the shape of the space distribution of the flux is almost independent of the source distribution, and is essentially given by the space distribution of the neutron flux in a source-free critical system. The formula (14.20′) is of particular importance in experimental investigations of the approach to criticality.

In such investigations, it is usual to introduce the *multiplication* M. This is defined as the ratio of the number of neutrons escaping from the system to the number emitted by the sources. If J_i is the total outward neutron current associated with the eigenfunction $\rho_i(\mathbf{r})$, then the multiplication is clearly given by

$$M = \sum_i \frac{1}{\gamma_i - 1} \iiint s(\mathbf{r}')\rho_i(\mathbf{r}')\,dV'.J_i \Big/ \iiint s(\mathbf{r})\,dV, \quad (14.24)$$

with an obvious simplification for near-critical systems ($\gamma_0 - 1 \ll 1$). Here

$$J_i = \iint_S \mathbf{j}_i(\mathbf{r}).\mathbf{dS}, \quad (14.25)$$

where $\mathbf{j}_i(\mathbf{r})$ is the neutron current corresponding to the eigenfunction $\rho_i(\mathbf{r})$ for the flux, $\mathbf{dS}$ is the vector whose magnitude equals the area element at dS and whose direction is that of the outward normal, and the integration is over the outer surface of the system, assumed convex.

14.8. Higher-order perturbations

Hitherto we have dealt only with first-order perturbations. In considering perturbations of higher order, it is necessary to take account of the fact that (as we have seen in Chapter VI) the normal derivative of the neutron flux generally has a logarithmic singularity at an interface. Thus, such quantities as

$$\iiint \rho(\mathbf{r})f(\mathbf{r})\,dV,$$

where $f(\mathbf{r})$ is a small perturbation such that $\iiint |f(\mathbf{r})|\,dV = \epsilon$, say, may contain not only terms in ϵ, ϵ^2, etc., but also terms in $\epsilon^2 \log \epsilon$, $\epsilon^3 \log \epsilon$, etc., and, if the change involved in the perturbation is proportional to ϵ, it is not always possible to expand the perturbation in the eigenvalues as, for instance, a Taylor series in ϵ. It is therefore better not to attempt to take the second variation of (14.1), but to apply a finite variation depending on the parameter ϵ, select the terms of each order of magnitude, and examine the behaviour of each group of terms as $\epsilon \to 0$. In order to ensure that possible cancellations of terms become apparent at an early stage, it is sometimes advantageous to modify the calculations slightly as compared with those of §§ 14.2 and 14.3. For

perturbation in $\beta(\mathbf{r})$ at $\mathbf{r}$ is equivalent to a source of strength proportional to $\rho_0(\mathbf{r})$, by (14.16).

If both $\beta(\mathbf{r})$ and $\alpha(\mathbf{r})$ are perturbed, the effects cannot be simulated by introducing an isotropic source, but they can be by introducing a certain anisotropic source. Thus, increasing $\alpha(\mathbf{r})$ by $\delta\alpha(\mathbf{r})$ is equivalent (if $\iiint \delta\alpha(\mathbf{r})\, dV$ is small compared with $\iiint \alpha(\mathbf{r})\, dV$) to introducing the source

$$s(\mathbf{r}, \Omega) = \left[\frac{\rho_0(\mathbf{r})}{4\pi} - \psi_0(\mathbf{r}, \Omega)\right] \delta\alpha(\mathbf{r}). \tag{14.23}$$

If $\beta(\mathbf{r})$ is unaltered, this means that only the scattering cross-section is altered, and not the capture and fission cross-sections; the alteration is such that the number of neutrons travelling in the direction Ω which are scattered out of this direction is increased by $\psi_0(\mathbf{r}, \Omega)\delta\alpha(\mathbf{r})$, where the suffix 0 indicates the angular distribution pertaining to the eigenfunction $\rho_0(\mathbf{r})$.

On specializing the treatment of anisotropic sources given in §§ 3.4 to 3.6 to the case of isotropic scattering and the constant cross-section approximation, and using (14.23), we easily obtain the corresponding expression for $\delta\psi_0(\mathbf{r}, \Omega)$ and hence also for $\delta\rho_0(\mathbf{r})$. The appearance of $\psi_0(\mathbf{r}, \Omega)$ in (14.23) and the fact that the formation of the adjoint equation involves the replacement of Ω by $-\Omega$ (see (4.22)) account for the appearance of the factor $g(\mathbf{r})$ before $\delta\alpha(\mathbf{r})$ in (14.13).

Of course, the formulae (14.20) and (14.21) are given here only for their usefulness in qualitative discussion. The numerical determination of $\delta\rho_0(\mathbf{r})$, etc., is more easily effected, except in special cases, by solving (4.15′) directly, rather than by expanding in eigenfunctions of (14.1).

14.7. The multiplication

The perturbation of a system may sometimes consist simply in introducing sources which were not present in the unperturbed system. If the perturbed system has a stationary distribution, then the unperturbed system must be subcritical, i.e. the lowest eigenvalue γ_0 must exceed unity. The neutron flux in the perturbed system is then given directly by (14.20). If the system is nearly critical, i.e. if $(\gamma_0 - 1) \ll 1$, only the first term in (14.20) need be taken into account, and the equation reduces to

$$\rho(\mathbf{r}) \cong \frac{4\pi \iiint s(\mathbf{r}')\rho_0(\mathbf{r}')\, dV'}{\gamma_0 - 1} \rho_0(\mathbf{r}). \tag{14.20′}$$

This shows that, for a nearly critical system, the magnitude of the neutron flux due to given sources varies inversely as the 'distance from

instance, if $\rho(\mathbf{r})$ and $\rho^\dagger(\mathbf{r})$ are the solutions of the two perturbed equations (14.1) and (14.2), and $\rho_0(\mathbf{r})$ and $\rho_0^\dagger(\mathbf{r})$ are those of the corresponding unperturbed equations, it is better to consider the expression for

$$\iiint [\rho(\mathbf{r})\rho_0^\dagger(\mathbf{r}) - \rho^\dagger(\mathbf{r})\rho_0(\mathbf{r})]\, dV$$

rather than that for

$$\iiint \rho_0^\dagger(\mathbf{r})[\rho(\mathbf{r}) - \rho_0(\mathbf{r})]\, dV,$$

as we should do if we followed the procedure of § 14.2.

We have seen that it is not necessary to know the perturbation in the eigenfunction in order to determine the first-order perturbation in the eigenvalue, i.e. the term proportional to ϵ in $\delta\gamma$. The same is true for the term proportional to $\epsilon^2 \log \epsilon$. However, the expression for the term proportional to ϵ^2 will in general involve the first-order perturbation $\delta\rho(\mathbf{r})$ in the eigenfunction, and so on.

Whilst first-order perturbation calculations are often used, higher-order calculations are unusual on account of their laboriousness.

Our discussion has been limited to problems of critical size. It is possible to apply perturbation theory also to problems with prescribed sources at finite or infinite distances, but the method in these cases is too obvious to require detailed discussion. An example of the application of higher-order perturbation methods to problems of the latter type is found in Davison's paper (12).

XV

THE VARIATIONAL METHOD

15.1. Critical-size problems: the general theory

In discussing the variational method, we shall discuss separately two kinds of problem, critical-size problems and those governed by an inhomogeneous equation. In both types the starting-point is the integral transport equation, as in the perturbation method, rather than the Boltzmann equation.

We first consider the application of the variational method to the solution of homogeneous integral equations. The general principle in this case is well known, but for completeness we shall develop the treatment *ab initio*. The integral equation (14.1) may be written as

$$\rho(\mathbf{r}) = \gamma \iiint K(\mathbf{r}' \to \mathbf{r})\rho(\mathbf{r}')\, dV', \tag{15.1}$$

where

$$K(\mathbf{r}' \to \mathbf{r}) = \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|^2} \frac{c(\mathbf{r}')}{l(\mathbf{r}')} e^{-\tau(\mathbf{r},\mathbf{r}')}. \tag{15.2}$$

If we assume for the moment that the system is of finite extent, the eigenvalues γ_j will form a discrete set.‡ If the corresponding eigenfunctions are normalized according to (14.17), they will, in view of the orthogonality relations, satisfy (14.18), and this, by (14.3), can be written

$$\iiint \rho_i(\mathbf{r})\rho_j^\dagger(\mathbf{r})\, dV = \delta_{ij}. \tag{15.3}$$

If the eigenfunctions $\rho_i(\mathbf{r})$ form a complete set, it is clear from a comparison of (15.3) and (15.1) that the expansion of the kernel $K(\mathbf{r}' \to \mathbf{r})$ in terms of eigenfunctions will be

$$K(\mathbf{r}' \to \mathbf{r}) = \sum_i \frac{1}{\gamma_i} \rho_i^\dagger(\mathbf{r}')\rho_i(\mathbf{r}). \tag{15.4}$$

Although it has not been postulated that the $\rho_i(\mathbf{r})$ form a complete set, it follows from a general theorem (Whittaker and Watson (**52**), p. 227) that the expansion (15.4) holds for a real symmetrical or 'nearly symmetrical'§ kernel, whether the $\rho_i(\mathbf{r})$ form a complete set or not.

‡ This theorem can be proved in two stages. One first proves it for a bounded kernel (**52**, p. 217), and then uses the obvious result that, if the original equation has only discrete eigenvalues, so has any iterated equation, and conversely.

§ By a 'nearly symmetrical' kernel we mean one which is symmetrical when the unknown function is taken as $\rho(\mathbf{r})[c(\mathbf{r})/l(\mathbf{r})]^{\frac{1}{2}}$ instead of $\rho(\mathbf{r})$; we avoid the more usual term 'symmetrizable by elementary means', on account of its length.

We now consider the functional

$$I(\tilde{\rho}) = \frac{\iiint dV \iiint dV' K(\mathbf{r}' \to \mathbf{r})\tilde{\rho}(\mathbf{r}')\tilde{\rho}^\dagger(\mathbf{r})}{\iiint \tilde{\rho}(\mathbf{r})\tilde{\rho}^\dagger(\mathbf{r})\, dV}, \tag{15.5}$$

where, similarly to (14.3),

$$\tilde{\rho}^\dagger(\mathbf{r}) = c(\mathbf{r})\tilde{\rho}(\mathbf{r})/l(\mathbf{r}), \tag{15.6}$$

and $\tilde{\rho}(\mathbf{r})$ is an arbitrary function of $\mathbf{r}$ which we shall call the *trial function*. Expanding $\tilde{\rho}(\mathbf{r})$ in terms of the eigenfunctions $\rho_i(\mathbf{r})$ we have

$$\tilde{\rho}(\mathbf{r}) = \sum_i a_i \rho_i(\mathbf{r}) + \omega(\mathbf{r}), \tag{15.7}$$

say, where $\omega(\mathbf{r})$ is orthogonal to every $\rho_i^\dagger(\mathbf{r})$, and therefore vanishes identically if the $\rho_i(\mathbf{r})$ form a complete set. On substituting (15.7) into (15.5) and using (15.3) and (15.4), (15.5) becomes

$$I(\tilde{\rho}) = \frac{\sum_i a_i^2/\gamma_i}{\sum_i a_i^2 + \iiint \omega^2(\mathbf{r})c(\mathbf{r})\, dV/l(\mathbf{r})}. \tag{15.8}$$

It is well known that, if the kernel is real and symmetrical (or nearly so, as defined in the last footnote), the eigenvalues cannot be complex, and it follows from physical considerations (see Chapter III) that no γ_i can be real and negative. The latter result also follows mathematically from the rapid decrease of the kernel $K(\mathbf{r}' \to \mathbf{r})$ when $|\mathbf{r}'-\mathbf{r}|$ increases, the position of $\mathbf{r}'$ and direction of $\mathbf{r}-\mathbf{r}'$ being fixed, but the proof is too lengthy to give here. All γ_i are therefore real and positive, and hence $I(\tilde{\rho})$ will take its maximum value if $\omega(\mathbf{r})$ and all a_i (except the one belonging to the smallest γ_i) vanish. If the smallest γ_i is γ_0, therefore,

$$\max I(\tilde{\rho}) = 1/\gamma_0, \tag{15.9}$$

and the trial function $\tilde{\rho}(\mathbf{r})$ which gives this maximum is the eigenfunction $\rho_0(\mathbf{r})$ corresponding to the lowest eigenvalue.

The formula (15.9) makes it possible to give an alternative mathematical proof of the statement made in Chapter III that this 'lowest eigenfunction' is everywhere positive. For, if $\rho_0(\mathbf{r})$ has a change of sign, we take for the trial function $\tilde{\rho}(\mathbf{r}) = |\rho_0(\mathbf{r})|$. Since $K(\mathbf{r}' \to \mathbf{r})$, $c(\mathbf{r})$ and $l(\mathbf{r})$ are non-negative, the denominator of $I(\tilde{\rho})$ will be the same as that of $I(\rho_0)$. In the numerator, the pairs of points $\mathbf{r}$ and $\mathbf{r}'$ for which $\rho_0(\mathbf{r})\rho_0(\mathbf{r}') > 0$ will give the same contribution to the two functionals, while if $\rho_0(\mathbf{r})\rho_0(\mathbf{r}') < 0$, the contribution to $I(\tilde{\rho})$ will be positive and that to $I(\rho_0)$ negative. Hence $I(\tilde{\rho}) > I(\rho_0)$, which is impossible by (15.9). The assumption that $\rho_0(\mathbf{r})$ changes sign is therefore false.

In the above derivation of (15.9), it was tacitly assumed that the series (15.4) and (15.7) were such that the operations involved were permissible. This assumption may be avoided by using a different derivation of (15.9). The trial function is represented as

$$\tilde{\rho}(\mathbf{r}) = \rho(\mathbf{r}) + \epsilon f(\mathbf{r}) \tag{15.10}$$

and substituted into (15.5), with the condition that $I(\tilde{\rho})$ is a maximum for $\tilde{\rho}(\mathbf{r}) = \rho(\mathbf{r})$. This implies that $\partial I(\tilde{\rho})/\partial\epsilon$ vanishes for $\epsilon = 0$; if this condition is satisfied independently of the form of $f(\mathbf{r})$, it follows from (15.2), (14.3), and (15.6) that $\rho(\mathbf{r})$ in (15.10) is a solution of (14.1). The solution of (14.1) is therefore equivalent to finding the extremum of (15.5). This derivation, however, shows only that $I(\tilde{\rho})$ has an extremum for $\tilde{\rho}(\mathbf{r}) = \rho(\mathbf{r})$, and an additional investigation is needed to find whether it is a maximum or a minimum if (15.9) is not derived by the use of (15.8).

We now consider systems extending to infinity, and make two assumptions: (1) that there is no supply of neutrons from infinity, (2) that, if c_0 is the largest value of $c(\mathbf{r})$ in the system, and c_∞ is its largest value in the media extending to infinity, then

$$c_0 > 1 > c_\infty; \tag{15.11}$$

this condition has been satisfied hitherto in applications of the method.

If the eigenvalue γ_t is smaller than $1/c_\infty$, it follows from the conditions at infinity that $\rho_t(\mathbf{r})$ decreases exponentially for $|\mathbf{r}| \to \infty$, and the integrals in $I(\rho_t)$ will therefore be absolutely convergent. It is consequently necessary to choose $\tilde{\rho}(\mathbf{r})$ in such a way that the integrals in (15.5) are absolutely convergent, and in this case it follows as for a finite system that $I(\tilde{\rho})$ is stationary for $\tilde{\rho}(\mathbf{r}) = \rho_0(\mathbf{r})$ (using (15.10)). The nature of the extremum is found as follows: we consider a sphere of large radius M and the functional $I_M(\tilde{\rho})$ obtained by omitting the contribution to $I(\tilde{\rho})$ from points outside this sphere. Since the integrals in $I(\tilde{\rho})$ are absolutely convergent,

$$\lim_{M\to\infty} I_M(\tilde{\rho}) = I(\tilde{\rho}).$$

However, for each finite M, $I_M(\tilde{\rho})$ is the functional $I(\tilde{\rho})$ for some finite system, and is therefore a maximum when $\tilde{\rho}$ is the 'lowest eigenfunction' of that system. Passing to the limit $M \to \infty$, we see that (15.9) holds for an infinite system also, at least when the assumptions (1) and (2) above are satisfied.

If the eigenvalue γ_t is greater than $1/c_\infty$, it is possible to find a solution of (14.1) for any such γ_t, provided that the infinite region where $c(\mathbf{r}) = c_\infty$ is fairly extended, and not (e.g.) a rod or a slab of constant thickness. The eigenvalues greater than $1/c_\infty$ therefore form a *continuous spectrum*

of eigenvalues, and $\gamma = 1/c_\infty$ is the lower edge of this continuous spectrum. If there are no discrete eigenvalues, then (15.9) gives this lower edge.

To apply (15.9) to the determination of the critical size, we recall that, for a critical system, $\gamma_0 = 1$. If the system depends on a parameter a, say, and the critical value a_0 is required, we first find $\gamma_0(a)$ for every a by means of (15.9); then a_0 is given by

$$\gamma_0(a_0) = 1. \tag{15.12}$$

It is important to notice that, if the relative deviations of the trial function $\tilde{\rho}(\mathbf{r})$ from the 'lowest eigenfunction' $\rho_0(\mathbf{r})$ are of the order of ϵ, the deviation of $I(\tilde{\rho})$ from $1/\gamma_0$ will be of the order ϵ^2. This follows from a comparison of (15.7) and (15.8). Consequently, even a poor trial function will give a fairly accurate value of γ_0, and so the use of (15.12) will give the critical size to quite good accuracy.

15.2. Examples of applications

The variational method was first used in neutron transport theory by Pryce (43) to determine the critical radius of a bare homogeneous sphere with $c-1$ comparable with unity. Since c is the same throughout the system, it is not necessary to use the parameter γ, c itself being employed. The integration over angles can be effected, since in a spherical system the 'lowest eigenfunction' is spherically symmetrical, and (14.1) then becomes, on putting $\rho(-r) = \rho(r)$ (see § 8.2),

$$\rho(r) = \frac{c}{2rl} \int_{-a}^{a} E_1\left(\frac{|r-r'|}{l}\right) r'\rho(r')\, dr',$$

and hence (15.9) gives

$$\frac{1}{c} = \max \frac{\int_{-a}^{a}\int_{-a}^{a} E_1(|r-r'|/l)\, r'\tilde{\rho}(r')\, r\tilde{\rho}(r)\, dr'dr}{2l\int_{-a}^{a} [r\tilde{\rho}(r)]^2\, dr}. \tag{15.13}$$

As trial function, Pryce took the simple expression

$$\tilde{\rho}(r) = 1-qr^2, \tag{15.14}$$

where q is the constant to be determined by variational methods. With this trial function the integrals in (15.13) can be expressed in terms of the functions $E_n(z)$ (see (10.59)), which are tabulated. The object of Pryce's calculations was to find a/l as a function of c rather than for a particular value of c, and the trial function (15.14) does not involve c, so that no appeal to (15.12) is necessary, and therefore none of the numerical work is wasted.

In order to find the accuracy of the results, Pryce repeated the calculations for a few values of c, using the improved trial function

$$\tilde{\rho}(r) = 1 - qr^2 - q'r^4, \tag{15.14'}$$

and found that, except for $c-1 \ll 1$, the results obtained from the two trial functions (15.14) and (15.14') differed only in the fourth significant figure. This confirms that the trial function (15.14) gives quite an accurate value of a/l unless $c-1 \ll 1$.

Fuchs (21) subsequently extended this work to the case of a reactive spherical core surrounded by a reflector of finite thickness with the same mean free path as the core. The trial function for the core was (15.14), while for the reflector it was

$$\tilde{\rho}(r) = A + B/r. \tag{15.15}$$

In order to have two disposable constants and not three, Fuchs stipulated that the function $\tilde{\rho}(r)$ should be continuous at the interface. The critical radius of the core was tabulated as a function of the reflector thickness and of the values of c in the core and in the reflector. As in Pryce's work, since c was not fixed in either medium and the trial function did not depend on c, no appeal to (15.12) was necessary and none of the numerical work was wasted. Though the trial function used was a poor one (for instance, it made no allowance for the rapid variation of $\rho(r)$ near the interface), the results for c_c-1 comparable with unity and the reflector thickness comparable with the core radius appear quite satisfactory.

Wilson (56) used a different trial function for the case of a large core ($c_c-1 \ll 1$) in an infinite reflector, again with the same mean free path as the core. On the basis of the arguments given in Chapter V (see equation (5.38)), he took as the trial function for the core the spherically symmetrical solution, regular at the origin, of the diffusion-approximation differential equation for the core, and as the trial function for the reflector the similarly appropriate solution for the reflector, leaving the ratio at the interface as the variable parameter; that is,

$$\tilde{\rho}(r) = \begin{cases} \dfrac{\sin \kappa r}{\kappa r} & \text{in the core,} \\ Ae^{-r/L}/r & \text{in the reflector.} \end{cases} \tag{15.16}$$

Since κ and L depend on the values of c in the core and in the reflector, it is not possible to interpret $\gamma c(\mathbf{r})$ as the value of $c(\mathbf{r})$ for another system of equal significance. It is therefore necessary to use (15.12), and this means that the calculations of $\gamma_0(a)$ for $a \neq a_0$ (the solution of (15.12)) do not contribute directly to the final result. In order to minimize this

wastage, it is important to use trial values of a as near a_0 as possible. Wilson accordingly found the first trial value of a by other methods and used the variational method only to improve this value.

The fact that some of the calculations are liable to be wasted is a disadvantage of Wilson's method, but this is usually compensated by the high accuracy obtainable by the use of his trial function. However, in the case of an infinite reflector with small capture ($1-c_r \ll c_c-1$), difficulties may occur since $\gamma = 1/c_r$ is the edge of the continuous spectrum, and consequently $\gamma_0(a)$ has a singularity at $a = a^*$, where a^* is the solution of $\gamma_0(a^*) = 1/c_r$. The variational method is, in fact, not really suitable for such cases, even though it may be made to give reasonable results with sufficient trouble.

It has been remarked in Chapter VIII that the most accurate results are obtained from the diffusion approximation by imposing a certain discontinuity on $\rho_{\text{diff}}(r)$ at the interface. The chief purpose of Wilson's calculations was to find an independent estimate of this discontinuity, and it was found to be very marked.

15.3. Simplifications for slowly varying trial functions

In the examples of the preceding section, the mean free path was the same throughout the system, and the geometry was very simple. If these conditions no longer hold, the integrals in (15.5) become rather unwieldy, and it is the difficulty of evaluating the variational integrals, even with simple trial functions, that places a limit on the practical applicability of the variational method.

However, it has been pointed out by Hitchcock (unpublished) that under certain conditions the properties of the kernel (15.2) can be used to halve the number of variables of integration, i.e. to replace the sixfold integral by simpler ones. This is possible if (1) the system is non-re-entrant and free from cavities and has boundaries of a simple shape (e.g. planes), all dimensions being large compared with the mean free paths involved, and (2) the trial function used in each medium is a slowly varying function whose analytical continuation is regular in all space.

We shall illustrate this by taking the case of a bare infinite slab. Here it is necessary to evaluate the double integral

$$I^* = \frac{1}{2l}\int_{-a}^{a}\int_{-a}^{a} E_1\left(\frac{|x-x'|}{l}\right)\tilde{\rho}(x)\tilde{\rho}(x')\,dxdx', \tag{15.17}$$

and we wish to represent this approximately as a sum of simple integrals

and integrated terms. Since the analytical continuation of $\rho(x)$ is assumed regular and slowly varying for all x, we can write

$$\frac{1}{2l}\int_{-a}^{a} E_1\left(\frac{|x-x'|}{l}\right)\tilde{\rho}(x')\,dx' = \frac{1}{2l}\int_{-\infty}^{\infty}\ldots - \frac{1}{2l}\int_{-\infty}^{-a}\ldots - \frac{1}{2l}\int_{a}^{\infty}\ldots. \quad (15.18)$$

Expanding $\tilde{\rho}(x')$ in the first term in powers of $x'-x$ and integrating term by term gives

$$\left(1+\frac{l^2}{3}\frac{d^2}{dx^2}+\frac{l^4}{5}\frac{d^4}{dx^4}+\ldots\right)\tilde{\rho}(x),$$

and so (15.17) can be written

$$\begin{aligned} I^* = \int_{-a}^{a}\tilde{\rho}(x)\left[1+\frac{l^2}{3}\frac{d^2}{dx^2}+\ldots\right]\tilde{\rho}(x)\,dx - \\ -\frac{1}{2l}\int_{-\infty}^{a}dx\int_{a}^{\infty}dx' E_1\left(\frac{x'-x}{l}\right)\tilde{\rho}(x)\tilde{\rho}(x') - \\ -\frac{1}{2l}\int_{-a}^{\infty}dx\int_{-\infty}^{-a}dx' E_1\left(\frac{x-x'}{l}\right)\tilde{\rho}(x)\tilde{\rho}(x') + \\ +\frac{1}{2l}\int_{-\infty}^{-a}dx\int_{a}^{\infty}dx' E_1\left(\frac{x'-x}{l}\right)\tilde{\rho}(x)\tilde{\rho}(x') + \\ +\frac{1}{2l}\int_{a}^{\infty}dx\int_{-\infty}^{-a}dx' E_1\left(\frac{x-x'}{l}\right)\tilde{\rho}(x)\tilde{\rho}(x'). \end{aligned} \quad (15.19)$$

Assuming that $a \gg l$ and that $\tilde{\rho}(x)$ varies only slowly, the last two terms in (15.19) can be neglected. In the remaining two double integrals, we expand $\tilde{\rho}(x)$ and $\tilde{\rho}(x')$ in powers of $x-a$ and $x'-a$ in the first, and of $x+a$ and $x'+a$ in the second, and integrate term by term. The final result is

$$\begin{aligned} I^* = \int_{-a}^{a}\tilde{\rho}(x)\left[1+\tfrac{1}{3}l^2\frac{d^2}{dx^2}+\tfrac{1}{5}l^4\frac{d^4}{dx^4}+\ldots\right]\tilde{\rho}(x)\,dx - \\ -\tfrac{1}{4}l\left\{\tilde{\rho}^2(a)+\tfrac{1}{2}l^2\left[2\tilde{\rho}(a)\frac{d^2}{da^2}\tilde{\rho}(a)-\left(\frac{d\tilde{\rho}(a)}{da}\right)^2\right]+ \right. \\ \left. +\tfrac{1}{3}l^4\sum_{m=0}^{4}(-1)^m\frac{d^m\tilde{\rho}(a)}{da^m}\frac{d^{4-m}\tilde{\rho}(a)}{da^{4-m}}+\ldots\right\}- \\ -\text{the same with } -a \text{ instead of } a + O(e^{-2a/l}). \end{aligned} \quad (15.20)$$

If $\tilde{\rho}(x)$ is in fact regular and slowly varying (in terms of l) for all x, then the series in (15.20) will converge rapidly, and it is sufficient to take only the first few terms of each. If $\tilde{\rho}(x)$ varies rapidly, however, a formal application of (15.20) would lead to divergent series, while, if $\tilde{\rho}(x)$ has a singularity for a finite x outside the interval considered but varies slowly in the interval, the series in question are asymptotic, i.e. the first few terms give a fair approximation.

The method described in this section could be applied to Wilson's problem (equation (15.16)), but, in general, problems where Hitchcock's conditions apply are somewhat rare.

15.4. Problems with inhomogeneous equations: the general theory

15.4.1. *Finite systems*

We now consider the inhomogeneous equation (4.15'), and rewrite it as

$$\rho(\mathbf{r}) = \iiint K(\mathbf{r}' \to \mathbf{r})\rho(\mathbf{r}')\, dV' + q(\mathbf{r}), \tag{15.21}$$

where $K(\mathbf{r}' \to \mathbf{r})$ is given by (15.2) and $q(\mathbf{r})$ is the flux of neutrons which come directly from the sources without collision.†

In applying the variational method to homogeneous equations, we saw that there was a general characteristic of the system (the lowest eigenvalue γ_0 or the critical size) which could be determined by this method much better than $\rho(\mathbf{r})$ itself. The same is true for inhomogeneous equations; the corresponding quantity in this case is

$$\iiint \rho^\dagger(\mathbf{r}) q(\mathbf{r})\, dV, \tag{15.22}$$

and we shall see in the next section that this very quantity is often of interest.

To determine (15.22) we introduce the functional

$$J(\tilde{\rho}) = \iiint \tilde{\rho}^\dagger(\mathbf{r}) \Big[\iiint \tilde{\rho}(\mathbf{r}') K(\mathbf{r}' \to \mathbf{r})\, dV' - \tilde{\rho}(\mathbf{r}) + 2q(\mathbf{r}) \Big]\, dV, \tag{15.23}$$

where $\tilde{\rho}(\mathbf{r})$ is again the trial function, and $\tilde{\rho}^\dagger(\mathbf{r})$ is defined by (15.6). The functional (15.23) clearly reduces to (15.22) if $\tilde{\rho}(\mathbf{r}) \equiv \rho(\mathbf{r})$. We shall now show that, firstly, for critical and subcritical systems the functional (15.23) has a finite upper bound equal to (15.22) and, secondly, for such systems (15.23) gives a fair approximation to (15.22) even for relatively poor trial functions. To do so, we represent $\tilde{\rho}(\mathbf{r})$ as

$$\tilde{\rho}(\mathbf{r}) = \rho(\mathbf{r}) + \delta\rho(\mathbf{r}), \tag{15.24}$$

† This use of $q(\mathbf{r})$ differs from that of earlier chapters.

where $\rho(\mathbf{r})$ is the exact solution of (15.21), and substitute this into (15.23), using (15.21) and assuming for the moment that

$$\iiint \tilde{\rho}^\dagger(\mathbf{r})\Big[\iiint \delta\rho(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'-\delta\rho(\mathbf{r})\Big]\,dV$$

$$=\iiint \delta\rho^\dagger(\mathbf{r})\Big[\iiint \tilde{\rho}(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'-\tilde{\rho}(\mathbf{r})\Big]\,dV. \quad (15.25)$$

We then find

$$J(\tilde{\rho})=\iiint \rho^\dagger(\mathbf{r})q(\mathbf{r})\,dV+\iiint \delta\rho^\dagger(\mathbf{r})\Big[\iiint \delta\rho(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'-\delta\rho(\mathbf{r})\Big]\,dV. \quad (15.26)$$

On the other hand, using $\delta\rho(\mathbf{r})$ as the trial function in (15.9), we find on the assumption that $\iiint \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})\,dV$ converges (it is positive, since $\delta\rho^\dagger(\mathbf{r})=c(\mathbf{r})\delta\rho(\mathbf{r})/l(\mathbf{r})$):

$$\iiint \delta\rho^\dagger(\mathbf{r})\iiint \delta\rho(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'\,dV \leqslant (1/\gamma_0)\iiint \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})\,dV, \quad (15.27)$$

where γ_0 is the lowest eigenvalue of the corresponding homogeneous equation.

The formula (15.9) has, it is true, been derived rigorously only for finite systems. However, (15.27) may be extended to infinite systems, since (15.9) gives for finite systems (denoting by γ_{0M} the lowest eigenvalue for a system of radius M):

$$\int\limits_{|\mathbf{r}|<M}\int\limits_{|\mathbf{r}|<M}\int\limits_{|\mathbf{r}|<M} \delta\rho^\dagger(\mathbf{r})\int\limits_{|\mathbf{r}'|<M}\int\limits_{|\mathbf{r}'|<M}\int\limits_{|\mathbf{r}'|<M} \delta\rho(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'\,dV \leqslant \frac{1}{\gamma_{0M}}\int\limits_{|\mathbf{r}|<M}\int\limits_{|\mathbf{r}|<M}\int\limits_{|\mathbf{r}|<M} \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})\,dV.$$

If $\rho_{0M}(\mathbf{r})$ is the 'lowest eigenfunction' for a finite system of radius M, then, by using in (15.9) the trial function $\tilde{\rho}(\mathbf{r})=\rho_{0M}(\mathbf{r})$ for $|\mathbf{r}|<M$, $=0$ for $|\mathbf{r}|>M$, we find that $1/\gamma_{0M}<1/\gamma_0$, so that the above inequality still holds if $1/\gamma_{0M}$ is replaced by $1/\gamma_0$. Since the integral $\iiint \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})dV$ is assumed absolutely convergent, we can pass to the limit $M\to\infty$, and this gives (15.27)

A comparison of (15.26) and (15.27) shows that for $\gamma_0 \geqslant 1$ (critical and subcritical systems) we have

$$\max J(\tilde{\rho})=\iiint \rho^\dagger(\mathbf{r})q(\mathbf{r})\,dV=J(\rho) \qquad (\gamma_0\geqslant 1), \quad (15.28)$$

and this confirms the first statement about the functional $J(\tilde{\rho})$ (15.23). For supercritical systems, however, (15.28) does not hold, and $J(\tilde{\rho})$ can be made as large as we please if $\delta\rho(\mathbf{r})$ is sufficiently large and nearly proportional to the 'lowest eigenfunction'.

The relation (15.26) also shows that, if

$$\tilde{\rho}(\mathbf{r})-\rho(\mathbf{r}) = O(\epsilon),$$

then

$$J(\tilde{\rho})-J(\rho) = O(\epsilon^2), \tag{15.29}$$

and this confirms the second statement about the functional (15.23).

To complete the proof of (15.28) and (15.29) it is still necessary to justify (15.25) and, if $\iiint \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})\,dV$ diverges, to test the applicability of (15.27). This is easily done for finite systems. Firstly, the convergence of (15.22) implies that

$$\iiint \rho^\dagger(\mathbf{r})\rho(\mathbf{r})\,dV \tag{15.30}$$

converges, since the divergence of the latter can be due only to singularities of $\rho(\mathbf{r})$, which in turn must be due to singularities of $q(\mathbf{r})$ which would imply the divergence of (15.22), since the leading terms of $\rho(\mathbf{r})$ and $q(\mathbf{r})$ at such singularities would be the same.

Next, the integral

$$\iiint \rho^\dagger(\mathbf{r}) \iiint K(\mathbf{r}' \to \mathbf{r})\rho(\mathbf{r}')\,dV'\,dV$$

must also converge absolutely, since (15.30) has a non-negative integrand and therefore converges absolutely, if at all.

It is clearly rational to use only trial functions such that the convergence of these integrals is as good as the convergence of those for the exact solution, and so

$$\iiint \tilde{\rho}^\dagger(\mathbf{r}) \iiint K(\mathbf{r}' \to \mathbf{r})\delta\rho(\mathbf{r}')\,dV'\,dV$$

will also converge absolutely. We can therefore interchange the order of integration, and since by (15.2), (15.6), etc.,

$$\tilde{\rho}^\dagger(\mathbf{r})K(\mathbf{r}' \to \mathbf{r})\delta\rho(\mathbf{r}') = \delta\rho^\dagger(\mathbf{r}')K(\mathbf{r} \to \mathbf{r}')\tilde{\rho}(\mathbf{r}),$$

we at once obtain (15.25).

It also follows from these arguments that for a finite system

$$\iiint \delta\rho^\dagger(\mathbf{r})\delta\rho(\mathbf{r})\,dV$$

converges, and this justifies (15.27). The proof of (15.28) and (15.29) is now complete for finite systems.

15.4.2. *Infinite systems. The edge of the continuous spectrum*

We now consider systems which extend to infinity. If in all regions which extend to infinity the capture is non-negligible ($c < 1$), then the analysis is the same as for finite systems, provided that $q(\mathbf{r})$ is quadratically integrable and there is no supply of neutrons from infinity. In these conditions, the neutron flux $\rho(\mathbf{r})$ in each region for which $c < 1$ will consist of a part proportional to $q(\mathbf{r})$ and a part which behaves, for

$|\mathbf{r}| \to \infty$, approximately as $\exp(-|\mathbf{r}|/L)$, where L is the diffusion length in the medium concerned. Since $q(\mathbf{r})$ is quadratically integrable, so is $\rho(\mathbf{r})$, and therefore (15.30) still converges, and consequently (15.25) and (15.27) hold.

If there is a supply of neutrons from infinity, this is equivalent to a source at an infinite distance; $q(\mathbf{r})$ in (15.21) therefore does not represent all the physically relevant sources, and (15.28) and (15.29) cannot be expected to be valid.

In the remaining case, where non-capturing media extend to infinity, the homogeneous equation (15.1) has a bounded stationary solution for any $\gamma > 1$, so that $\gamma = 1$ is the edge of the continuous spectrum.† As before, we are chiefly interested in the solution of (15.21) which does not involve a supply of neutrons from infinity. This means that $\rho(\mathbf{r})$ is bounded at infinity; but, since for a non-capturing medium the current is zero for any constant flux, $\rho(\mathbf{r})$ need not vanish at infinity. The integral (15.30) therefore diverges in general, which complicates the analysis; we shall, however, discuss this case further, since it is one of the most important applications of the variational method. In doing so, we assume that

(i) the corresponding homogeneous equation has no non-trivial solution bounded at infinity;
(ii) for any absolutely integrable $q(\mathbf{r})$, (15.21) has a solution bounded at infinity, which is non-negative if $q(\mathbf{r})$ is non-negative;
(iii) the system is not supercritical, i.e. (15.1) has no eigenvalues less than unity.

In view of (i), the bounded solution referred to in (ii) is unique, and our earlier remarks about non-capturing media imply that this solution is the one which involves no supply of neutrons from infinity, i.e. the solution required.

If the flux increases indefinitely with $|\mathbf{r}|$, there must be a supply of neutrons from infinity, and so the assumption (i) implies that in the absence of such a supply the homogeneous equation has no non-trivial solution, that is, the system is 'effectively subcritical', and $\gamma = 1$ is not an eigenvalue, even though it is the edge of the continuous spectrum.

† For simplicity, we disregard the case where (for instance) a rod or a slab of finite thickness extends to infinity. In such cases the effect of a non-capturing body is approximately the same as that of a slightly larger body with a small capture. It is also possible that such a rod (etc.) can be multiplying ($c > 1$) without rendering the problem physically meaningless. In the remainder of this section we consider only systems such that $\gamma = 1$ is the edge of the continuous spectrum; this replaces the condition $c(\infty) = 1$.

The solution for a system which is subcritical in the strict sense, with no supply of neutrons from infinity, can always be obtained by means of the Neumann series

$$\rho(\mathbf{r}) = q(\mathbf{r})+\Lambda q(\mathbf{r})+\Lambda^2 q(\mathbf{r})+..., \tag{15.31}$$

where

$$\Lambda\phi(\mathbf{r}) = \iiint K(\mathbf{r}'\to\mathbf{r})\phi(\mathbf{r}')\,dV'. \tag{15.32}$$

The arguments of the preceding paragraphs suggest that (15.31) is true in the present case also, and this may be proved as follows. Since $q(\mathbf{r})$ is assumed absolutely integrable, it is sufficient to consider only non-negative $q(\mathbf{r})$, and then $\rho(\mathbf{r})$ is also non-negative, by assumption (ii). Iterating equation (15.21) n times, we have

$$\rho(\mathbf{r}) = \Lambda^n\rho(\mathbf{r})+\sum_{m=0}^{n-1}\Lambda^m q(\mathbf{r}),$$

and since $K(\mathbf{r}'\to\mathbf{r})$ is also non-negative, this implies that

$$\rho(\mathbf{r}) \geqslant \sum_{m=0}^{n-1}\Lambda^m q(\mathbf{r}).$$

Thus, for fixed $\mathbf{r}$, the partial sums of the series $\sum\Lambda^m q(\mathbf{r})$ are bounded, and the series therefore converges, since its terms are positive. The series clearly satisfies (15.21), and nowhere exceeds $\rho(\mathbf{r})$; since $\rho(\mathbf{r})$ is the only bounded solution, the sum of the series in question must be identical with $\rho(\mathbf{r})$. This proves (15.31) for the present case.

We now return to (15.25) and (15.26). We write for brevity

$$\left.\begin{aligned} &\tilde{q}(\mathbf{r}) = \tilde{\rho}(\mathbf{r})-\Lambda\tilde{\rho}(\mathbf{r});\ \delta q(\mathbf{r}) = \delta\rho(\mathbf{r})-\Lambda\delta\rho(\mathbf{r}),\ \text{etc.}\\ &\text{and}\\ &(\phi_1,\phi_2) = \iiint \phi_1^\dagger(\mathbf{r})\phi_2(\mathbf{r})\,dV = \iiint \{c(\mathbf{r})/l(\mathbf{r})\}\phi_1(\mathbf{r})\phi_2(\mathbf{r})\,dV \end{aligned}\right\}, \tag{15.33}$$

and introduce the following additional assumption:

(iv) the trial functions used are such that the corresponding $\tilde{q}(\mathbf{r})$ (see (15.33)) is absolutely and quadratically integrable; this implies that $\tilde{\rho}(\mathbf{r})$ can be reconstructed from $\tilde{q}(\mathbf{r})$ by (15.31).

The actual free term $q(\mathbf{r})$ of the equation to be solved possesses the same properties, because of the assumptions made previously. We have assumed the convergence of (15.22), and this implies (see remark after (15.30)) that $q(\mathbf{r})$ has no singularity capable of causing (q,q) to diverge; the latter then converges by the absolute integrability of $q(\mathbf{r})$ (assumption (ii)). Since $\rho(\mathbf{r})$ is bounded at infinity, it follows that (15.22) converges absolutely. It can be deduced from assumption (iv), by a similar argument, that $(\tilde{q},\tilde{\rho})$ converges absolutely.

Replacing now $\delta\rho(\mathbf{r})$ in (15.27) by $q(\mathbf{r})$, we see that $(q, \Lambda q)$ converges absolutely, since (15.27) is valid whenever the right side converges. Next, if we start with the functional $I_n(\tilde{\rho}) = (\tilde{\rho}, \Lambda^n\tilde{\rho})/(\tilde{\rho}, \tilde{\rho})$ instead of $I(\tilde{\rho}) = (\tilde{\rho}, \Lambda\tilde{\rho})/(\tilde{\rho}, \tilde{\rho})$ (15.5), and proceed as in the derivation of (15.27), we can show that, if (q, q) converges, each of the integrals $(q, \Lambda^n q)$ converges absolutely. Further, $(q, \Lambda^{n+m}q) = (\Lambda^n q, \Lambda^m q)$, and by assumption (iv) all the above conclusions hold when $q(\mathbf{r})$ is replaced by $\tilde{q}(\mathbf{r})$. Using Schwarz's inequality, we see that the integrals $(\Lambda^n q, \Lambda^m\tilde{q})$ also converge absolutely. All the conditions of Hopf's lemma (6.68) are therefore satisfied, and so

$$(\tilde{\rho}, \delta q) = (\delta q, \sum \Lambda^n\tilde{q}) = (\tilde{q}, \sum \Lambda^n \delta q) = (\tilde{q}, \delta\rho), \tag{15.34}$$

which is (15.25). That is, (15.25) holds whenever the assumptions (i)–(iv) are satisfied, whether or not (ρ, ρ) converges; and then (15.26) follows.

It is possible to prove (15.34) without using Hopf's lemma. To do so, we start with non-negative δq and $\tilde{q}$ and show the permissibility of interchanging the order of summation and integration; then we represent any δq and $\tilde{q}$ as the differences of non-negative functions and use the absolute convergence of all the integrals involved.

It is necessary to examine the sign of the second term in (15.26). This cannot be done by means of (15.27) as for finite systems, since the integral $(\delta\rho, \delta\rho)$ may not now converge. However, if we use the definition of $\delta q(\mathbf{r})$ (15.33) and take $\delta\rho(\mathbf{r})$ from (15.31), and prove the permissibility of interchanging summation and integration as for Hopf's lemma, we can rewrite (15.26) as

$$J(\tilde{\rho}) = (\rho, q) - \sum_{n=0}^{\infty} (\delta q, \Lambda^n \delta q). \tag{15.35}$$

The absence of negative eigenvalues γ implies that $(\delta q, \Lambda^n \delta q)$ is positive. The sum $\sum(\delta q, \Lambda^n \delta q)$ is therefore non-negative, and (15.28) follows.

15.4.3. *An alternative functional*

Instead of $J(\tilde{\rho})$, a closely related functional $H(\tilde{\rho})$ is sometimes used; it is obtained as follows. We take the trial function in $J(\tilde{\rho})$ in the form

$$\tilde{\rho}(\mathbf{r}) = Af(\mathbf{r}), \tag{15.36}$$

where A is an adjustable constant and $f(\mathbf{r})$ is normalized in some manner, and perform the variation of $\tilde{\rho}(\mathbf{r})$ in two stages, first varying A with $f(\mathbf{r})$ fixed, and then varying $f(\mathbf{r})$ with $A = A(f)$ already determined. Substituting (15.36) into (15.23), we have

$$J(Af) = 2AX(f) - A^2Y(f),$$

where $X(f)$ and $Y(f)$ are integrals depending on $f(\mathbf{r})$ but not on A. Making this expression a maximum with respect to A, we find

$$\max_A J(Af) = [X(f)]^2/Y(f) = [AX(f)]^2/A^2Y(f),$$

which, on replacing $AX(f)$ and $A^2Y(f)$ by their explicit forms, becomes

$$\max_A J(\tilde{\rho}) = \frac{\left[\iiint \tilde{\rho}^\dagger(\mathbf{r})q(\mathbf{r})\,dV\right]^2}{\iiint \tilde{\rho}^\dagger(\mathbf{r})\left[\tilde{\rho}(\mathbf{r}) - \iiint \tilde{\rho}(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'\right]dV} = H(\tilde{\rho}), \tag{15.37}$$

say. On comparing (15.37) with (15.28), we see that whenever the latter holds we have also

$$\max H(\tilde{\rho}) = \iiint \rho^\dagger(\mathbf{r})q(\mathbf{r})\,dV = H(\rho), \tag{15.38}$$

and, similarly to (15.29),

$$H(\tilde{\rho}) - H(\rho) = O(\epsilon^2) \quad \text{if} \quad \tilde{\rho}(\mathbf{r}) - \rho(\mathbf{r}) = O(\epsilon). \tag{15.39}$$

Thus the functional $H(\tilde{\rho})$ is applicable under the same conditions as $J(\tilde{\rho})$, and if the trial function has the form (15.36), it will give the same value of (ρ, q) and the same optimum shape (represented by $f(\mathbf{r})$) of the trial function $\tilde{\rho}(\mathbf{r})$.

If the trial function is linear in the adjustable parameters, i.e. if

$$\tilde{\rho}(\mathbf{r}) = \sum_i A_i f_i(\mathbf{r}), \tag{15.40}$$

where the $f_i(\mathbf{r})$ are chosen beforehand, then the maximization of the functional $J(\tilde{\rho})$ will clearly lead to a system of equations linear in the A_i. Kourganoff† (27, p. 142) considers this as a superiority of $J(\tilde{\rho})$ over $H(\tilde{\rho})$, but in fact the latter has the same property: after cancellation, the equation

$$\frac{\partial}{\partial A_i}\left\{\frac{\left[\sum_j X_j A_j\right]^2}{\sum_{j,k} Y_{jk} A_j A_k}\right\} = 0,$$

where the X_j and Y_{jk} are the appropriate integrals, reduces to

$$\sum_k Y_{ik} A_k \sum_j X_j A_j = X_i \sum_{j,k} Y_{jk} A_j A_k.$$

On applying these for all i and eliminating $\sum Y_{jk} A_j A_k$, we have

$$\left[X_i \sum_k Y_{ik} A_k - X_i \sum_k Y_{ik} A_k\right] \cdot \sum X_j A_j = 0,$$

and to satisfy these last equations we must put

$$\sum_j Y_{ij} A_j = C X_i, \tag{15.41}$$

† It should be noted that what we call $J(\tilde{\rho})$ is what Kourganoff calls $-H(\tilde{\rho})$, and our $H(\tilde{\rho})$ is the reciprocal of his $I(\tilde{\rho})$.

where C is an arbitrary common multiplier. (We reject solutions such that $\sum X_j A_j = 0$, since they lead to a minimum of $H(\tilde{\rho})$ and not a maximum.) Had we used the functional $J(\tilde{\rho})$, we should have obtained the same equations with $C = 1$, and the fact that C is arbitrary when $H(\tilde{\rho})$ is used is really an advantage.

The functional $H(\tilde{\rho})$ is homogeneous in $\tilde{\rho}(\mathbf{r})$ (see (15.37)), and consequently the value of (ρ, q) given by (15.38) is independent of C. If, therefore, we want $\rho(\mathbf{r})$ itself as well as (ρ, q), the latter can be determined first and C then chosen in the light of our knowledge of (ρ, q). This will be illustrated in the following section.

15.5. Examples of applications

15.5.1. *LeCaine's variational solution of Milne's problem*

As our first example of the use of the variational technique discussed above, we shall take LeCaine's solution (28) of Milne's problem for a non-capturing medium ($c = 1$). With the mean free path as unit of length, the corresponding integral equation (6.1) is

$$\rho(x) = \tfrac{1}{2} \int_0^\infty \rho(x') E_1(|x-x'|)\, dx'. \tag{15.42}$$

For $c = 1$, $\rho(x)$ is asymptotically linear, and to reduce the problem to one with no supply of neutrons from infinity we put

$$\rho(x) = \rho'(\infty)[x+\rho_1(x)], \tag{15.43}$$

where $\rho_1(x)$ is bounded at infinity. Substituting (15.43) into (15.42), we have

$$\rho_1(x) = \tfrac{1}{2} \int_0^\infty \rho_1(x') E_1(|x-x'|)\, dx' + \tfrac{1}{2} E_3(x), \tag{15.44}$$

where $E_3(x)$ is given by (10.59) with $n = 3$; and we shall be chiefly interested in $\rho_1(\infty)$. This may be found as follows. If $\rho(x_0 \to x)$ is the flux at x due to a plane source of unit strength at x_0, with no supply of neutrons from infinity, then the solution of (15.44) is

$$\rho_1(x) = \int_0^\infty \rho(x_0 \to x) \tfrac{1}{2} E_3(x_0)\, dx_0.$$

Interchanging x and x_0, applying the optical reciprocity theorem (4.26) in the form (6.77), and taking the limit as $x_0 \to \infty$, we have

$$\rho_1(\infty) = \tfrac{1}{2} \int_0^\infty E_3(x) \lim_{x_0 \to \infty} \rho(x_0 \to x)\, dx. \tag{15.45}$$

The quantity $\lim_{x_0 \to \infty} \rho(x_0 \to x)$ is clearly the $\rho(x)$ of (15.42), divided by

$|j(0)|$, and since, for $c = l = 1$, $\rho'(\infty) = 3|j(0)|$ (see (6.38)), we have from (15.43) and (15.45)

$$\rho_1(\infty) = \tfrac{3}{2}\int_0^\infty [x+\rho_1(x)]E_3(x)\,dx = \tfrac{3}{8}+3\int_0^\infty \rho_1(x).\tfrac{1}{2}E_3(x)\,dx. \quad (15.45')$$

The integral in (15.45') is of the same form as (15.22). The assumptions (i)–(iii) of § 15.4.2 are satisfied by the system considered, and (iv) is satisfied by any asymptotically constant trial function $\tilde{\rho}_1(x)$. Thus (15.38) gives

$$\rho_1(\infty) = \tfrac{3}{8}+\tfrac{3}{2}\max \frac{\left[\int_0^\infty \tilde{\rho}_1(x)E_3(x)\,dx\right]^2}{\int_0^\infty \tilde{\rho}_1(x)\left[2\tilde{\rho}_1(x)-\int_0^\infty \tilde{\rho}_1(y)E_1(|x-y|)\,dy\right]dx}, \quad (15.46)$$

provided that $\tilde{\rho}_1'(\infty) = 0$. If $\tilde{\rho}_1(x)$ is a constant, the formula (15.46) gives

$$\rho_1(\infty) \cong \tfrac{3}{8}+\tfrac{1}{3} = \tfrac{17}{24} = 0{\cdot}7083,$$

which deviates from the correct value (0·7104) by only 0·3 per cent. LeCaine took the trial function

$$\tilde{\rho}_1(x) = A+B_2E_2(x)+B_3E_3(x), \quad (15.47)$$

since this expression has the correct asymptotic behaviour at infinity

$$\tilde{\rho}_1(x) \sim \rho_1(\infty)+O(e^{-x}/x),$$

and the correct behaviour near the origin (a logarithmic singularity in the derivative). This trial function gave $\rho_1(\infty)$ correct to eight significant figures. Choosing the constant C in (15.41) to make A in (15.47) equal $\rho_1(\infty)$ from (15.46), and using the optimum values of B_2/A and B_3/A as determined by LeCaine, we find

$$\rho_1(x) = 0{\cdot}710446-0{\cdot}243608E_2(x)+0{\cdot}224409E_3(x). \quad (15.48)$$

At $x = 0$ this differs from the correct value $\rho_1(0) = 1/\sqrt{3}$ (6.47) by less than 0·3 per cent., and the error decreases extremely rapidly as x increases. Thus, though the variational method is primarily intended to give the value of the integral (15.22), it also gives a good approximation to the unknown function itself, in certain circumstances.

LeCaine has also extended these calculations to the case of $c < 1$ (29), and used the values of $\rho(x)$ obtained in order to construct the $\rho_q(x)$ of § 6.5. Sykes (46) has given a convenient form of the results for $1-c \ll 1$.

15.5.2. *Marshak's determination of the linear extrapolation length of a black sphere*

Marshak (36) has determined by the variational method the linear extrapolation length (see § 8.5) at the surface of a black sphere in an

infinite non-capturing medium. In this case the equation (4.15′) becomes, on integration over angles,

$$r\rho(r) = \tfrac{1}{2}\int_a^\infty dr'r'\rho(r')\{E_1(|r-r'|)-E_1[(r^2-a^2)^{\frac{1}{2}}+(r'^2-a^2)^{\frac{1}{2}}]\}, \tag{15.49}$$

where a is the radius of the black sphere, and we take $l = c = 1$ for the surrounding medium. It follows from (5.38) and the definition (6.24), since $\nabla^2\rho_{\text{diff}} = 0$ for $c = 1$, that at large distances the solution has the form

$$\rho(r) = \rho(\infty)[1-\{a^2/r(a+\lambda)\}+O(e^{-r})], \tag{15.50}$$

where λ is the linear extrapolation length in units of the mean free path. In view of (15.50), we put

$$r\rho(r) = \rho(\infty)[r-\rho_2(r)], \tag{15.51}$$

and our chief object will be to find $\rho_2(\infty)$. Substituting (15.51) into (15.49), we obtain

$$\left.\begin{aligned} \rho_2(r) &= \int_a^\infty \rho_2(r')K(r,r')\,dr' + q_2(r) \\ \text{where} \\ K(r,r') &= \tfrac{1}{2}\{E_1(|r-r'|)-E_1[(r^2-a^2)^{\frac{1}{2}}+(r'^2-a^2)^{\frac{1}{2}}]\} \\ q_2(r) &= \tfrac{1}{2}\{aE_2(r-a)+E_3[(r^2-a^2)^{\frac{1}{2}}]-E_3(r-a)\} \end{aligned}\right\} \quad (r > a). \tag{15.52}$$

It is convenient to extend the definitions of $\rho_2(r)$ and $q_2(r)$ to $r < a$ by putting

$$\left.\begin{aligned} \rho_2(r) &= \tfrac{1}{2}\int_a^\infty \rho_2(r')\{E_1(|r-r'|)-E_1(r+r')\}\,dr' \\ q_2(r) &= 0 \end{aligned}\right\} \quad (0 < r < a), \tag{15.53}$$

and to introduce the abbreviations

$$\left.\begin{aligned} \Lambda\phi(r) &= \tfrac{1}{2}\int_0^\infty \phi(r')\{E_1(|r-r'|)-E_1(r+r')\}\,dr' \\ \Delta_1\phi(r) &= \tfrac{1}{2}\int_0^a \phi(r')\{E_1(|r-r'|)-E_1(r+r')\}\,dr' \\ \Delta_2\phi(r) &= \begin{cases} \tfrac{1}{2}\int_a^\infty \phi(r')\{E_1[(r^2-a^2)^{\frac{1}{2}}+(r'^2-a^2)^{\frac{1}{2}}]-E_1(r+r')\}\,dr' & (r > a) \\ 0 & (r < a) \end{cases} \end{aligned}\right\}. \tag{15.54}$$

With this notation (15.52) and (15.53) become

$$\rho_2(r) = (\Lambda - \Delta_1 - \Delta_2)\rho_2(r) + q_2(r). \tag{15.55}$$

It is now necessary to express $\rho_2(\infty)$ in terms of the integral (15.22). The optical reciprocity theorem is here not very useful, and we shall use an alternative procedure. If we have an equation of the form

$$\phi(r) = \Lambda\phi(r) + \Lambda s(r), \tag{15.56}$$

where Λ is defined by (15.54), then its solution bounded at infinity can always be interpreted as r times the neutron flux in an infinite non-capturing medium due to sources of strength $s(r)/4\pi r$ per unit time, volume and solid angle (see § 8.2). For an infinite non-capturing medium, however, the integrated source strength must equal the outward current integrated over the surface of an infinitely large sphere. The current at large distances from the sources is, on our interpretation of $\phi(r)$,

$$\mathbf{j}(r) = -\tfrac{1}{3}\,\mathbf{grad}[\phi(r)/r].$$

This follows from (8.14), since $L^2(1-c)/l^2 = \frac{1}{3}$ for $c = 1$.

After some simple algebra, we have

$$\phi(\infty) = 3\int_0^\infty rs(r)\,dr,$$

or, since $\Lambda r = r$,

$$\phi(\infty) = 3\int_0^\infty r\Lambda s(r)\,dr. \tag{15.57}$$

This formula could have been used instead of the optical reciprocity theorem to derive (15.45), as has been remarked by Marshak (36).

Applying (15.57) to (15.55), we find

$$\rho_2(\infty) = 3\int_0^\infty rq_2(r)\,dr - 3\int_0^\infty r\Delta_1\rho_2(r)\,dr - 3\int_0^\infty r\Delta_2\rho_2(r)\,dr. \tag{15.58}$$

The first term on the right of (15.58) is easily found to be $\frac{3}{4}a^2$. The second term, on interchanging the order of integration, becomes $\int_0^a r\rho_2(r)\,dr$. Substituting from (15.53) and combining with the third term, we obtain

$$\rho_2(\infty) = \tfrac{3}{4}a^2 - 3\int_a^\infty \rho_2(r)q_2(r)\,dr. \tag{15.59}$$

We now return to (15.52). Though the quantity $\rho(r)$ is bounded at infinity, the homogeneous equation corresponding to (15.52) is an equation for $r\rho(r)$, and the latter cannot be bounded at infinity unless it vanishes identically. The kernel $K(r, r')$ is not of the form discussed in § 15.4.2, but it is symmetrical and positive, and $\int_0^\infty K(r, r')\, dr'$ is less than unity for any finite r. The kernel therefore possesses the properties which were used in § 15.4.2, and thus (15.28) holds if the trial function $\tilde{\rho}_2(r)$ is asymptotically constant (assumption (iv)). Hence, expressing $\rho_2(\infty)$ in terms of λ by (15.50), we have finally

$$\frac{a^2}{a+\lambda} = \frac{3a^2}{4} - 3 \max \frac{\left[\int_a^\infty \tilde{\rho}_2(r) q_2(r)\, dr\right]^2}{\int_a^\infty \tilde{\rho}_2(r)\left[\tilde{\rho}_2(r) - \int_a^\infty \tilde{\rho}_2(r') K(r, r')\, dr\right] dr'}, \tag{15.60}$$

provided that $\tilde{\rho}_2'(\infty) = 0$.

Marshak used the trial function $\tilde{\rho}_2(r) =$ constant. Even this gave values of λ in excellent agreement with those found by Davison (12), who used a combination of the perturbation and spherical harmonics methods.

15.5.3. *The neutron flux near a free surface*

The two foregoing examples were concerned with systems extending to infinity. Although the variational method has not yet been much applied to inhomogeneous equations for finite systems, we shall give two instances where its use would be very profitable. We shall also illustrate the precautions which are sometimes necessary.

We first consider a bare plane slab of critical thickness; the problem is to find the behaviour of the neutron flux $\rho(x)$ near the free surfaces $x = \pm a$. We know from previous work (6.54) that the derivative of the flux has a logarithmic singularity there, that is, near $x = a$ (for instance)

$$\frac{d\rho}{dx} = K_0 \log(a-x) + K_1 + O[(a-x)\log(a-x)], \tag{15.61}$$

and the formula (6.54) shows how to determine K_0. If the value of K_1 is also needed, we can proceed as follows. Since $\rho(x)$ must be an even function of x, equation (4.15) for this case can be written:

$$\rho(x) = \frac{c}{2} \int_0^a \rho(x')[E_1(|x-x'|) + E_1(x+x')]\, dx', \tag{15.62}$$

where the mean free path is again the unit of length. Differentiating this equation,

$$\frac{d\rho(x)}{dx} = \frac{c}{2}\int_0^a \frac{d\rho(x')}{dx'}[E_1(|x-x'|)-E_1(x+x')]\,dx' - \frac{c}{2}[E_1(a-x)-E_1(a+x)]\rho(a), \tag{15.63}$$

so that the quantity

$$\lim_{x\to a}\left\{\frac{d\rho(x)}{dx}+\frac{c}{2}\rho(a)[E_1(a-x)-E_1(a+x)]\right\}$$

(and hence also K_1 in (15.61)) is directly expressible in terms of the integral (15.22) for the equation (15.63). The applicability of (15.28) is shown by the fact that the homogeneous equation corresponding to (15.63) is the same as that for r times the neutron flux in a sphere (§ 8.2). Since the lowest eigenvalue for a sphere of radius a is greater than that for a slab of the same half-thickness, the system where the neutron flux is governed by (15.63) is subcritical. Thus (15.28) is valid.

We next consider the same problem for a bare sphere. It might seem that the transformation given in § 8.2 could be used, and then the method for the plane case could be followed. However, the equation (15.62) then becomes

$$r\rho(r) = \frac{c}{2}\int_0^a r'\rho(r')[E_1(|r-r'|)-E_1(r+r')]\,dr', \tag{15.62'}$$

while (15.63) is replaced by

$$\frac{d}{dr}[r\rho(r)] = \frac{c}{2}\int_0^a \frac{d}{dr'}[r'\rho(r')][E_1(|r-r'|)+E_1(r+r')]\,dr' - \\ -\frac{c}{2}a\rho(a)[E_1(a-x)+E_1(a+x)]. \tag{15.63'}$$

The homogeneous equation corresponding to (15.63′) is (15.62). Since the lowest eigenvalue for a slab is smaller than that for a sphere, the equation (15.63′) (with the value of c determined from (15.62′)) governs the neutron flux in a supercritical system, and the formula (15.28) is therefore inapplicable.

This difficulty may be avoided by using an alternative equation for $d\rho(r)/dr$, such that its lowest eigenvalue is greater than that of (15.62′). Such an equation may be constructed as follows. We start with the

three-dimensional equation for $\rho(r)$ and rewrite it for convenience as

$$\rho(r) = \frac{1}{4\pi} \iiint_{\text{all space}} c(r')\rho(r')e^{-|\mathbf{r}-\mathbf{r}'|} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2},$$

where $c(r') = c$ for $r' < a$, $= 0$ for $r' > a$. Differentiating this with respect to a Cartesian coordinate (x, say), we have

$$\frac{x}{r}\frac{d\rho(r)}{dr} = \frac{1}{4\pi} \iiint_{\text{all space}} \frac{x'}{r'}\frac{d}{dr'}[c(r')\rho(r')]e^{-|\mathbf{r}-\mathbf{r}'|} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}.$$

Integrating over angles, cancelling x/r and using $dc(r')/dr' = -c\delta(r'-a)$, we obtain

$$r\frac{d\rho(r)}{dr} = \frac{c}{2}\int_0^a r'\frac{d\rho(r')}{dr'} K_p(r,r')\,dr' - \frac{c}{2}a\rho(a)K_p(r,a), \qquad (15.64)$$

where

$$K_p(r,r') = \int_{|r-r'|}^{r+r'} \frac{r^2+r'^2-R^2}{2rr'}\frac{e^{-R}}{R}\,dR$$

$$= \frac{r^2+r'^2}{2rr'}\{E_1(|r-r'|)-E_1(r+r')\} - \frac{1+|r-r'|}{2rr'}e^{-|r-r'|} + \frac{1+r+r'}{2rr'}e^{-(r+r')}.$$

The equation (15.64) is of the required form, since the quantity

$$\lim_{r\to a}\left[r\frac{d\rho(r)}{dr} + \frac{c}{2}a\rho(a)K_p(r,a)\right],$$

and therefore also the constant K_1 in the formula corresponding to (15.61), are expressible directly in terms of the integral (15.22) for the equation (15.64). The lowest eigenvalue c for the homogeneous equation corresponding to (15.64) is for a flux positive in one hemisphere and negative in the other, and will certainly be greater than the eigenvalue for a flux positive throughout the sphere. That is, (15.64) may be interpreted as the equation governing the neutron flux in a subcritical system, and then (15.28) is applicable.

15.6. Miscellaneous remarks

In all the examples given above, we have dealt with simple systems. For more complex systems, the integrals involved rapidly become very unwieldy. The technique described in § 15.3 is useful in some circumstances, but it has not been applied to inhomogeneous equations, and in general the variational method is successful only when a very accurate result is required for a fairly simple system.

It has been mentioned in § 15.4 that the variational method is chiefly intended to give a single general characteristic of the solution (for a homogeneous equation, the lowest eigenvalue; for an inhomogeneous equation, the integral (15.22)) rather than the behaviour of $\rho(\mathbf{r})$. However, we have seen that in the examples considered, particularly that of § 15.5.1, a good approximation to $\rho(\mathbf{r})$ itself is sometimes obtained by means of the variational method.

Several alternative techniques have been suggested for the latter purpose. The one which has been expounded above was suggested by Placzek and developed by Marshak, LeCaine, and Davison. Another has been used by Kourganoff (27), who considered the case $c = 1$. For this case, the Boltzmann equation (4.4) gives, on integration over angles, a first-order differential equation for the current $\mathbf{j}(\mathbf{r})$. If the system is so symmetrical that $\mathbf{j}(\mathbf{r})$ depends on only one position coordinate, this equation can be integrated at once, and thus $\mathbf{j}(\mathbf{r})$ is known. On the other hand, for any assumed neutron flux $\tilde{\rho}(\mathbf{r})$, the corresponding current can be determined from (4.17). Kourganoff's method consists in minimizing the mean square deviation of this from the correct $\mathbf{j}(\mathbf{r})$.

This technique, however, is less convenient than the one which we have discussed previously, and does not give the same accuracy for the same effort. This is not surprising, as it makes less use of information about the nature of the exact solution. Other alternative techniques appear even less promising, and we shall not consider them here.

It may be asked whether the integral (15.22) is the only general characteristic of the solution of an inhomogeneous equation which is given to accuracy $O(\epsilon^2)$, or whether some other functional would give another expression to the same accuracy. No such functional has yet been discovered. If, however, it is required to determine, instead of (15.22), the integral

$$(\rho, q_1) = \iiint \rho^\dagger(\mathbf{r}) q_1(\mathbf{r})\, dV, \tag{15.65}$$

where $q_1(\mathbf{r})$ is some function other than the free term of (15.21), it is sometimes possible to proceed as follows. Let the system be finite and subcritical, and suppose that $q(\mathbf{r})$ (the free term of (15.21)) and $q_1(\mathbf{r})$ are quadratically integrable; let $\rho_1(\mathbf{r})$ be the solution of (15.21) with $q(\mathbf{r})$ replaced by $q_1(\mathbf{r})$. Similarly to (15.25) we have $(\rho, q_1) = (\rho_1, q)$, and hence

$$(g\rho+\rho_1, gq+q_1)-(g\rho-\rho_1, gq-q_1) = 4g(\rho, q_1), \tag{15.66}$$

where g is any constant. If (ρ_1, q_1) is comparable with $g^2(\rho, q)$, and the trial functions $\tilde{\rho}(\mathbf{r})$ and $\tilde{\rho}_1(\mathbf{r})$ differ from the correct solutions by a

quantity of order ϵ, then (15.65) is given correct to order ϵ^2 by determining $(g\rho+\rho_1, gq+q_1)$ and $(g\rho-\rho_1, gq-q_1)$ from (15.28). Unless the ratio $(\rho_1, q_1)/(\rho, q)$ can be estimated beforehand, the best choice of g is given by

$$(q_1, q_1) = g^2(q, q). \tag{15.67}$$

However, there are two limitations which seriously restrict the usefulness of (15.66). Firstly, even if g is taken from (15.67), it is not always certain that (ρ_1, q_1) will be comparable with $g^2(\rho, q)$; secondly, (15.66) involves the subtraction of two quantities determined by the variational method, and hence the sign of the error in (ρ, q_1) is not known.

XVI

THE ITERATION AND MONTE CARLO METHODS

16.1. The iteration method: a general survey

THE methods described in the last two chapters (the perturbational and variational methods) were chiefly concerned with finding some general characteristic of the system, such as the critical size of a source-free system. The information they gave about the actual distribution of neutrons was not so accurate as that obtainable by other methods. We now consider the iteration method, whose main object is the contrary one of determining the neutron distribution itself. The basis of the method is as follows. Let the equation to be solved be, for instance, (15.21), and define for any trial solution $\tilde{\rho}(\mathbf{r})$ the function

$$\tilde{\rho}_{\mathrm{It}}(\mathbf{r}) = \iiint K(\mathbf{r}' \to \mathbf{r})\tilde{\rho}(\mathbf{r}')\, dV' + q(\mathbf{r}). \tag{16.1}$$

Let the system be finite and subcritical, the exact solution of (15.21) being $\rho(\mathbf{r})$, and the ith eigenfunction of the corresponding homogeneous equation being $\rho_i(\mathbf{r})$. We assume that the $\rho_i(\mathbf{r})$ form a complete set. The trial function $\tilde{\rho}(\mathbf{r})$ can then be represented as

$$\left.\begin{aligned} \tilde{\rho}(\mathbf{r}) &= \rho(\mathbf{r}) + \textstyle\sum a_i \rho_i(\mathbf{r}) \\ \text{and hence} \quad \tilde{\rho}_{\mathrm{It}}(\mathbf{r}) &= \rho(\mathbf{r}) + \textstyle\sum (a_i/\gamma_i)\rho_i(\mathbf{r}) \end{aligned}\right\}. \tag{16.2}$$

Since for a subcritical system all the γ_i are greater than unity, a comparison of the last two formulae shows that the *iterated solution* $\tilde{\rho}_{\mathrm{It}}(\mathbf{r})$ differs from the exact solution by less than the trial function $\rho(\mathbf{r})$. Consequently, a repeated application of (16.1) will improve the accuracy of the solution indefinitely. It can be shown similarly that for a critical source-free system the iterated solution is a better approximation to the exact solution than is the original trial function. The same arguments can be extended to the case where the $\rho_i(\mathbf{r})$ do not form a complete set, and to the case of infinite systems, in the same way as for the variational method. For supercritical systems the formulae (16.2) show that the iterated solution may differ from the exact solution by more than the initial trial function, but the stationary solutions for supercritical systems are never of direct interest in problems of neutron transport.

The iteration may be performed either numerically or analytically, but we shall discuss only analytic iteration. If the kernel and the trial function are sufficiently simple, the first iterated solution may be

expressible in terms of tabulated functions. It is very unusual, however, for this to be true of the second and higher iterated solutions, and this effectively limits analytical work to one iteration only. In order to obtain the best results, we should therefore use a trial function which is itself a good approximation. Further, if the deviations of the trial function from the exact solution are very localized, these will appear as a contribution chiefly from high-order eigenfunctions (i.e. from high values of γ_i), and the formulae (16.2) show that such deviations are reduced by iteration much more than smoother deviations which are expressible by means of the low-order eigenfunctions.

It follows that the best trial functions for use with the iteration method are the solutions obtained by one of the methods discussed previously, namely, the diffusion approximation, the spherical harmonics method, the variational method (Placzek's form), etc. These solutions are given in terms of relatively simple functions, so that it is not excessively laborious to perform one iteration, and they deviate appreciably from the exact solution only in the immediate neighbourhood of the boundaries, i.e. in restricted regions. The improvement that can be achieved by a single iteration should therefore be significant, especially near the boundaries. For instance, iteration of LeCaine's variational solution of Milne's problem (15.48) reduces the error at the free surface from 0·3 per cent. to 0·02 per cent.

16.2. The combination of the iteration method with the discrete ordinates method or the spherical harmonics method

We have already encountered the iteration method, in effect, in combination with the method of discrete ordinates in § 13.5; for, in deriving (13.25), we approximated the neutron flux $\rho(x) \equiv \psi_0(x)$ in the Boltzmann equation (10.1) by the trial function (13.24), and integrated the resulting differential equation. In order to complete the iteration in the sense of (16.1), it is sufficient to integrate (13.25) over all μ. This gives

$$\tilde{\rho}_{\mathrm{it}}(x) = 2\pi \int_{-1}^{1} \left[F(\mu) e^{-x/\mu} + \frac{c}{2} \sum_s \frac{A_s}{1+\mu\gamma_s} e^{x\gamma_s/\mu} \right] d\mu, \qquad (16.3)$$

which, according to the arguments of § 16.1, should be a noticeably better approximation than (13.24) itself.

The integral (16.3) will usually be expressible in terms of tabulated functions. This may be seen as follows. Let $x = 0$ be the free surface, with the medium in $x > 0$. Then, for $\mu > 0$, we can rewrite (13.25) for

the medium adjacent to the free surface as

$$\psi(x,\mu) = \frac{c}{2}\sum_s \frac{A_s}{1+\mu\gamma_s}[e^{x\gamma_s/\mu} - e^{-x/\mu}],$$

where $F(\mu)$ has been determined from the appropriate boundary conditions (§ 13.3). The contribution to (16.3) in this medium which arises from $\mu > 0$ is therefore

$$c\pi \sum_s A_s e^{x\gamma_s/\mu} \int_1^\infty \frac{dt}{t}\frac{1}{\gamma_s+t}[1-e^{-x(\gamma_s+t)/\mu}],$$

where we have put $t = 1/\mu$. The integrals in this formula are easily expressed in terms of $E_1(z)$ by means of the relation

$$\int_1^\infty \frac{dt}{t}\frac{1-e^{-\xi(t+a)}}{t+a} = \frac{1}{a}[\log(a+1)+E_1\{\xi(a+1)\}-e^{-\xi a}E_1(\xi)].$$

The contributions from $\mu < 0$ and from more distant media can likewise be expressed in terms of $E_1(z)$.

The above discussion of the discrete ordinates method can be immediately extended to the spherical harmonics method. In the case of plane symmetry, for instance, the angular distribution is normally considered as given by (10.2) in this method. However, it is also possible to start with $\psi_0(x)$ as given by the P_N approximation of the spherical harmonics method, substitute it into the Boltzmann equation (10.1) and integrate, using the appropriate boundary conditions. The result is of the same general form as (13.25). This is the iterated angular distribution, and it is clearly a much better approximation to the exact result than the formula (10.2), in the P_N approximation. On integrating this iterated angular distribution over all angles, we get for the iterated neutron flux an expression of exactly the same form as (16.3). The same is true of other geometries, although the expressions for the iterated angular distribution and the iterated flux are more complicated. The iterated angular distribution in general is no longer elementary, though sometimes expressible in terms of tabulated functions, while the iterated flux may require a numerical integration.

16.3. Iteration depending on a parameter. Miscellaneous remarks

We have seen that the most profitable use of the iteration method is to improve by a single iteration (or sometimes only half an iteration) the results of other methods. Before the other methods were fully developed,

however, much attention was given to purely iterative processes. The problem is then to find from general considerations the best possible trial function to be submitted to a single iteration. The most interesting suggestion in this respect is that of iteration depending on a parameter. In this case, we start with a trial function $\tilde{\rho}(\mathbf{r})$ depending on one or more parameters; having iterated it, we choose the parameters, either so that $\tilde{\rho}_{\mathrm{It}}(\mathbf{r})$ coincides with $\tilde{\rho}(\mathbf{r})$ at some fixed point or points, or so that some integral or integrals of $\tilde{\rho}_{\mathrm{It}}(\mathbf{r})$ equal those of $\tilde{\rho}(\mathbf{r})$. Results of surprising accuracy have been obtained in this way. The limitations of the method are, firstly, the arbitrariness in the choice of points or integrals; secondly, that the results of solving a simple problem in this manner do not indicate how to proceed with more complex problems. We shall therefore not discuss this method in detail; a fairly complete survey of the results obtained is given by Kourganoff (27).

In one type of problem, however, these disadvantages are slight. Suppose that it is desired to determine the lowest eigenvalue and the corresponding eigenfunction of (15.1). We put, as in (16.1),

$$\tilde{\rho}_{\mathrm{It}}(\mathbf{r}) = \bar{\gamma}\Lambda\tilde{\rho}(\mathbf{r}), \tag{16.4}$$

where

$$\Lambda\tilde{\rho}(\mathbf{r}) = \iiint K(\mathbf{r}' \to \mathbf{r})\tilde{\rho}(\mathbf{r}')\, dV',$$

and $\bar{\gamma}$ is the trial value of γ_0; proceeding as in the derivation of (16.2), multiplying the result by $\rho_0^{\dagger}(\mathbf{r})$ and integrating, using the orthogonality relations (15.3), we have

$$(\rho_0, \tilde{\rho}_{\mathrm{It}})/(\rho_0, \tilde{\rho}) = \bar{\gamma}/\gamma_0,$$

in the notation of (15.33), and therefore, for $\bar{\gamma} = \gamma_0$, the integral $(\tilde{\rho}_{\mathrm{It}} - \tilde{\rho}, \rho_0)$ is zero. Since $\rho_0^{\dagger}(\mathbf{r})$ is non-negative $\tilde{\rho}_{\mathrm{It}}(\mathbf{r}) - \tilde{\rho}(\mathbf{r})$, i.e. $(1 - \gamma_0\Lambda)\tilde{\rho}(\mathbf{r})$, must change sign. On the other hand, $\tilde{\rho}(\mathbf{r})$ would clearly be taken non-negative, and since the kernel is non-negative, so is $\Lambda\tilde{\rho}(\mathbf{r})$. Thus $(\tilde{\rho}(\mathbf{r})/\Lambda\tilde{\rho}(\mathbf{r})) - \gamma_0$ changes sign, and γ_0 lies between the limits

$$\min\{\tilde{\rho}(\mathbf{r})/\Lambda\tilde{\rho}(\mathbf{r})\} \leqslant \gamma_0 \leqslant \max\{\tilde{\rho}(\mathbf{r})/\Lambda\tilde{\rho}(\mathbf{r})\}. \tag{16.5}$$

In using the method of iteration depending on a parameter, we should therefore select the parameter so that the ratio

$$\max\frac{\tilde{\rho}(\mathbf{r})}{\Lambda\tilde{\rho}(\mathbf{r})} \Big/ \min\frac{\tilde{\rho}(\mathbf{r})}{\Lambda\tilde{\rho}(\mathbf{r})} \tag{16.6}$$

is as small as possible.

16.4. The Monte Carlo method: a general survey

The methods discussed hitherto have been analytical methods, or at least largely so. We finally consider a purely numerical method, the so-

called *Monte Carlo method*. A detailed discussion of this method is beyond our scope, being rather a matter of statistical theory, and we shall give only a general outline of the method and some qualitative comments. In fact, Monte Carlo is very seldom advantageous in dealing with the constant cross-section approximation and isotropic scattering; it becomes useful when the variation of the cross-sections and the anisotropy of scattering have to be taken into account, and particularly when the energy variation of the cross-sections is extremely irregular. For simplicity, however, we shall first describe its application to the case of constant cross-sections and isotropic scattering.

The basic idea of the method is as follows. In the variational method we form a problem in the calculus of variations whose analytic solution would lead to the equation we actually wish to solve. This problem is then attempted by numerical methods or trial-and-error methods. In the Monte Carlo method we form a problem in probability whose analytic solution would lead to the equation we actually wish to solve. This problem is then attempted by an analogue experiment, using tables of random numbers, or a fast computing machine to produce such numbers.

Suppose that it is desired to find the lowest eigenvalue c_0 for a bare non-re-entrant homogeneous body. We first choose an integer or simple rational number c' which is somewhat greater than the expected value of c_0. Next, we divide the area under the curve $y = e^{-x/l}$ into n equal parts and choose a point on the curve over each part, i.e. we choose n lengths R_j $(j = 1, 2, ..., n)$ such that $l\log(n/j) < R_j < l\log(n/[j-1])$, where l is the mean free path. We also divide the directions Ω into m sectors each of solid angle $4\pi/m$, and choose a direction Ω_i $(i = 1, 2, ..., m)$ in each sector.

We now take a set of N_0 test particles distributed through the body in any manner. For each test particle, we first take a random number between 1 and m. If this number is i, then Ω_i is taken as the direction of motion of the test particle. We next take a random number between 1 and n. If this number is j, then R_j is taken as the distance travelled by the test particle. If the point distant R_j from the initial point in the direction Ω_i is outside the system, the particle is lost; if it is inside, the particle is regarded as having a collision at the point. If c' is integral, the particle is replaced by c' new ones, each with weight c/c', where c is a parameter to be determined, and each of these new test particles is treated like the original ones. If c' is not integral, a third random number is needed to decide whether the number of new particles is the integer

above or below c', but in either case the weight of each will be c/c'. When a test particle of the sth generation, of weight $(c/c')^s$, has a collision, the weight of each resulting particle is $(c/c')^{s+1}$. If N_s is the total number of particles in the sth generation, the total weight of these is

$$W_s = (c/c')^s N_s. \tag{16.7}$$

Let $w_s(\mathbf{r})\,dV$ be the expectation of the weight of the test particles of the sth generation in the volume element dV at $\mathbf{r}$. Then, if m and n are large, and sums are approximated by integrals, we obtain the same relation between $w_{s+1}(\mathbf{r})$ and $w_s(\mathbf{r})$ as between $\tilde{\rho}_{1t}(\mathbf{r})$ and $\tilde{\rho}(\mathbf{r})$ (see (16.4)). In practice one would normally take m and n between 10 and 100. This confirms that the procedure described is in fact an analogue experiment, provided that we deal with W_s rather than N_s, and shows that the Monte Carlo procedure may be regarded as one way of effecting the iteration. It follows from (16.2) that, for sufficiently large s, $w_s(\mathbf{r})$ is proportional to $\rho_0(\mathbf{r})$, and then W_s will oscillate about a constant value if c is chosen equal to c_0. If $c > c_0$, W_s will increase exponentially, and if $c < c_0$ it will decrease exponentially.

The procedure described above gives an idea of the Monte Carlo method as applied in its simplest form; to be of practical utility, it must be considerably improved. Problems of finding the value, corresponding to criticality, of some parameter are usually solved in Monte Carlo by estimating the multiplication rate per generation of the distribution of neutrons for certain values of the parameter, and then obtaining by interpolation an indirect estimate of the quantity required. In the example given in the previous paragraph, the parameter could be carried through the Monte Carlo calculation, and the necessity of several separate experiments was avoided. This is not always the case, however, for example in the determination of critical size.

Two stages can be seen to exist in the application of the Monte Carlo method to these problems. The first is to produce in the system a distribution of neutrons which is truly representative of the correct one. The second is to estimate from this the multiplication rate. The first stage is accomplished by the process of tracing neutron histories as described above. However, as in the method of finding the eigenvalues of a matrix by forming successive powers of the matrix, to which the above process bears a close analogy, much labour can be saved by starting the process with a distribution as near the correct one as possible. The second stage is accomplished, in the simple example above, by dividing the total weight of the neutrons present in the system at some generation by that

at the preceding generation. Much more efficient estimators than this can be found, and in some cases the expected weight of the neutrons can be easily calculated exactly.

Such methods as the above can, of course, be extended at once to re-entrant and composite systems, anisotropic scattering, and cross-sections varying with energy.

16.5. Comparison of the Monte Carlo and iteration methods

To examine the logical foundation of the Monte Carlo method, we shall compare it with the iteration method in somewhat greater detail. The former amounts to alternate iteration and replacement of the iterated solution $\tilde{\rho}_{\mathrm{It},s}(\mathbf{r})$ by an approximate expression of the form

$$\left(\frac{c}{c'}\right)^s \sum_{i=1}^{N_s} \delta(\mathbf{r}-\mathbf{r}_{i,s}). \tag{16.8}$$

This replacement introduces a certain random error whenever it is made. However, whereas ordinary iteration (whether analytical or numerical) is restricted to a few steps at most, the Monte Carlo method can be carried to a very high number of iterations, and this usually outweighs the disadvantage of introducing a random error at each step. The reason for this may be seen as follows. We expand the deviation of the initial distribution from the correct solution $\rho_0(\mathbf{r})$, and the additional random error introduced at each step, in a series of the form (16.2), and consider separately the contributions due to the lower and higher eigenvalues. The former we call the 'smooth deviation' and the latter the 'high harmonics'. It follows from (16.2) that the high harmonics are greatly reduced at each iteration and will disappear almost entirely after the first few iterations; moreover, they are often much reduced by numerical smoothing before iteration begins. The smooth deviation is therefore more important. On the other hand, the approximation of $\tilde{\rho}_{\mathrm{It},s}(\mathbf{r})$ by (16.8), although it may introduce an appreciable amount of the high harmonics, is very unlikely to add a noticeable amount of smooth deviation, and only that which is present in the initial distribution will affect the situation. However, since the Monte Carlo method can be used to obtain very high iterations, the smooth deviation in the initial distribution will also be suppressed. In fact, as regards the spatial distribution, it is relatively unimportant what the initial distribution is, though, as was mentioned above, such information as is available should naturally be used, so as to reduce the amount of subsequent labour necessary. In energy-dependent problems, however, the total energy

spectrum may take many generations to lose its initial smooth deviation ('bias'), and in these problems it is often profitable to do a certain amount of preliminary work to find this spectrum as accurately as possible.

16.6. Statistical fluctuations and the choice of c'

The method we have described is a statistical process, and therefore the number N_s of test particles in the sth generation is subject to statistical fluctuation. This, by the theory of probability, will be of the order of $\sqrt{N_s}$, and the probable relative error will be of the order of $1/\sqrt{N_s}$. In order to avoid a freak result, the initial number of particles N_0 should therefore be fairly large. Again, if c' is taken less than c_0, it follows from the discussion of (16.7) that N_s would tend to decrease, giving an increasing probable relative error. This is clearly undesirable; we should rather wish the accuracy and reliability of the results to increase with s, and this is why c' should be greater than c_0. In fact, we could start with a few neutrons and use these to get rid of the initial bias, later increasing the number of neutrons to improve the accuracy of the estimation.

16.7. Reduction of numerical work by using the symmetry of the system, etc.

In order to obtain reasonable results from Monte Carlo without great effort, it is necessary to eliminate all numerical work which does not contribute directly to the accuracy and reliability of the final result. This is true of all numerical calculation, but it is especially important in the present case. The following points are particularly to be noted:

(i) The samples should not be unnecessarily large. For instance, if the accuracy is limited by the test particle population in one part of the system, the population in other parts should not much exceed that corresponding to the given accuracy.

(ii) The chain of events traced should not be unnecessarily long. For instance, if the result of a series of events can be found analytically, this should be done.

(iii) Quantities which are irrelevant by the symmetry of the system should be ignored.

We first illustrate (iii). Suppose that the lowest eigenvalue for a bare sphere is required. For a spherically symmetrical system, the only relevant position coordinate is the radial one. Further, to find the next radial coordinate r_{s+1} from the coordinate r_s and the distance travelled R, we need only know the value of $\mu = \Omega \cdot \mathbf{r}_s/r_s$, and not Ω itself. The

division of the directions Ω into m sectors of equal solid angle is thus reduced to the division of the interval $(-1, 1)$ of μ into m segments, each of length $2/m$. The actual determination and recording of the successive r_s can then easily be done graphically, using a single sheet of paper which at each step is regarded as the plane through the origin, $\mathbf{r}_s$ and $\mathbf{r}_{s+1}$. This can also be expressed by saying that a spherically symmetric system with isotropic scattering is equivalent to a plane system with anisotropic scattering which is a function of position, a fact which is very useful in the Monte Carlo method, but has no particular value for the analytical methods.

16.8. Reduction of numerical work by the use of analytical methods

We next illustrate (ii) of the preceding section. The situation covered by this may arise, for instance, in the case of an inhomogeneous spherical core in an infinite homogeneous reflector, the Monte Carlo method being adopted because of the complexity of the core. This problem is quite frequent when the variation of cross-sections with energy has to be taken into account. If now a test particle enters the reflector, there may be a great many events before it returns to the core or is captured in the reflector. However, the probability of return to the core and the expected weight at that time are easily determined analytically by solving the problem of a point source outside a black sphere to find the current into the sphere. (The sphere is treated as black because a particle that enters it is returned to the Monte Carlo programme and is lost from the reflector.) The current thus obtained gives either the probability of return to the core or the product of the probability and the probable weight at the time of return, according as we use c_r or c'_r. If the core is spherically symmetrical, however complex, the position of the point of return is irrelevant, and then the point source can be replaced by a shell source. The diffusion approximation solution for a shell source round a concentric black sphere is quite simple, and if the source is far enough from the sphere this solution will be very accurate also.

Further, if the source is far enough from the sphere, the angular distribution of test particles entering the sphere will be nearly independent of the radius of the source shell, and can be determined once and for all, either analytically or by a Monte Carlo calculation.

To combine the results of these auxiliary calculations with the main Monte Carlo programme, the procedure is as follows. Let a be the radius of the core, and b the radius of a sphere such that $b-a$ is large enough,

compared with the mean free path in the reflector, to justify the use of the diffusion approximation. While the test particle remains in the sphere of radius b, we use the normal Monte Carlo procedure as given in §§ 16.4 and 16.7. When it leaves this sphere, we record the radial coordinate of the next point where it has a collision, and then find from the diffusion approximation solution the probability of its returning to the core. Whether the particle returns to the core, and, if so, the angle at which it does so, are decided by chance, knowing the probabilities concerned. The weight of the particle is given by the diffusion approximation, as described above. After this, we can revert to the normal Monte Carlo procedure until a test particle next leaves the sphere of radius b.

In § 16.4 we dealt with the total weight W_s of the test particles of the sth generation. The question therefore arises, to which generation we should allot a particle whose history between leaving the sphere of radius b and returning to the core was not followed in detail, but estimated as above. However, the time spent in the reflector and the number of collisions undergone there cannot have any effect on the stationary distribution in the core.† It is thus a matter of convenience to which generation we allot test particles returning to the core after leaving the sphere of radius b; the most simple and economical procedure is evidently to allot them to the next generation after the one in which they left that sphere. Other situations where (ii) of § 16.7 applies can be dealt with similarly.

† This follows, since if $\rho(\mathbf{r}, t) = \rho_0(\mathbf{r})$ is a stationary solution of

$$\rho(\mathbf{r}, t) = \iiint dV' \int_{-\infty}^{t} K_1(\mathbf{r}' \to \mathbf{r}, t-t')\rho(\mathbf{r}', t')\, dt',$$

where $\int_0^\infty K_1(\mathbf{r}' \to \mathbf{r}, t)\, dt = \int_0^\infty K_2(\mathbf{r}' \to \mathbf{r}, t)\, dt$, then it is also a stationary solution of

$$\rho(\mathbf{r}, t) = \iiint dV' \int_{-\infty}^{t} K_2(\mathbf{r}' \to \mathbf{r}, t-t')\rho(\mathbf{r}', t')\, dt'.$$

Thus, once the system has reached a stable state, any *census parameter* can be used to find whether it is subcritical, critical, or supercritical. That is, we may use neutrons which have travelled for a given time, or a given distance, or have undergone a given number of collisions, or etc. However, unless the neutrons are all of the same energy, the manner of determining the stationary distribution depends on the census parameter. It has been shown by Fermi and Richtmyer (**20**) that, if the census parameter is of the form

$$\tau = \int_0^t g[v(t)]\, dt,$$

where $g(v)$ is any function satisfying certain general conditions, the distribution observed at a census must be divided by $g(v)$ to obtain the actual stationary distribution.

16.9. Reduction of sample sizes. Importance sampling

We now consider point (i) of § 16.7, namely the necessity of avoiding large samples. Suppose we desire to find the thickness a of a slab of slightly capturing material with a uniform irradiation from $x < 0$ on to the surface $x = 0$, such that the neutron current out of the face $x = a$ is some fraction 10^{-p} of that into $x = 0$, where p is a positive integer. This problem is easily solved analytically in the constant cross-section approximation, but if the cross-section is variable with energy the Monte Carlo method may prove superior.

To obtain an answer of reasonable reliability, we should allow at least 100 test particles to traverse the slab. If no special device is used, it is therefore necessary to start with 10^{p+2} particles, which is quite impracticable even for $p = 3$. To avoid this difficulty, we divide the slab into p layers, bounded by $x = x_1$, $x = x_2$, etc. Starting with a test particle population of, say, 1,000 falling on the plane $x = 0$, we replace each particle by 10 particles every time it crosses a plane $x = x_j$ in the direction of x increasing, and decimate it by chance every time it crosses one of these planes in the direction of x decreasing. This will clearly make reasonable the size of the samples to be handled. Of course, the factor ten is used only for the sake of illustration; any other integer will serve equally well.

In this example the shape and position of the surfaces at which we change the number of particles are obvious, but this is not always so. However, these surfaces can easily be determined as a by-product of the Monte Carlo procedure. Let there be a subcritical system with a source at S, for which we require the neutron flux at a distant point P, the geometry being too complex for us to choose the surfaces *a priori*. Then we start with a reasonable number of particles at S and use the normal Monte Carlo procedure until the size of the sample is reduced by capture in a ratio decided beforehand. We then draw a convex surface through the furthest points reached at that time, and take this as the first surface at which the number of test particles is changed; the other surfaces are similarly determined.

This procedure has the disadvantage that the size of the sample is increased not only where it is necessary, but also in regions whence P is even less accessible than from S, and thereby we increase the work done without improving the accuracy and reliability of the answer. A better method is as follows. The solution of the integral equation at P due to the source at S is the same as the solution of the adjoint equation at S due to the source at P. This follows by comparing the Neumann

series for the two quantities term by term, since Neumann series may always be used for subcritical systems. The Monte Carlo method can be applied to the adjoint equation without difficulty; in fact, in the constant cross-section approximation the only difference is that the weights are adjusted by adding the factor $c(\mathbf{r})/l(\mathbf{r})$ where appropriate (cf. (14.3)).

We now start with a relatively small 'exploratory' sample at P and solve the adjoint equation in the manner previously explained. Let P_1, P_2, etc., be the consecutive surfaces about P at which we change the number of test particles, and let P_{k+1} enclose S, while P_k does not. We stop at P_k and begin from S with an ordinary 'working' sample, using the Monte Carlo method on the original equation, regardless of the depletion of the sample, until P_k is reached. We then change the number of particles and continue until P_{k-1} is reached, and so on. It is clear that no numerical effort is wasted in the second part of this procedure, and although there was some wastage in the construction of the surfaces P_j, the small size of the exploratory sample renders this unimportant compared with the wastage in the cruder method described previously.

Such a procedure is called *importance sampling*, and the solution of the adjoint equation thus constructed is called the *importance function*. It sometimes happens that this function can be found analytically to a rather rough approximation which is still accurate enough for this purpose. In this case the Monte Carlo method is used only for the main part of the calculation, and there is no wastage of numerical work.

If there is no analytical approximation available, it may be advantageous to use *repeated importance sampling*. Here we start with a very small sample at S and construct a very crude solution of the original equation, using this as an importance function for the solution of the adjoint equation. Such a procedure gives a more accurate importance function for the problem considered, and reduces the wastage of numerical work still further, while adding no great complication to the effort involved. It has, however, not been used in practice to any appreciable extent.

16.10. An alternative procedure

There is another possibility, besides importance sampling, which may be of use, though it has not yet been tried in practice. We first prove an auxiliary theorem.

Let the system be subcritical and the integral equation inhomogeneous, and let the entire domain of integration in (15.21) be divided into two

parts V_A and V_B. We consider the equations

$$\left.\begin{aligned}\rho_A(\mathbf{r}) &= \iiint_{V_A} \rho_A(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV' + q(\mathbf{r})\\ \rho_B(\mathbf{r}) &= \iiint_{V_A+V_B} \rho_B(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV' + \iiint_{V_B} \rho_A(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r})\,dV'\end{aligned}\right\}. \quad (16.9)$$

Adding these, we see that $\rho_A(\mathbf{r})+\rho_B(\mathbf{r})$ satisfies (15.21), and since the solution is unique for a subcritical system, we have

$$\rho(\mathbf{r}) = \rho_A(\mathbf{r})+\rho_B(\mathbf{r}). \quad (16.10)$$

Thus, if in a certain sub-region of V_B (V', say) $\rho_A(\mathbf{r})$ is negligible in comparison with $\rho_B(\mathbf{r})$, then in so far as the determination of $\rho(\mathbf{r})$ in V' is concerned we can replace the solution of (15.21) by that of the second of (16.9). This may be called the *displacement of sources theorem*.

If now it is desired to evaluate the integral

$$\iiint_{V_A+V_B} p(\mathbf{r})\rho(\mathbf{r})\,dV,$$

and if $\rho_f(\mathbf{r})$ is the solution of (15.21) with $f(\mathbf{r})$ instead of $q(\mathbf{r})$, and $\rho_p^\dagger(\mathbf{r})$ that of the adjoint equation with free term $p(\mathbf{r})$, then clearly

$$\iiint_{V_A+V_B} \rho_f(\mathbf{r})p(\mathbf{r})\,dV = \iiint_{V_A+V_B} \rho_p^\dagger(\mathbf{r})f(\mathbf{r})\,dV.$$

Choosing $\rho_f(\mathbf{r})$ as $\rho_B(\mathbf{r})$ and combining with (16.9) and (16.10), we obtain

$$\begin{aligned}&\iiint_{V_A+V_B} p(\mathbf{r})\rho(\mathbf{r})\,dV\\ &\quad= \iiint_{V_A+V_B} p(\mathbf{r})\rho_A(\mathbf{r})\,dV + \iiint_{V_B} dV' \iiint_{V_A+V_B} dV\rho_p^\dagger(\mathbf{r})\rho_A(\mathbf{r}')K(\mathbf{r}'\to\mathbf{r}). \quad (16.11)\end{aligned}$$

The application of this result to Monte Carlo is evident. If the flux at P ($\mathbf{r}=\mathbf{r}_p$) is required due to a point source at S ($\mathbf{r}=\mathbf{r}_s$), we take $p(\mathbf{r}) = \delta(\mathbf{r}-\mathbf{r}_p)$, $q(\mathbf{r}) = K(\mathbf{r}_s\to\mathbf{r})$, and V_A the volume inside one of the surfaces S_j constructed by starting from S. Then, under the conditions in which it is necessary to eliminate part of a sample which is too large, the first term on the right side of (16.11) will be negligible. The second term can be constructed by starting from S and P at the same time and continuing until the regions covered begin to overlap. The main contribution to (16.11) comes from the first overlap region, and once the overlap has occurred no further surfaces of discontinuity need be introduced; the ordinary Monte Carlo procedure can be followed (§ 16.4) until the test particle population of both samples has died out.

We have discussed the procedure based on (16.11) as an alternative to importance sampling. By a slight modification, the two methods can be used in conjunction, but we shall not consider this possibility in detail.

16.11. Conclusion

The following general features of the Monte Carlo method may be noted in conclusion:

(1) The Monte Carlo method may be regarded as in a way complementary to the analytical methods, in the sense that where the former is most useful the latter are least useful, and conversely. The Monte Carlo method is at its best in dealing with small systems whose geometrical dimensions are comparable with the mean free paths involved. For large systems, the Monte Carlo method may easily prove unworkable, unless some device such as that of § 16.8 is employed, while the analytical methods are at their best for large systems.

(2) Since the Monte Carlo method consists of a statistical analogue experiment, the errors occurring in it are of a fundamentally different nature from those in a conventional numerical approach; many applications of the latter produce exactly the same result, which will usually differ slightly from the true solution. If, however, we conduct a Monte Carlo experiment many times, we shall obtain a different result each time. It is fairly certain that the mean of these will tend to the correct solution, but any one result will have a certain probability of being considerably in error. Although this probabiltiy is never zero, it can always be made less than the probability of a human error such as an arithmetical mistake or a misuse of a formula. Further, since the Monte Carlo method is usually applied when the only other alternative is conventional iteration, it is generally possible to apply a single such iteration to the result of the Monte Carlo procedure. If this result is a 'freak', the fact will become evident when the iteration is performed.

(3) Monte Carlo methods in their simplest form are often inefficient, and an increase in accuracy of 10^p requires the expenditure of 10^{2p} as much labour, if recourse is had only to replication of the experiments. We have indicated, fairly generally, several ways of overcoming these objections, but developments are continually being made in this direction, and the reader must refer to the current literature on the subject to obtain a more complete picture.

Note added in proof. Appendix B gives a description of another numerical method for the solution of neutron transport problems, recently published by Carlson (58).

XVII

ANISOTROPIC SCATTERING

17.1. Preliminary results

THE last chapter concludes our discussion of the principal methods used in the case of constant cross-sections and scattering isotropic in the L system. We now examine the case where the constant cross-section approximation is still valid, but the anisotropy of scattering in the L system has to be taken into account. That is, we return to equation (4.3), and now assume that $f(\Omega' \to \Omega)$, though a function of $\Omega \cdot \Omega'$ only, is no longer a constant. It is often convenient to represent this function as an expansion in Legendre polynomials:

$$f(\Omega' \to \Omega) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) b_n P_n(\Omega \cdot \Omega'), \tag{17.1}$$

where, from the normalization condition (2.6) and the definition (4.1), we have $b_0 = 1$.

We first consider the form of equation (4.3) in the case of plane symmetry, with $f(\Omega' \to \Omega)$ given by (17.1). If μ and μ' are the x-components of Ω and Ω', and ϕ is the angle between the projections of Ω and Ω' on a plane $x =$ constant, we have

$$d\Omega' = d\mu' d\phi$$

and

$$\Omega \cdot \Omega' = \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi,$$

and hence, from the properties of Legendre polynomials (see Whittaker and Watson (52), p. 327),

$$P_n(\Omega \cdot \Omega') = P_n(\mu) P_n(\mu') + \text{terms in } \cos r\phi \quad (r = 1, 2, \ldots, n). \tag{17.2}$$

For the plane case, according to (10.2),

$$\psi(\mathbf{r}, \Omega) = \psi(x, \mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) \psi_n(x) P_n(\mu).$$

Combining these formulae, we find, since

$$\int_{-1}^{1} P_n(\mu') P_{n'}(\mu')\, d\mu' = \frac{2\delta_{nn'}}{2n+1},$$

that the equation (4.3) reduces to

$$\mu\frac{\partial\psi(x,\mu)}{\partial x}+\frac{\psi(x,\mu)}{l}=\frac{c}{4\pi l}\sum_{n=0}^{\infty}(2n+1)b_n\psi_n(x)P_n(\mu)+S(x,\mu). \tag{17.3}$$

In a similar manner we have for a general geometry, in the notation of (12.5),

$$\boldsymbol{\Omega}\cdot\mathbf{grad}\,\psi(\mathbf{r},\boldsymbol{\Omega})+\frac{\psi(\mathbf{r},\boldsymbol{\Omega})}{l}=\frac{c}{4\pi l}\sum_{n=0}^{\infty}(2n+1)b_n[\Psi_n]_{U=1}+S(\mathbf{r},\boldsymbol{\Omega}). \tag{17.3'}$$

We give below the values of the coefficients b_n for some typical cases.

If the scattering is isotropic in the C system, and the neutron flux is required, we have on combining (2.16) and (4.1):

$$[f(\Omega'\to\Omega)]_{\mathrm{el},M}=\frac{(M+1)^2}{4\pi Mv'^2}\int_{\frac{M-1}{M+1}v'}^{v'}v\,dv\,\delta\left(\Omega\cdot\Omega'-\frac{(M+1)v^2-(M-1)v'^2}{2vv'}\right). \tag{17.4}$$

Making the substitution

$$v=\frac{(M^2-1+s^2)^{\frac{1}{2}}+s}{M+1}v',$$

using the identity

$$\delta(\mu_1-\mu_2)=\tfrac{1}{2}\sum_{n=0}^{\infty}(2n+1)P_n(\mu_1)P_n(\mu_2),$$

and comparing the resulting expression with (17.1), we find at once

$$[b_n]_{\mathrm{el},M}=\int_{-1}^{1}\frac{[(M^2-1+s^2)^{\frac{1}{2}}+s]^2}{2M(M^2-1+s^2)^{\frac{1}{2}}}P_n(s)\,ds. \tag{17.5}$$

If the scattering is isotropic in the C system, but the neutron energy flux is required, we use instead of (4.1) the equation†

$$f_{\mathrm{ef}}(\Omega'\to\Omega)=(c/c_{\mathrm{ef}})\int(v^2/v'^4)f(v'\Omega'\to v\Omega)\,dv$$

(see § 4.3, formula (4.10)), where c_{ef} is so defined as to give the usual normalization condition:

$$\int\int f_{\mathrm{ef}}(\Omega'\to\Omega)\,d\Omega=1.$$

Continuing as in the derivation of (17.5), we easily obtain

$$[b_{n,\mathrm{ef}}]_{\mathrm{el},M}=\int_{-1}^{1}\frac{[(M^2-1+s^2)^{\frac{1}{2}}+s]^4}{2M(M^2+1)(M^2-1+s^2)^{\frac{1}{2}}}P_n(s)\,ds, \tag{17.6}$$

† The suffix 'ef' denotes energy flux.

and so on. For the case of hydrogen ($M = 1$), the formulae (17.5) and (17.6) become

$$[b_n]_{\mathrm{el},H} = 2\int_0^1 sP_n(s)\,ds \tag{17.5'}$$

and

$$[b_{n,\mathrm{ef}}]_{\mathrm{el},H} = 4\int_0^1 s^3P_n(s)\,ds. \tag{17.6'}$$

If the medium is *not* hydrogen, we see by expanding the first factor in the integrands of (17.5) and (17.6) in powers of s that the coefficient of s^k is of the order of M^{-k}. It then follows from the orthogonality properties of Legendre polynomials that

$$[b_n]_{\mathrm{el},M} = O(M^{-n}), \tag{17.7}$$

and the same is true of $b_{n,\mathrm{ef}}$.

In the case of heavy nuclei, the scattering may not be isotropic in the C system, but it is usually given by the first few terms of (1.1) (see § 1.2.2). The coefficients $g_n(E)$ in that formula vary rapidly with energy, and so the constant cross-section approximation will not in general be applicable. If, however, only a narrow range of neutron energies is occupied by neutrons, so that the $g_n(E)$ do not vary appreciably within that range, the approximation is still valid. In this case, going from the C system to the L system, we have for the contribution to b_n the expression

$$[b_n] = g_n[1+O(1/M^2)].$$

It is evident that in all the above cases except that given by (17.5'), namely when the neutron flux in hydrogen is required, the coefficients b_n decrease rapidly with n. We can therefore obtain a good approximation by terminating (17.1) after a few terms, that is, by replacing it by

$$f(\Omega'\to\Omega) = \frac{1}{4\pi}\Big[1+\sum_{n=1}^{t}(2n+1)b_nP_n(\Omega\,.\,\Omega')\Big], \tag{17.8}$$

where t is a small integer.

If (17.8) is not a good approximation, but there is plane symmetry, the form of (4.3) can be somewhat simplified as follows. We start, as in the derivation of (17.3), by characterizing Ω and Ω' in terms of μ, μ', and ϕ, and introduce an auxiliary function $f(\mu'\to\mu)$, defined thus:

$$f(\mu'\to\mu) = \int_0^{2\pi} f(\Omega'\to\Omega)\,d\phi. \tag{17.9}$$

Then, if in the expression

$$\iint f(\Omega'\to\Omega)\psi(x,\mu')\,d\Omega'$$

we integrate first over ϕ, the equation (4.3) reduces to

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \frac{\psi(x,\mu)}{l} = \frac{c}{l} \int_{-1}^{1} \psi(x,\mu') f(\mu' \to \mu)\, d\mu'. \qquad (17.10)$$

Using (17.1) and (17.2), we can express (17.9) in the form:

$$f(\mu' \to \mu) = \tfrac{1}{2} \sum_{n=0}^{\infty} (2n+1) b_n P_n(\mu) P_n(\mu'). \qquad (17.11)$$

However, when (17.8) is not a good approximation, it is often simpler to use (17.9) directly. For instance, in the case where the neutron flux in hydrogen is required, the formula (17.4) gives

$$[f(\Omega' \to \Omega)]_{\mathrm{el},H} = \frac{1}{2\pi} [|\Omega \cdot \Omega'| + \Omega \cdot \Omega'],$$

and hence

$$[f(\mu' \to \mu)]_{\mathrm{el},H}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \{|\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi| + \mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi\}\, d\phi$$

$$= \begin{cases} |\mu\mu'| + \mu\mu' & \text{if } \mu^2+\mu'^2 > 1 \\ 2\mu\mu' \left(1 - \frac{1}{\pi} \cos^{-1} \frac{\mu\mu'}{(1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}}\right) + \frac{2}{\pi}(1-\mu^2-\mu'^2)^{\frac{1}{2}} & \text{if } \mu^2+\mu'^2 < 1 \end{cases}. \qquad (17.12)$$

17.2. The integral equation for the neutron flux

We have seen in Chapter IV that in the case of isotropic scattering the equation (4.3) can be reduced to an integral equation for the neutron flux $\rho(\mathbf{r}) = \psi_0(\mathbf{r})$ only (see (4.15′)). In the case of anisotropic scattering this is in general no longer possible. However, if the system is non-re-entrant with a total mean free path everywhere the same, and if $f(\Omega' \to \Omega)$ in each region of the system is a polynomial in $\Omega \cdot \Omega'$, i.e. (17.8) holds, then it is possible to reduce (4.3) to a form rather similar to (4.15).

We shall show this first for a bare homogeneous non-re-entrant source-free body for which $f(\Omega' \to \Omega)$ is linear in $\Omega \cdot \Omega'$. In this case the right side of (17.3′) becomes

$$\frac{c}{4\pi l} [\rho(\mathbf{r}) + 3b_1 \Omega \cdot \mathbf{j}(\mathbf{r})].$$

Proceeding as in the derivation of (4.16) we then obtain

$$\psi(\mathbf{r},\Omega) = \frac{c}{4\pi l}\int_0^{R_0(\mathbf{r},\Omega)} [\rho(\mathbf{r}-R\Omega)+3b_1\Omega\,.\,\mathbf{j}(\mathbf{r}-R\Omega)]e^{-R/l}\,dR, \quad (17.13)$$

where $R_0(\mathbf{r},\Omega)$ is the value of R for which $\mathbf{r}-R\Omega$ lies on the surface of the body. Integrating over all Ω, putting $\mathbf{r}-R\Omega=\mathbf{r}'$ and using the fact that $dRd\Omega = dV'/|\mathbf{r}-\mathbf{r}'|^2$, we have

$$\rho(\mathbf{r}) = \frac{c}{4\pi l}\left\{\iiint_V \rho(\mathbf{r}')\frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^2}\,dV' + \right.$$
$$\left. +3b_1\iiint_V (\mathbf{r}-\mathbf{r}')\,.\,\mathbf{j}(\mathbf{r}')\frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^3}\,dV'\right\}. \quad (17.14)$$

We consider the second integral in greater detail. Using Gauss's theorem, and the result

$$(\mathbf{r}-\mathbf{r}')\frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{grad}_{\mathbf{r}'}\int_{|\mathbf{r}-\mathbf{r}'|}^{\infty}\frac{e^{-s/l}\,ds}{s^2} = \mathbf{grad}_{\mathbf{r}'}\left[\frac{1}{|\mathbf{r}-\mathbf{r}'|}E_2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l}\right)\right],$$

we find that the second term in the braces in (17.14) is

$$-3b_1\iiint_V \frac{dV'}{|\mathbf{r}-\mathbf{r}'|}E_2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l}\right)\mathrm{div}\,\mathbf{j}(\mathbf{r}')+3b_1\iint_S E_2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l}\right)\frac{\mathbf{j}(\mathbf{r}')\,.\,\mathbf{dS}}{|\mathbf{r}-\mathbf{r}'|},$$

where S is the surface of the body and $\mathbf{dS}$ is a vector having the direction of the outward normal and the magnitude of the area element. Integrating (17.3′) over all angles gives

$$\mathrm{div}\,\mathbf{j}(\mathbf{r}) = (c-1)\rho(\mathbf{r})/l,$$

and thus the equation (17.14) reduces to

$$\rho(\mathbf{r}) = \frac{c}{4\pi l}\left\{\iiint_V \rho(\mathbf{r}')\left[\frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|}+\frac{3b_1(1-c)}{l}E_2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l}\right)\right]\frac{dV'}{|\mathbf{r}-\mathbf{r}'|}+\right.$$
$$\left.+3b_1\iint_S E_2\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l}\right)\frac{\mathbf{j}(\mathbf{r}')\,.\,\mathbf{dS}}{|\mathbf{r}-\mathbf{r}'|}\right\}. \quad (17.15)$$

This procedure is easily extended to any case where $f(\Omega'\to\Omega)$ is a polynomial of finite order in $\Omega\,.\,\Omega'$. Thus, denoting the $[\Psi_n]_{U=1}$ of (12.5) and (17.3′) by $\Psi_n(\mathbf{r},\Omega)$, taking a general scattering law of the form (17.8), and continuing as in the derivation of (17.14), we find

$$\rho(\mathbf{r}) = \frac{c}{4\pi l}\iiint_V \frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^2}\sum_{n=0}^{t}(2n+1)b_n\Psi_n\left(\mathbf{r}',\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right)dV',$$

or, since $\Psi_n(\mathbf{r}, \Omega)$ is a homogeneous polynomial of order n in the components of Ω,

$$\rho(\mathbf{r}) = \frac{c}{4\pi l} \sum_{n=0}^{t} (2n+1) b_n \iiint_V \Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}') \frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^{n+2}} dV'. \tag{17.14'}$$

We now consider a particular term on the right side of the expression (17.14'). By Euler's theorem,

$$n\Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}') = (\mathbf{r}-\mathbf{r}') \cdot \mathbf{grad}_{\mathbf{r}} \Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}'),$$

while $$(\mathbf{r}-\mathbf{r}') \frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^{n+2}} = \mathbf{grad}_{\mathbf{r}'} \left[\frac{1}{|\mathbf{r}-\mathbf{r}'|^n} E_{n+1}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l} \right) \right].$$

Thus, using Gauss's theorem, we see that the term in question reduces to

$$-\frac{(2n+1)cb_n}{4\pi l n} \iiint_V \frac{1}{|\mathbf{r}-\mathbf{r}'|^n} E_{n+1}\left(\frac{|\mathbf{r}-\mathbf{r}'|}{l} \right) \mathrm{div}_{\mathbf{r}'} \mathbf{grad}_{\mathbf{r}} \Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}')\, dV' +$$
$$+\text{surface terms.}$$

Next, since $\nabla_{\mathbf{r}}^2 \Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}') = 0$, we have

$$\mathrm{div}_{\mathbf{r}'} \mathbf{grad}_{\mathbf{r}} \Psi_n(\mathbf{r}', \mathbf{r}-\mathbf{r}') = [\mathrm{div}_{\mathbf{r}'} \mathbf{grad}_{\mathbf{U}} \Psi_n(\mathbf{r}', \mathbf{U})]_{\mathbf{U}=\mathbf{r}-\mathbf{r}'}. \tag{17.16}$$

By separating in (17.3') the spherical harmonics of order $n-1$, as was done in the derivation of (12.11), we can easily express the right side of (17.16) in terms of $\Psi_{n-1}(\mathbf{r}', \mathbf{r}-\mathbf{r}')$, $\Psi_{n-2}(\mathbf{r}', \mathbf{r}-\mathbf{r}')$, etc., and thus eliminate all the higher-order spherical harmonics, obtaining finally

$$\rho(\mathbf{r}) = \iiint_V \rho(\mathbf{r}') K(|\mathbf{r}-\mathbf{r}'|)\, dV' + \text{surface terms.} \tag{17.15'}$$

The above discussion has assumed a single homogeneous non-re-entrant body, but it is evident that it applies also to the case where $c, b_1, b_2, \ldots, b_t$ vary in the system, provided that it is non-re-entrant and that the total mean free path is everywhere the same, so that the probability for a neutron to reach $\mathbf{r}$ from $\mathbf{r}'$ without collision depends only on the distance $|\mathbf{r}-\mathbf{r}'|$ and not on the direction of $\mathbf{r}-\mathbf{r}'$. In dealing with such a system, of course, Gauss's theorem is applied separately to each region, so that the surface terms will include contributions from the interfaces as well as from the free surfaces. We have also assumed that sources are absent, but this too is immaterial and was introduced only for simplicity.

If the system has some symmetry, it is easier to transform (17.14') to (17.15') by using the symmetry from the start. For instance, for an

infinite plane slab the equation (17.14′) has the form

$$\rho(x) = \frac{c}{4\pi l} \sum_{n=0}^{t} (2n+1) b_n \iiint_V \psi_n(x') P_n\left(\frac{x-x'}{|\mathbf{r}-\mathbf{r}'|}\right) \frac{e^{-|\mathbf{r}-\mathbf{r}'|/l}}{|\mathbf{r}-\mathbf{r}'|^2} \, dV'.$$

Integrating over the planes $x' =$ constant, we have

$$\rho(x) = \frac{c}{2l} \sum_{n=0}^{t} (2n+1) b_n \sum_{m=0}^{n} a_{m,n} \int_{x_1}^{x_2} \psi_n(x') [\mathrm{sign}(x-x')]^m E_{m+1}\left(\frac{|x-x'|}{l}\right) dx', \tag{17.14″}$$

where the $a_{m,n}$ are defined by $P_n(\mu) = \sum a_{m,n} \mu^m$, while x_1 and x_2 correspond to the surfaces of the slab. We consider the terms for a particular $n \neq 0$ and integrate by parts. Since

$$\sum_{m=0}^{n} a_{m,n} \int_{-\infty}^{\infty} [\mathrm{sign}(x-x')]^m \, E_{m+1}\left(\frac{|x-x'|}{l}\right) dx'$$

is proportional to $\int_{-1}^{1} P_n(\mu) \, d\mu$, and so vanishes for all $n \geqslant 1$, the integrated term cannot involve $\psi_n(x)$, and the result of the integration by parts is

$$-\sum_{m=0}^{n} a_{m,n} \int_{x_1}^{x_2} \frac{d\psi_n(x')}{dx'} [\mathrm{sign}(x-x')]^{m+1} \, E_{m+2}\left(\frac{|x-x'|}{l}\right) l \, dx' +$$

$$+ \text{terms in } \psi_n(x_1) \text{ and } \psi_n(x_2) \text{ only,}$$

while, multiplying (17.3) by $P_{n-1}(\mu)$ and integrating, we can express $d\psi_n(x')/dx'$ in terms of $\psi_{n-1}(x')$ and $d\psi_{n-2}(x')/dx'$. Proceeding thus, we can easily show that in the case of a plane slab (17.15′) has the form

$$\rho(x) = \int_{x_1}^{x_2} \rho(x') \sum_{n=0}^{t} A_n \, E_{2n+1}\left(\frac{|x-x'|}{l}\right) dx' + \text{surface terms.} \tag{17.15″}$$

Although (17.15″) was derived on the assumption of a bare slab, the analysis is the same for any system with plane symmetry, and for such a system it is no longer necessary to assume the total mean free path constant throughout the system. This is because one can always work in terms of optical thickness in the plane case in the constant cross-section approximation, whether or not the scattering is isotropic.

For spherically symmetric systems, since the kernel $K(|\mathbf{r}-\mathbf{r}'|)$ of the equation (17.15′) depends only on $|\mathbf{r}-\mathbf{r}'|$, it can be shown as in § 8.2 that the kernel of the equation satisfied by $r\rho(r)$ in the spherical

case is the same as that of the equation satisfied by $\rho(x)$ in the appropriate plane case, and similarly that the surface terms are related in the same way. However, the auxiliary conditions which determine the surface values of the higher moments may not be the same (they are obtained by applying (17.13) at the boundary of the system, multiplying by $P_{n,m}(\Omega)$ and integrating over Ω), and consequently it is no longer true, as it was in the case of isotropic scattering, that a spherically symmetric problem with a constant mean free path is equivalent to the corresponding plane problem.

17.3. An infinite source-free medium

17.3.1 *The formal solution*

In dealing with methods of solution of the equations derived above, it is natural to begin with the case of an infinite homogeneous source-free medium. The neutron distribution in such a medium can be regarded as the superposition of plane waves, and so it is sufficient to solve (17.10), assuming that $\psi(x,\mu)$ is of the form

$$\psi(x,\mu) = A(\mu)e^{x/L}. \tag{17.17}$$

Substituting this in (17.10), we have

$$\left(\mu\frac{l}{L}+1\right)A(\mu) = c\int_{-1}^{1} A(\mu')f(\mu'\to\mu)\,d\mu', \tag{17.18}$$

where L is a variable parameter to be chosen so that (17.18) has a solution; its value is, of course, the *diffusion length* (see § 5.2).

We first consider the case where $f(\Omega'\to\Omega)$ is a polynomial of comparatively low order in $\Omega.\Omega'$, i.e. where (17.8) holds. Using (17.11), and writing

$$A_n = \int_{-1}^{1} A(\mu')P_n(\mu')\,d\mu', \tag{17.19}$$

we can rewrite (17.18) as

$$\left(\mu\frac{l}{L}+1\right)A(\mu) = \frac{c}{2}\sum_{n=0}^{t}(2n+1)b_nA_nP_n(\mu).$$

Hence, determining $A(\mu)$ and substituting into (17.19), we have

$$A_n = \frac{c}{2}\sum_{m=0}^{t}(2m+1)b_mA_m\int_{-1}^{1}\frac{P_n(\mu)P_m(\mu)}{1+\mu l/L}\,d\mu \quad (n=0,1,\ldots,t),$$

and hence L is the solution of the determinantal equation

$$\left| \delta_{nm} - \frac{c}{2}(2m+1)b_m \int_{-1}^{1} \frac{P_n(\mu)P_m(\mu)\,d\mu}{1+\mu l/L} \right| = 0. \tag{17.20}$$

17.3.2. *The case where c is close to unity. The transport mean free path and cross-section*

If $c = 1$, then $L = \infty$ is clearly a solution of (17.20). Thus, for c close to unity, there should be a solution $L = L_0$ which tends to infinity as $c \to 1$. We put $l/L_0 = \nu$ and expand the integrals in (17.20) in powers of ν. The determinantal equation then becomes

$$\begin{vmatrix} 1-c(1+\frac{1}{3}\nu^2+\frac{1}{5}\nu^4+\ldots) & b_1 c\nu(1+\frac{2}{5}\nu^2+\ldots) & -\frac{2}{5}b_2 c\nu^2(1+\frac{6}{7}\nu^2+\ldots) & \ldots \\ \frac{1}{3}c\nu(1+\frac{2}{5}\nu^2+\frac{3}{7}\nu^4+\ldots) & 1-b_1 c(1+\frac{2}{5}\nu^2+\ldots) & \frac{2}{5}b_2 c\nu(1+\frac{6}{7}\nu^2+\ldots) & \ldots \\ -\frac{2}{15}c\nu^2(1+\frac{5}{7}\nu^2+\ldots) & \frac{2}{5}b_1 c\nu(1+\frac{5}{7}\nu^2+\ldots) & 1-b_2 c(1+\frac{11}{21}\nu^2+\ldots) & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0.$$

It is evident that, for small ν, $1-c$ is of the order of ν^2. Dividing the first row and column by ν and collecting the terms which remain finite when ν tends to zero (only even powers of ν can enter the final expression), we obtain:

$$\begin{vmatrix} \frac{1-c}{\nu^2} - \frac{c}{3} & b_1 c \\ \frac{c}{3} & 1-b_1 c \end{vmatrix} = O(\nu^2),$$

and hence also

$$L_0 = \frac{l}{\nu} = \frac{l}{\sqrt{3}(1-c)^{\frac{1}{2}}(1-b_1)^{\frac{1}{2}}}[1+O(1-c)]. \tag{17.21}$$

Transforming this expression similarly to (5.15 a), we can rewrite it as

$$L_0 = \left[\frac{l_s l_c}{3(1-b_1)}\right]^{\frac{1}{2}}[1+O(1-c)]. \tag{17.22}$$

In the above derivation we have imposed no restriction on the value of any b_n, though of course none of them can exceed unity, since $f(\Omega' \to \Omega)$ is non-negative. Thus, irrespective of the values of the b_n, for small $1-c$ the leading term in the diffusion length L_0 depends only on b_1, while $b_2, b_3, \ldots, b_l$ enter only in the correction term.

The quantity

$$l_{tr} = l_s/(1-b_1) \tag{17.23}$$

is often used; it is called the *transport mean free path*. In terms of this, the expression for the diffusion length can be written

$$L_0 = \left(\frac{l_{tr} l_c}{3}\right)^{\frac{1}{2}} [1+O(1-c)]. \tag{17.24}$$

By analogy with formula (1.3), we introduce also the *transport cross-section*, defined by

$$\sigma_{tr} = (1-b_1)\sigma_s. \tag{17.25}$$

By (17.1), the quantity b_1 can be expressed as

$$b_1 = \int\int f(\Omega' \to \Omega)\Omega \,.\, \Omega' \, d\Omega, \tag{17.26}$$

and is consequently known as *the mean cosine of the scattering angle*.

On comparing (17.24), 17.25), and (5.16 b), one can say that for the case $|1-c| \ll 1$ the effect of anisotropic scattering is equivalent to a relative reduction of the scattering cross-section by an amount equal to the mean cosine of the scattering angle.†

17.3.3. *The case where c is not close to unity*

If $1-c$ is small, but not negligibly small, the formula (17.21) can be improved by including terms up to the order of $1-c$, but not those of order $(1-c)^2$, or in general by including terms up to order $(1-c)^n$, where $n \geqslant 1$. For larger values of $|1-c|$, L can be determined directly from (17.20), if (17.8) holds and t is fairly small. The integrals in (17.20) are elementary, and the resulting transcendental equation for L can be solved numerically.

If t in (17.8) is rather large, or if (17.8) cannot be used for any finite t, there is no advantage in reducing (17.18) to (17.20), and the former should be treated as an integral equation and solved by an approximate method. The most frequently used method for this purpose is that of discrete ordinates, which we shall discuss in section 17.5.3 below. Both the variational method (Chapter XV) and the iteration method (Chapter XVI) are also suitable for use with (17.18). Kuščer (unpublished) has used Fredholm series to solve (17.18) for some cases, but this method is not generally profitable.

† The importance of the transport mean free path and cross-section lies in the fact that experimentally it is often easier to determine l_{tr} than l_s and b_1 themselves. It has been suggested that the case of anisotropic scattering might be approximately treated by replacing l_s by l_{tr} and assuming isotropic scattering; this is sometimes called the *transport approximation*. It would be exact if $f(\Omega' \to \Omega)$ were of the form

$$f(\Omega' \to \Omega) = (1/4\pi)[(1-b_1)+4\pi b_1 \, \delta(\Omega \,.\, \Omega'-1)],$$

i.e. if $b_n = b_1$ for $n \geqslant 1$. However, the approximation leads in general to rather poor results, as we might expect.

17.3.4. *The number of roots of the equation for the diffusion length*

Hitherto we have discussed the determination of a particular eigenvalue of (17.18), and of a particular root of (17.20). The question arises whether this root is unique. If a particular value of L satisfies (17.20), then so does $-L$, so that we are really concerned with pairs of roots. Further, as has been remarked in § 2.3.3, only those solutions are physically permissible for which $\psi(\mathbf{r}, \Omega)$ does not increase faster than $e^{|\mathbf{r}|/l}$ as $|\mathbf{r}| \to \infty$. Hence, for physically permissible solutions of the form (17.17), l/L must lie in the strip $-1 < \mathrm{re}(l/L) < 1$. However, all the integrals in (17.20) converge in the complex L-plane cut along the real axis from $L = -l$ to $L = l$. It is therefore reasonable to investigate the number of pairs of roots in the cut L-plane. In the case of isotropic scattering, the equation (17.20) becomes

$$1 - \frac{c}{2} \int_{-1}^{1} \frac{d\mu}{1+\mu l/L} = 0,$$

and we have seen in § 5.1 that this equation has just one pair of roots. In the case of anisotropic scattering this is no longer necessarily true; examples have been constructed in which $f(\Omega' \to \Omega)$ is non-negative and linear in $\Omega \cdot \Omega'$, but c is so large that b_1 can be found such that (17.20) has two pairs of roots in the cut plane. Further, for any c, if t in (17.8) is large enough, $b_1, b_2, \ldots, b_t$ can be chosen so that (17.20) has any prescribed number k of pairs of roots ($k \geqslant 1$), though $f(\Omega' \to \Omega)$ remains non-negative (Davison (59)). However, these situations have not yet been met with in practical applications; all practical problems hitherto considered have been such that (17.20) has just one pair of roots, as in the case of isotropic scattering. We summarize below the available information on this topic:

When all the b_n ($n \geqslant 1$) are negligibly small, the situation is identical with isotropic scattering. As the b_n increase, a point is reached when two more roots appear at the ends of the cut, i.e. at $L = \pm l$, and begin to move away from the cut along the real axis. When the b_n increase further, two more roots appear at the ends of the cut, and so on. However, unless c is very large, all these extra roots are close to the ends of the cut: if we call the roots of (17.20) $\pm L$, $\pm L_1$, etc., with $1/L^2 < 1/L_1^2 < 1/L_2^2$, etc., then only $e^{x/L}$ can vary reasonably slowly, while e^{x/L_m} for $m \geqslant 1$ increases, for $x \to \infty$, almost as fast as $e^{x/l}$. Thus, as we shall see later, the extra roots are unimportant, even if they exist.

If the equation (17.20) has only two roots, then, by taking an arbitrary

superposition of plane waves of the type (17.17) and determining the corresponding flux, we can readily see that the flux will be a solution, regular in all space, of

$$(\nabla^2 - 1/L^2)\rho(\mathbf{r}) = 0, \tag{17.27}$$

just as in the case of isotropic scattering. If there are two pairs of roots, this equation becomes

$$(\nabla^2 - 1/L_1^2)(\nabla^2 - 1/L^2)\rho(\mathbf{r}) = 0, \tag{17.27'}$$

and so on.

17.4. Other exact solutions

17.4.1. *An infinite medium containing sources*

The exact solutions have been found (see Chapters V and VI) for the case of isotropic scattering in an infinite medium containing sources and in a semi-infinite medium. We now consider the corresponding problems for anisotropic scattering. If (17.8) does not hold, then the exact solution is not known even for an infinite source-free medium, so that we shall attempt to generalize the results of Chapters V and VI only to the cases where (17.8) holds. Further, we shall only outline the procedure involved, since the multiplicity of parameters makes it impracticable to write out the complete solution.

We first take the case of an isotropic point source in an infinite anisotropically scattering medium. Here the integral equation derived in § 17.2 can be used. It can be assumed without loss of generality that there is no supply of neutrons from infinity. The problem is therefore spherically symmetric, and all moments of $\psi(r,\mu)$ other than $\psi_0(r)$ vanish at $r = 0$, at least so far as neutrons that have been scattered are concerned. The contribution due to neutrons coming directly from the sources should, of course, be included if all neutrons are considered. This contribution to any $\psi_n(r)$ is, however, simply $se^{-r/l}/r^2$, where s is the number of neutrons emitted by the source in unit time and solid angle. Thus the determination of the surface terms due to a small sphere around $r = 0$ offers no difficulty. The outer surface of the system is at infinity, and the term due to this vanishes. Thus, in the case where, for instance, $f(\Omega' \to \Omega)$ is linear in $\Omega.\Omega'$, extending the definition of $\rho(r)$ to $r < 0$ by putting $\rho(-r) = \rho(r)$, we obtain

$$r\rho(r) = \frac{c}{2l}\int_{-\infty}^{\infty} r'\rho(r')\left[E_1\left(\frac{|r-r'|}{l}\right) + 3b_1(1-c)E_3\left(\frac{|r-r'|}{l}\right)\right]dr' + \\ + s\left[\frac{e^{-|r|/l}}{r} - \frac{3b_1 c}{l}E_2\left(\frac{|r|}{l}\right)\operatorname{sign} r\right]. \tag{17.28}$$

Similarly, we can obtain the equation [(17.28′)] for $r\rho(r)$ when $f(\Omega'\to\Omega)$ takes the more general form (17.8), and these equations can then be solved by the Fourier transform method, as in § 5.3. If, as is normally the case, (17.20) has only one pair of roots, the solution of (17.28′) will be of the form

$$\rho(r) = (A/r)e^{-r/L}+O(e^{-r/l}), \tag{17.29}$$

where the first term represents $\rho_{as}(r)$ and the second $\rho_{tr}(r)$. If (17.20) has two pairs of roots, say, then (17.29) will be replaced by

$$\rho(r) = (A/r)e^{-r/L}+(A_1/r)e^{-r/L_1}+O(e^{-r/l}). \tag{17.29′}$$

In this case the first two terms of (17.29′) may be regarded as $\rho_{as}(r)$ and the third as $\rho_{tr}(r)$, so that $\rho_{as}(r)$ now satisfies (17.27′), while $\rho_{tr}(r)$ is still of the order of $e^{-r/l}$. However, in view of what we have said about the values of L_1, L_2, etc., at the end of § 17.3.4, it is more natural to regard only the first term in (17.29′) as $\rho_{as}(r)$. In this case, $\rho_{as}(r)$ still satisfies (17.27), but $\rho_{tr}(r)$ is now of the order of e^{-r/L_1} rather than $e^{-r/l}$.

The solution of the problem of an anisotropic source in an infinite anisotropically scattering medium can be reduced by using (5.30) and the method of § 5.4 to that of a spherically symmetric problem, which can be solved as for an isotropic point source. The solution for a line, surface, or distributed source can be obtained by superposition, and $\rho(r)$ can again be separated into $\rho_{as}(r)$ and $\rho_{tr}(r)$.

17.4.2. *An infinite source-free half-space*

Suppose now that the medium occupies the half-space $x > 0$, with $x = 0$ a free surface, there being no sources in the medium; that is, we have Milne's problem for anisotropic scattering. In this case the surface values of the higher moments $\psi_1(0)$, $\psi_2(0)$,..., $\psi_t(0)$ are no longer elementary, but represent t additional constants which have to be determined when equation (17.15″) is solved.

A possible procedure for this purpose has been developed by Tait (unpublished); it consists of an adaptation of the Wiener–Hopf method. We extend the definition of $\rho(x)$ to $x < 0$ by regarding the half-space $x < 0$ as filled with purely absorbing matter ($c = 0$) of the same mean free path as that in $x > 0$. Let $F^+(p)$ and $F^-(p)$ be the Fourier transforms of $\rho(x)$ for $x > 0$ and $x < 0$ respectively, as defined in § 6.1. Taking the Fourier transform of (17.15″), we have

$$\begin{aligned} F^-(p)+F^+(p) = F^+(p)\{(\bar{A}_1 p^{-1}+\bar{A}_3 p^{-3}+\ldots+\bar{A}_{2t+1}p^{-2t-1})\tan^{-1}pl+ \\ +(\bar{B}_2 p^{-2}+\ldots+\bar{B}_{2t}p^{-2t})\}+ \\ +\sum_{n=1}^{t}\psi_n(0)\times\text{an expression analogous to }\{\ \}, \end{aligned} \tag{17.30}$$

where $\bar{A}_1,\ldots,\bar{B}_2,\ldots$ are certain coefficients depending on $b_1, b_2,\ldots, b_t$. Th calculation of these coefficients is simplified if, instead of eliminatin $\psi_n(x')$ from(17.14″) and then taking the Fourier transform, we do th latter first and then eliminate the transforms of the $\psi_n(x')$, because the we do not need to find the coefficients in (17.15″).

We now introduce two auxiliary functions $H^+(p)$ and $H^-(p)$, defined a follows: $H^+(p)$ is the coefficient of $\tan^{-1}pl$ on the right of equation (17.30 and $H^-(p)$ is defined by

$$H^-(p)+H^+(p) = H^+(p)\{(\bar{A}_1 p^{-1}+\ldots+\bar{A}_{2t+1}p^{-2t-1})\tan^{-1}pl+(\bar{B}_2 p^{-2}+\ldots+\bar{B}_{2t}p^{-2t})\} \quad (17.31)$$

evidently $H^-(p)$ is expressible directly in terms of $F^-(p)$, the constant $\psi_n(0)$, and known functions. The equation (17.31) can be solved by th usual Wiener–Hopf method (see § 6.1); we find

$$H^+(p) = C(p)\omega(p),$$

where $C(p)$ is a polynomial containing $t+1$ constants $C_1, C_2,\ldots, C_{t+1}$, say, and $\omega(p)$ can be found by means of the Cauchy integral. We sti have to determine the $2t+1$ constants $\psi_1(0),\ldots,\psi_t(0)$, $C_1,\ldots,C_{t+1}$. Le $p_1, p_2,\ldots, p_t$ be the roots of $\bar{A}_1 p^{-1}+\ldots+\bar{A}_{2t+1}p^{-2t-1} = 0$ in the upper hal plane $\mathrm{im}\, p \geqslant 0$. $F^+(p)$ is regular at these roots, and so the expression fo $H^+(p_n)$ $(n = 1, 2,\ldots, t)$ will not involve $F^+(p_n)$ and will be equal to certain linear combination of the $\psi_{n'}(0)$. This gives t linear equation between the constants to be determined. Another t linear equations ar obtained from $H^-(-p_n)$, and a normalization condition completes th number needed to determine all the constants.

One of the earliest applications of this method was to the case wher $f(\boldsymbol{\Omega}' \to \boldsymbol{\Omega})$ is linear in $\boldsymbol{\Omega}.\boldsymbol{\Omega}'$. We quote the value obtained for th extrapolated end-point z_0, defined in (6.23), when $1-c$ is small (Daviso (10); Marshak (37)):

$$z_0/l = \frac{0{\cdot}7104}{c(1-b_1)} - \frac{1-c}{c(1-b_1)}[0{\cdot}508b_1 - 0{\cdot}156b_1^2] + O\left[\frac{(1-c)^2}{(1-b_1)^2}\right]. \quad (17.32$$

Most other calculations have dealt with some particular problem, in view of the many parameters involved. Tait (47) has considered the albed problem for the neutron energy flux in hydrogen, using the constan cross-section approximation and terminating (17.1) after five terms i.e. using (17.8) with $t = 4$. We shall refer presently to some qualitativ results of Tait's calculations.

The chief limitation of the Wiener–Hopf method applied to anisotropic scattering lies in the awkwardness of the integrals that must be evaluated, which compels a great deal of numerical work. If an answer of a given accuracy is required, it is usually better to use a numerical method (see the following section).

17.5. Approximate methods

17.5.1. *The diffusion approximation and the Serber–Wilson method*

All the approximate methods discussed earlier in this book for the case of isotropic scattering can be extended to the case of anisotropic scattering, though their relative advantages and disadvantages may be somewhat different in the latter case. The logical basis of the diffusion approximation is as before (see §§ 17.3 and 17.4.1). However, the information available regarding the best boundary conditions to be imposed in solving (17.27) is much more restricted than in the case of isotropic scattering. In particular, the connexion between the solutions of the plane and spherical problems (see § 8.2) no longer holds, and the value of the extrapolated end-point determined for the plane case is no longer necessarily applicable to other geometries. In fact, a comparison with other methods indicates that such application may now lead to a considerable error, and this is a serious limitation of the diffusion approximation for anisotropic scattering.

In the Serber–Wilson method (see § 9.1), the boundary conditions are replaced by certain integral conditions, and the disadvantage just mentioned does not apply. Of course, the method is applicable only to spherically symmetric systems, as in the case of isotropic scattering. To formulate the integral conditions in question, the integral equation for $\rho(r)$ need be applied only at the centre of the system, and this makes it possible to rewrite it in a much simpler form, and to dispense with the condition that the mean free path is constant throughout the system. For instance, if the system consists of a core of radius a in an infinite reflector, with $f(\Omega' \to \Omega)$ linear in $\Omega . \Omega'$ for both media, while the values of l, c, and b_1 for the core and the reflector are denoted by the suffixes 1 and 2 respectively, the equation (17.13) at $r = 0$ gives

$$\rho(0) = 4\pi\psi(0, \Omega) = \frac{c_1}{l_1}\int_0^a [\rho(r) - 3b_{11}j(r)]e^{-r/l_1}\,dr + \\ + \frac{c_2}{l_2}e^{-a/l_1}\int_a^\infty [\rho(r) - 3b_{12}j(r)]e^{-(r-a)/l_2}\,dr, \quad (17.33)$$

while the equation $\operatorname{div}\mathbf{j}(\mathbf{r}) = (c-1)\rho(\mathbf{r})/l$ reduces in the case of spherical symmetry to

$$\frac{d}{dr}[r^2 j(r)] = \frac{c-1}{l} r^2 \rho(r).$$

Thus, eliminating $j(r)$ from (17.33), we get

$$\rho(0) = \frac{c_1}{l_1}\int_0^a \left[\rho(r) + \frac{3b_{11}(1-c_1)}{l_1 r^2}\int_0^r r'^2\rho(r')\,dr'\right] e^{-r/l_1}\,dr +$$

$$+\frac{c_2}{l_2}e^{-a/l_1}\int_a^\infty \left[\rho(r) - \frac{3b_{12}(1-c_2)}{l_2 r^2}\int_r^\infty r'^2\rho(r')\,dr'\right] e^{-(r-a)/l_2}\,dr; \quad (17.34)$$

since $\rho(r)$ is here the spherically symmetric solution of (17.27), the inner integrals $\int r'^2\rho(r')\,dr'$ are elementary. Starting with (17.34), we can proceed as in the case of isotropic scattering. The same method is applicable to systems with a larger number of layers and $f(\Omega' \to \Omega)$ of the more general form (17.8).

17.5.2. *The spherical harmonics method*

One of the most powerful methods for dealing with the case of anisotropic scattering is the spherical harmonics method. Here the procedure developed for the case of isotropic scattering can be applied immediately. The equation (10.1) for the plane case is now, of course, replaced by (17.3), so that the equations (10.4) become

$$(n+1)\psi'_{n+1}(x) + n\psi'_{n-1}(x) + (2n+1)\frac{1-cb_n}{l}\psi_n(x) = 0, \quad (17.35)$$

with similar alterations in the equations for other geometries. For the most general geometry, the equation (12.11) is now replaced by

$$\operatorname{div}_{\mathbf{U}}\mathbf{grad}_{\mathbf{r}}\Psi_{n+1} + \frac{2n+1}{l}[1-cb_n]\Psi_n +$$

$$+[(2n-1)\mathbf{U}\cdot\mathbf{grad}_{\mathbf{r}}\Psi_{n-1} - U^2\operatorname{div}_{\mathbf{U}}\mathbf{grad}_{\mathbf{r}}\Psi_{n-1}] = 0. \quad (17.35')$$

The functions $G_n(\nu) \equiv G_n(\nu, c)$ are consequently replaced by

$$G_n^*(\nu) \equiv G_n^*(\nu, c, b_1, b_2, \ldots)$$

defined by the recurrence relation

$$(n+1)G_{n+1}^*(\nu) + \{(2n+1)/\nu\}(1-cb_n)G_n^*(\nu) + nG_{n-1}^*(\nu) = 0$$

and $G_*^0(\nu) = 1$. The zeros ν_i of $G_{N+1}^*(\nu)$ (where N is the order of the approximation) and the corresponding values of $G_n^*(\nu_i)$ must, of course, be worked out afresh for each set of c, b_1, b_2,..., since no tabulation of $G_n^*(\nu_i)$ exists. However, the numerical work involved is not excessive, while the general properties of the ν_i can be analysed as in § 10.2.

The boundary conditions in the spherical harmonics method are either purely local or are obtained by eliminating the solution for a black medium, and are unaffected by the isotropy or otherwise of the scattering.

If we work in the P_N approximation, all b_n for $n > N$ are neglected, and thus, unless (17.8) holds with $t \leqslant N$, there will be a distortion of the scattering law. It was at one time thought that this made the spherical harmonics method inaccurate for the case $t > N$. However, since the higher moments are small, the corresponding values of b_n are unimportant, and the order of approximation used need not normally be much higher than would be used in the case of isotropic scattering, even if the b_n thus neglected are by no means negligible *a priori*. This was confirmed, for instance, by the work of Tait (47) on the neutron energy flux in hydrogen, referred to in § 17.4.2. The calculations were originally done by the Wiener–Hopf method, and then repeated, using the spherical harmonics method in the P_1 and P_7 approximations. The P_7 results were practically indistinguishable from the Wiener–Hopf results, which is not surprising since they involved no distortion of the scattering law, but the P_1 results also were in remarkable agreement with the Wiener–Hopf results, although the distortion of $f(\Omega' \to \Omega)$ in this case was drastic, $|\mu|^3+\mu^3$ being approximated by $\frac{1}{4}+\frac{3}{4}\mu$.

The above remarks refer, of course, only to the conventional form of the spherical harmonics method. The extension of Yvon's modification to anisotropic scattering is referred to on p. 173.

17.5.3. *The discrete ordinates method*

The discrete ordinates method has hitherto been restricted to problems with plane symmetry, but for these its extension to anisotropic scattering is straightforward. We start again with the equation (17.10) and approximate it by

$$\mu_i \psi_i'(x)+\psi_i(x)/l = (c/l) \sum_j a_j f_{i,j} \psi_j(x), \tag{17.36}$$

where $f_{i,j} = f(\mu_j \to \mu_i)$ and $\psi_i(x) = \psi(x, \mu_i)$. The solution of the equations (17.36) has, of course, the same form as for isotropic scattering (13.3). The permissible values of γ_s in (13.3) will obviously be given by the roots of the determinantal equation

$$|ca_j f_{i,j} - \delta_{ij}(1+\gamma_s \mu_i)| = 0, \tag{17.37}$$

which follows at once from the extension of (13.4) to anisotropic scattering. If (17.8) holds with t much smaller than the number of ordinates used in (13.1), the equation (17.37) can be put into a simple form similar to (13.6). If, for instance, $f(\Omega' \to \Omega)$ is linear in $\Omega \cdot \Omega'$, then as in the derivation of (17.20) we obtain

$$\begin{vmatrix} 1-\frac{c}{2}\sum_i \frac{a_i}{1+\gamma_s\mu_i} & -\frac{3b_1c}{2}\sum_i \frac{a_i\mu_i}{1+\gamma_s\mu_i} \\ -\frac{c}{2}\sum_i \frac{a_i\mu_i}{1+\gamma_s\mu_i} & 1-\frac{3b_1c}{2}\sum_i \frac{a_i\mu_i^2}{1+\gamma_s\mu_i} \end{vmatrix} = 0, \qquad (17.38)$$

etc. However, unless t in (17.8) is small compared with the number of ordinates used, there is no advantage in transforming (17.37) to the form (17.38).

Apart from the derivation of (17.38), no use has been made of the assumption that (17.8) holds. This suggests that, when $f(\mu' \to \mu)$ varies rapidly or irregularly, so that the convergence of (17.1) is poor, the discrete ordinates method may be more useful than the spherical harmonics method. In such cases, however, it is necessary to be careful in the choice of summation formula (13.1). The choice advocated for the case of isotropic scattering (Gauss for critical-size problems and double-Gauss for the angular distribution emerging—see § 13.6), presupposes that the integrand is smooth, i.e. that (17.8) holds. If $f(\mu' \to \mu)$ is irregular, one should use a type of summation formula given by Bückner (7, pp. 111 ff.). If, for instance, $f(\mu' \to \mu)$ has the form (17.12), then $\partial^2 f(\mu' \to \mu)/\partial\mu'^2$ is discontinuous, and is infinite for $\mu' = \pm(1-\mu^2)^{\frac{1}{2}}$. In this case, according to Bückner, little can be gained by using a summation formula more complex than

$$\int_{-1}^{1} f(x)\,dx = \frac{2}{N}\sum_{j=-\frac{1}{2}N}^{\frac{1}{2}N-1} f\left(\frac{2j+1}{N}\right). \qquad (17.39)$$

Bückner's conclusions, however, were based on tentative, not rigorous, arguments.

Another more important advantage of the discrete ordinates method lies in the simplicity of the result for certain problems. For instance, since the boundary conditions at the free surface are the same as for isotropic scattering, the angular distribution of neutrons emerging from a half-space which is infinite and homogeneous is still given by the formula (13.19), as for isotropic scattering. The only difference is that the values of γ_s are different. Further, all the zeros and poles of (13.19)

lie to the right of $\mu = 0$, so that the formula (13.19) is applicable for the emerging neutrons ($\mu < 0$), whether or not μ is one of the ordinates used in the summation. Except in the case of isotropic scattering, it is much easier to achieve a given accuracy by increasing the number of ordinates in (13.19) than by increasing the accuracy with which the integrals of the Wiener–Hopf method are evaluated.

PART III

ENERGY-DEPENDENT PROBLEMS WITH SPECTRUM REGENERATION

XVIII

A GENERAL SURVEY OF ENERGY-DEPENDENT PROBLEMS

18.1. Slowing-down and spectrum-regenerating media

We now turn to problems where the assumptions of the constant cross-section approximation are no longer valid. That is, either the variation of the cross-sections in question with energy has to be taken into account, or the neutron energy spectrum is of interest in itself.

It is evident that media containing thermally fissile material behave in a way fundamentally different from those containing no fissile material. In the former case, a neutron of any energy has a probability of causing a fission and thus generating neutrons whose energies are at the top of the range considered. There is therefore a certain regeneration of the neutron spectrum, and we should expect the existence of some equilibrium spectrum, and hence of a solution of (2.4′), for an infinite source-free medium. If, however, fissile material is absent, a fast neutron will certainly lose energy in a collision, and if fast neutrons exist in the medium they must be supplied by sources, inside or outside the medium. That is, the equation (2.4′) has no solution in this kind of infinite source-free medium.

It is convenient to express this distinction more mathematically. Any solution for an infinite source-free medium can be expressed as a superposition of plane waves, i.e. of solutions of the form

$$N(\mathbf{r}, v\Omega) = A(v, \mu)e^{x/L}, \tag{18.1}$$

where L is some suitably chosen parameter of the dimensions of a length; according to § 2.3.3, L should satisfy the inequality

$$-1/\max l_{\text{tot}}(v) < \text{re}(1/L) < 1/\max l_{\text{tot}}(v). \tag{18.2}$$

Substituting (18.1) into (2.4'), we readily obtain

$$\left(1+\mu\frac{l(v)}{L}\right)vA(v,\mu) = l(v)\int\int\int \frac{v'c(v')}{l(v')}f(v'\Omega'\to v\Omega)A(v',\mu')\,dv'd\Omega'. \tag{18.3}$$

If scattering is isotropic in the L system, i.e. if $f(v'\Omega'\to v\Omega)$ has the form (2.23), $A(v,\mu)$ should clearly be of the form

$$A(v,\mu) = \frac{l(v)}{v}\left[1+\frac{\mu l(v)}{L}\right]^{-1}B(v). \tag{18.4}$$

Substituting this into (18.3) and effecting the integration over angles, we find

$$B(v) = \int B(v')c(v')f(v'\to v)\frac{l(v')}{2L}\log\frac{L+l(v')}{L-l(v')}\,dv'. \tag{18.3'}$$

Thus, if (2.4') has a solution in an infinite source-free medium, then there is a value of L satisfying (18.2) such that (18.3) (or (18.3') if (2.23) holds) has a solution.

We shall refer to media for which (18.3) (or (18.3') if appropriate) has a solution for L satisfying (18.2) as *spectrum-regenerating media*. If, on the other hand, (18.3) or (18.3') has no solution when L satisfies (18.2), we shall speak of *slowing-down media*.

Spectrum-regenerating media can be further classified according to the permissible eigenvalues L. If at least one of these eigenvalues is purely imaginary, the equation (2.4') can have a periodic solution in an infinite source-free medium. In a periodic solution, multiplication predominates over capture for every $\mathbf{r}$. We shall therefore call such media *multiplying media*. If no eigenvalue L is purely imaginary, then a solution of (2.4') in an infinite source-free medium presupposes a supply of neutrons from infinity, and in this case we shall speak of a *capturing medium*.

18.2. Media which do not degrade the spectrum

The above definition of spectrum-regenerating media in terms of the properties of equation (18.3) is rather inexact in one respect. If there is a medium containing no fissile material, but containing elastically scattering infinitely heavy nuclei, their contribution to $f(v'\Omega'\to v\Omega)$ from elastic scattering will be proportional to $\delta(v'-v)$ (see (2.18)). With fissile material absent and none of the nuclei infinitely heavy, the equation (18.3) is of the Volterra type, as may be seen from the discussion of $f(v'\Omega'\to v\Omega)$ in § 2.2. However, in the case mentioned just now, although the limits of integration are of the Volterra type, the usual

conditions on the behaviour of the kernel for $v' = v$ are not satisfied; in fact, unlike a Volterra equation, the equation (18.3) may have a solution for some $c(v)$ and L. The medium in question may therefore be spectrum-regenerating in the sense of § 18.1, although there is no regeneration in the sense of neutrons gaining energy. The term 'not degrading the spectrum indefinitely' would therefore have been more appropriate. However, it is usually a matter of indifference mathematically whether the medium is actually spectrum-regenerating, provided that it is non-multiplying, or merely does not degrade the spectrum indefinitely. We shall therefore use the former term for simplicity, in accordance with the properties of (18.3) rather than with the physical situation.

The above example imagined the presence of infinitely heavy nuclei in the medium. Although all actual nuclei have finite masses, the heavier ones may often be regarded as infinitely heavy, even when the medium is slowing-down and not spectrum-regenerating. Suppose, for instance, that we have a finite body composed of slowing-down material and irradiated by neutrons from outside, and that we require the neutron distribution close to the exposed surfaces. If the body contains some fairly heavy nuclei which scatter elastically in the C system, the neutron energy loss in the L system in each collision with these nuclei will be very small (cf. (2.11)), and many such collisions are needed to reduce the energy of a neutron appreciably. If, therefore, the number of neutrons in a given region which have undergone this large number of collisions is small, the neutron distribution in this region can be determined by neglecting the energy loss in such collisions, i.e. by treating the masses of the heavy nuclei as infinite. This may amount, from what we have said earlier in this section, to replacing a slowing-down medium by one which does not degrade the spectrum indefinitely, or, as we have agreed to call it, a spectrum-regenerating medium.

These remarks do not envisage any approximation in (18.3). We refer to an approximation in (2.4′) which is justified near sources, boundaries, etc., and to the properties of the modified equation (18.3) derived from this approximate form of (2.4′).

18.3. Spectrum-regeneration problems and slowing-down problems

The classification of media introduced in § 18.1 and discussed in § 18.2 leads directly to a classification of the problems that may be encountered. For, although the system concerned may contain both spectrum-regenerating and slowing-down media, the neutron distribution required

is always that in one or other type of medium. We shall speak of *spectrum-regeneration problems* and *slowing-down problems*, as the case may be.

In the constant cross-section approximation, the equation corresponding to (18.3) (or to (18.3′)) always has a solution. Spectrum-regeneration problems are therefore more akin to those discussed earlier in this book. For this reason we shall first consider spectrum-regeneration problems, leaving the discussion of slowing-down problems until Part IV. We shall also postpone until Part IV (§ 23.5) a certain method of dealing with spectrum-regeneration problems which depends on a preliminary solution of a slowing-down problem.

18.4. The main methods of solution of spectrum-regeneration problems

There are three possible approaches to the solution of spectrum-regeneration problems. These are:

(1) to subdivide the energy range concerned into a finite number of sub-ranges, within each of which the relevant cross-sections are approximately constant, and to treat the neutrons in each sub-range in the same manner as in the constant cross-section approximation;

(2) to select beforehand a set of simple orthogonal functions of the energy alone, such as a set of weighted orthogonal polynomials in v, and to expand the neutron density $N(\mathbf{r}, v\Omega)$ in terms of these, subsequently working with the coefficients in this expansion;

(3) to reduce the problem to one with a solution in two stages, the first being the solution of a set of one-velocity problems, and the second that of an equation involving only the energy.

Which of these approaches is used depends, of course, on the nature of the problem. We shall discuss them in turn in the next three chapters.

XIX

MULTI-GROUP THEORY

19.1. A general survey

THE first of the three approaches described above is called *two-group theory, three-group theory*, etc., depending on the number of energy sub-ranges used. The neutrons of the ith energy sub-range are called the ith group. If the total number of groups is unimportant, we shall speak of *multi-group theory*. This term is sometimes used to denote any calculation in which the variation of the cross-sections with energy is taken into account, but we shall not use it in that sense.

The division of the energy range into sub-ranges arises in some problems quite naturally. For example, neutrons are often divided into thermal and fast groups, because for many substances the cross-sections for thermal neutrons are markedly different from those for fast neutrons. Similarly, where a threshold process is involved, which cannot take place below a given energy, the threshold energy often forms a natural division between two groups. However, if many groups are used, it is better to take them from the curves showing the variation of cross-sections with energy, without requiring a simple physical interpretation of the groups.

In discussing multi-group theory, we shall assume scattering isotropic in the L system, i.e. we assume in §§ 19.2 to 19.6 that $f(v'\Omega' \to v\Omega)$ in (2.4′) has the form $(1/4\pi)f(v' \to v)$ (see (2.23)). In § 19.7 we shall make some remarks about the more general case where (2.23) is not assumed.

19.2. The assumptions and basic equations of multi-group theory

19.2.1. *The formal derivation of the equations*

As we have already indicated, in using m-group theory it is assumed that the entire energy range, and therefore the entire velocity range, can be divided into m intervals such that, if v is the ith interval,

$$l_{\text{tot}}(v) = l_i, \quad c(v) = c_i \quad \text{for} \quad v_{i-1} < v < v_i, \tag{19.1}$$

where l_i and c_i are some constants, while v_{i-1} and v_i are the lower and upper limits of the values of v in the ith group. In other words, the energy dependence of the total mean free path, and of the mean number of secondaries per collision, is approximated by means of step functions.

The assumptions (19.1), however, are not sufficient, and some further assumption must be made before the Boltzmann equation (2.4′) takes a tractable form. This may be done in several ways. The simplest mathematically, though not very convincing physically, is to postulate that each of the integrals

$$\int_{v_{i-1}}^{v_i} f(v' \to v)\, dv$$

can be approximated by a step function similar to (19.1), i.e. that

$$\int_{v_{i-1}}^{v_i} f(v' \to v)\, dv = f_{j \to i} \quad \text{for} \quad v_{j-1} < v' < v_j, \tag{19.2}$$

where $f_{j \to i}$ are some constants. By the normalization condition (2.24), these constants must satisfy the relation

$$\sum_i f_{j \to i} = 1. \tag{19.3}$$

If we assume (19.2) to hold, then, on integrating (2.4′) over the ith group of v and putting for brevity

$$\left.\begin{aligned} \psi_i(\mathbf{r}, \Omega) &= \int_{v_{i-1}}^{v_i} vN(\mathbf{r}, v\Omega)\, dv \\ S_i(\mathbf{r}, \Omega) &= \int_{v_{i-1}}^{v_i} S(\mathbf{r}, v\Omega)\, dv \end{aligned}\right\}, \tag{19.4}$$

we obtain, as in § 4.1,

$$\Omega \cdot \mathbf{grad}\, \psi_i(\mathbf{r}, \Omega) + \frac{\psi_i(\mathbf{r}, \Omega)}{l_i} = \sum \frac{c_j f_{j \to i}}{4\pi l_j} \int\int \psi_j(\mathbf{r}, \Omega)\, d\Omega + S_i(\mathbf{r}, \Omega). \tag{19.5}$$

If the assumptions underlying equation (2.30) also hold, i.e. if the sources are isotropic and we consider a single non-re-entrant homogeneous body, then by putting

$$\rho_i(\mathbf{r}) = \int_{v_{i-1}}^{v_i} vn(\mathbf{r}, v)\, dv$$

and proceeding as in the derivation of (19.5), we obtain from (2.30)

$$\rho_i(\mathbf{r}) = \frac{1}{4\pi} \int\int\int \frac{dV'}{|\mathbf{r} - \mathbf{r}'|^2} e^{-|\mathbf{r}-\mathbf{r}'|/l_i} \left[4\pi S_i(\mathbf{r}') + \sum \frac{c_j f_{j \to i}}{l_j} \rho_j(\mathbf{r}') \right], \tag{19.6}$$

while the equation (2.32) gives, in an obvious notation,

$$Q_i(\mathbf{r}) = S_i(\mathbf{r}) + \frac{1}{4\pi}\iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\sum_j e^{-|\mathbf{r}-\mathbf{r}'|/l_j}\frac{c_j f_{j\to i}}{l_j} Q_j(\mathbf{r}'), \quad (19.7)$$

and so on.

19.2.2. *The validity of the multi-group approximation for an infinite source-free medium*

The range of validity of the equations (19.5), etc., is not reflected in the above derivation, since the latter is based on the assumption (19.2), which is often open to objection. The most probable result of a collision is usually elastic scattering, and if a neutron is scattered elastically (in the C system) by a nucleus of mass M, its final velocity v will lie in the interval $[(M-1)/(M+1)]v' \leqslant v \leqslant v'$ (see (2.16)). Thus the actual value of the integral

$$\int_{v_{i-1}}^{v_i} f(v' \to v)\, dv$$

for v' in the interval $v_i < v' < v_{i+1}$ will be by no means constant; it will be quite large for $v_i < v' < [(M+1)/(M-1)]v_i$ and noticeably smaller for $[(M+1)/(M-1)]v_i < v' < v_{i+1}$. In fact, if there are no nuclei of mass less than M, and inelastic scattering is absent, the integral vanishes in the latter sub-interval.

This objection usually renders the equations (19.5), etc., inapplicable in slowing-down problems, but for spectrum-regeneration problems these equations can be derived from other assumptions, provided that $f_{j\to i}$ is suitably redefined. The assumption (19.1) is, of course, retained.

We first investigate under what conditions, and with what choice of $f_{j\to i}$, the equations (19.5), etc., lead to the correct result (i.e. the correct solution of (2.4′)) for an infinite source-free medium, nothing being assumed about the form of $f(v' \to v)$. This has partly been done in § 18.1. Let us consider the solution of (2.4′) corresponding to a particular eigenvalue, assumed non-degenerate. Using the assumption (2.23), (18.1), (18.4) and (19.1), and combining the contributions of all plane waves for the same value of L, we obtain

$$N(\mathbf{r}, v\boldsymbol{\Omega}) = \frac{l_i}{v}B(v)\iint G(\boldsymbol{\Omega}_0)\frac{L}{L+l_i\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}_0}e^{\mathbf{r}\cdot\boldsymbol{\Omega}_0/L}\, d\Omega_0 \quad \text{for } v_{i-1} < v < v_i, \quad (19.8)$$

where $\boldsymbol{\Omega}_0$ characterizes the orientation of the various plane waves, and $G(\boldsymbol{\Omega}_0)$ is an arbitrary function depending on the superposition of these

waves. The equation (18.3′) reduces, on the assumptions (19.1), to

$$B(v) = \sum_j \frac{c_j l_j}{2L} \log \frac{L+l_j}{L-l_j} \int_{v_{j-1}}^{v_j} B(v') f(v' \to v)\, dv'. \tag{19.9}$$

Putting now

$$\left.\begin{aligned} B_j &= \int_{v_{j-1}}^{v_j} B(v)\, dv \\ \text{and} \qquad f_{j\to i} &= \frac{1}{B_j} \int_{v_{i-1}}^{v_i} dv \int_{v_{j-1}}^{v_j} dv' B(v') f(v' \to v) \end{aligned}\right\}, \tag{19.10}$$

and using

$$\sum_i f_{j\to i} = \frac{1}{B_j} \int_{v_{j-1}}^{v_j} dv' B(v') \int_0^\infty f(v' \to v)\, dv = \frac{B_j}{B_j} = 1$$

in agreement with (19.3), we integrate (19.9) over the ith interval of v, obtaining

$$B_i = \sum_j \frac{c_j l_j}{2L} \log \frac{L+l_j}{L-l_j} f_{j\to i} B_j. \tag{19.11}$$

From the definition (19.4), the expression for $\psi_i(\mathbf{r}, \Omega)$ corresponding to (19.8) is obtained by replacing $B(v)/v$ in the latter formula by B_i. Substituting these $\psi_i(\mathbf{r}, \Omega)$ into (19.5), we arrive at an equation for the B_i which is identical with (19.11), since we are at present considering the source-free case. Thus, for an infinite source-free medium satisfying (19.1) and (2.23), the $\psi_i(\mathbf{r}, \Omega)$ defined by (19.4) in fact satisfy (19.5), regardless of any assumptions about $f(v' \to v)$, provided that only solutions for one particular L are considered, and that the $f_{j\to i}$ are chosen in accordance with (19.10).

19.2.3. *The validity of the multi-group approximation for finite systems. General conclusions*

Suppose we have a critical system consisting of a slightly multiplying core surrounded by capturing or slowing-down material; the fact that the core is multiplying means that the corresponding equation (18.3′) has at least one pair of purely imaginary eigenvalues

$$L = \pm i/\kappa. \tag{19.12}$$

The term *slightly* multiplying means that (i) there are no purely imaginary roots other than (19.12), (ii) κ is very small, i.e.

$$\left.\begin{aligned} &\kappa l(v) \ll 1 \\ \text{and} \qquad &\kappa |L'| \ll 1 \end{aligned}\right\}, \tag{19.13}$$

where L' is any eigenvalue, other than L, of (18.3′) for the core.

Let the value of $f(v' \to v)$ for the sth medium be $f_s(v' \to v)$, and let $B(v)$ be the solution of (18.3') for the core material, belonging to the eigenvalue (19.12). We define $f_{s,j\to i}$ by

$$f_{s,j\to i} = \frac{1}{B_j} \int_{v_{i-1}}^{v_i} dv \int_{v_{j-1}}^{v_j} dv' B(v') f_s(v' \to v), \tag{19.10'}$$

where B_j is given by the first of (19.10). We assume that none of the $f_s(v' \to v)$ differs greatly from the delta function $\delta(v'-v)$:

$$f_s(v' \to v) \cong \delta(v'-v), \tag{19.14}$$

so that none of the $f_{s,j\to i}$ differs greatly from δ_{ji}.

We seek a solution of (2.4') in the form

$$N(\mathbf{r}, v\Omega) = \{B(v)/vB_i\}\psi_i(\mathbf{r}, \Omega) + \delta N(\mathbf{r}, v\Omega) \quad \text{for } v_{i-1} < v < v_i, \tag{19.15}$$

where $\psi_i(\mathbf{r}, \Omega)$ are the solutions of (19.5), and we inquire whether $\delta N(\mathbf{r}, v\Omega)$ in this formula can be regarded as a negligibly small correction.

Let us first consider the situation in the interior of the core. By analogy with the conclusions of the constant cross-section approximation, it is seen that the solution in the core can be regarded as a superposition of infinite-medium solutions for the various eigenvalues L and of the contributions due to boundary effects. At the boundary of the core the contributions from the different types of term should be roughly comparable. As we move into the core, the contribution from the infinite-medium solution for the eigenvalue (19.12) should increase, while the contributions of the other terms decrease as $\exp[-d|\mathrm{re}(1/L')|]$ and $\exp[-d/\max l(v)]$ respectively, where d is the distance from the core boundary. Since the system is assumed to be critical or nearly so, the geometrical dimensions will be of the order of $1/\kappa$. The inequalities (19.13) then imply that in the interior of the core all terms except the infinite-medium solution for the eigenvalue (19.12) are quite negligible. The latter term, by the arguments of § 19.2.2, is given just by the first term of (19.15). Thus δN in (19.15) is completely negligible in the interior of the core.

We now estimate the value of δN in the rest of the system. Starting from the interior of the core and moving outwards, the various $\rho_i(\mathbf{r}) \equiv \int\int \psi_i(\mathbf{r}, \Omega)\, d\Omega$ will no longer remain in the same ratio as we approach the boundary and cross it into the surrounding medium.

Consequently, if we substitute the first term of (19.15) for $N(\mathbf{r}, v\Omega)$ in (2.4'), the right side of the latter will not be exactly proportional to $B(v)$, and so the first term of (19.15) alone is no longer an exact solution of (2.4'). However, if

$$\sum_j \frac{c_j}{4\pi l_j B_j} \int_{v_{j-1}}^{v_j} dv' B(v') f(v' \to v) \iint \psi_j(\mathbf{r}, \Omega)\, d\Omega \qquad (19.16)$$

were proportional to $B(v)$ within each interval (the constant of proportionality need not be the same for every interval), then on dividing (2.4') by $B(v)/vB_i$ for $v_{i-1} < v < v_i$ we should have (19.5); that is, the first term of (19.15) would still be a solution of (2.4'). In other words, the deviation of (19.16) from strict proportionality to $B(v)$ in each interval is the only reason for the appearance of δN in (19.15). However, since $f(v' \to v)$ is assumed not to differ greatly from $\delta(v'-v)$, the deviations of (19.16) from proportionality to $B(v)$ in each interval should be small. The free term of the inhomogeneous equation for $\delta N(\mathbf{r}, v\Omega)$ is therefore small everywhere, and though $\delta N(\mathbf{r}, v\Omega)$ may increase as we move outwards, it will do so very slowly. It is thus reasonable to expect that δN will not become comparable with the first term in (19.15) until distances from the core are reached where the total neutron population is too small to affect noticeably the solution in the core.

Thus, though (19.5) is not strictly equivalent to (2.4') with $l(v)$ and $c(v)$ given by (19.1), it is a good first approximation under the conditions imposed (see (19.13), etc.).

We can express somewhat differently the conclusion that under these conditions δN remains small beyond the point where $N(\mathbf{r}, v\Omega)$ ceases to be represented by the infinite-medium solution for the eigenvalue (19.12). For it implies that the spectrum within each group remains approximately independent of position, beyond the point where the total spectrum ceases to be so.

It should be remarked that the assumption (19.14) was introduced only to ensure this approximate conservation of the spectrum within each group, i.e. to ensure the approximate proportionality of (19.16) to $B(v)$ within each interval. However, the assumption (19.14) is not the only way of ensuring this property. For instance, instead of (19.14) we could have assumed that

$$f_s(v' \to v) = \gamma_{s,j\to i} B(v)(1+x) + \bar{\gamma}_{s,j\to i} \bar{f}_s(v' \to v)$$

$$\text{for } v_{j-1} < v' < v_j,\ v_{i-1} < v < v_i, \qquad (19.14')$$

where $\bar{f}_s(v' \to v)$ satisfies the assumption (19.14), x is a small quantity, and $\gamma_{s,j\to i}$ and $\bar{\gamma}_{s,j\to i}$ are constants such that $f_s(v' \to v)$ satisfies the normalization condition (2.24).

Similarly, if the spectrum of the neutrons in a group can be determined from other considerations, as for thermal neutrons, no restriction need be imposed on $f_s(v' \to v)$ for v in this group.

Finally, if the assumption (19.2) holds for the neutrons in some groups, and (19.14′) for the neutrons in the other groups, then the arguments of § 19.2.1 can be applied to the former and those of the present section to the latter. The equations (19.5), etc., can thus be justified for all the groups. This shows that for spectrum-regeneration problems multi-group theory may be applicable under a variety of conditions.

It sometimes happens that for one group neither of the assumptions underlying (19.5) is satisfied, but that the only processes affecting the neutrons in this group are slowing-down processes. In this case the group concerned can be dealt with by solving a slowing-down problem and using its solution as the source term in the next group. We shall not discuss this method until after the consideration of slowing-down problems, in Chapter XXIII; at present we shall use only the simpler form of multi-group theory expressed in the equations (19.5), (19.6), etc.

If it is desired to use the formulae (19.10), (19.10′), etc., to evaluate the constants $f_{j\to i}$, $B(v)$ must be known. The latter will be determined in § 22.2. Meanwhile, we shall regard the $f_{j\to i}$ as given numerically, which is often the case.

19.2.4. *The matrix form of the multi-group equations*

Before discussing the solution of the equations (19.5), (19.6), etc., the subject of their transcription may be considered. In order to simplify the algebraic manipulation of these equations, it is often convenient to introduce matrix notation. We shall use the m-dimensional column vectors† $\boldsymbol{\psi}(\mathbf{r}, \boldsymbol{\Omega})$, $\boldsymbol{\rho}(\mathbf{r})$, etc., with the components $\psi_i(\mathbf{r}, \boldsymbol{\Omega})$, $\rho_i(\mathbf{r})$, etc., respectively, and the diagonal matrices c, l, l^{-1}, etc., with components along the diagonal c_i, l_i, l_i^{-1}, etc., respectively, and finally the matrix f whose element in the jth column and ith row is $f_{j\to i}$. In this notation, with the usual rule of matrix multiplication, the equations (19.5), (19.6),

† These m-dimensional column and row vectors are denoted by bold-face italic or Greek letters (with the exception of $\boldsymbol{\Omega}$), while bold-face roman letters and $\boldsymbol{\Omega}$ denote physical vectors. We denote by $\mathbf{j}$, unless otherwise stated, a column vector whose components are physical vectors.

(19.7), etc., can be rewritten‡

$$\boldsymbol{\Omega}.\mathbf{grad}\,\psi(\mathbf{r},\boldsymbol{\Omega})+\mathfrak{l}^{-1}\psi(\mathbf{r},\boldsymbol{\Omega}) = \frac{1}{4\pi}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\int\!\!\int \psi(\mathbf{r},\boldsymbol{\Omega}')\,d\boldsymbol{\Omega}'+S(\mathbf{r},\boldsymbol{\Omega}), \tag{19.5'}$$

$$\rho(\mathbf{r}) = \frac{1}{4\pi}\int\!\!\int\!\!\int \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\exp(-\mathfrak{l}^{-1}|\mathbf{r}-\mathbf{r}'|)[\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\rho(\mathbf{r}')+4\pi S(\mathbf{r}')], \tag{19.6'}$$

$$Q(\mathbf{r}) = \frac{1}{4\pi}\int\!\!\int\!\!\int \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\exp(-\mathfrak{l}^{-1}|\mathbf{r}-\mathbf{r}'|)Q(\mathbf{r}')+S(\mathbf{r}), \tag{19.7'}$$

and so on. In this notation, the kernel of the equation (19.7′) differs from that of (19.6′) only by the order of the factors. Since $\mathfrak{f}$ does not in general commute with the other factors, the two kernels are not the same. Of course, all the diagonal matrices commute.

In the equation (19.6′) and (19.7′), we have assumed that the body concerned is single, homogeneous, and non-re-entrant. If this assumption does not hold, the equations mentioned must be replaced by

$$\rho(\mathbf{r}) = (1/4\pi)\int\!\!\int\!\!\int \{dV'/|\mathbf{r}-\mathbf{r}'|^2\}\exp[-\mathfrak{t}(\mathbf{r},\mathbf{r}')]\times \times[\mathfrak{f}(\mathbf{r}')\mathfrak{c}(\mathbf{r}')\mathfrak{l}^{-1}(\mathbf{r}')\rho(\mathbf{r}')+4\pi S(\mathbf{r}')] \tag{19.17}$$

and

$$Q(\mathbf{r}) = (1/4\pi)\int\!\!\int\!\!\int \{dV'/|\mathbf{r}-\mathbf{r}'|^2\}\mathfrak{f}(\mathbf{r})\mathfrak{c}(\mathbf{r})\mathfrak{l}^{-1}(\mathbf{r})\exp[-\mathfrak{t}(\mathbf{r},\mathbf{r}')]Q(\mathbf{r}')+S(\mathbf{r}), \tag{19.18}$$

where $\mathfrak{t}(\mathbf{r},\mathbf{r}')$ is a diagonal matrix with elements $\tau_i(\mathbf{r},\mathbf{r}')$; see (2.31′) and (2.38′).

The corresponding homogeneous equations under variable composition are obtained by omitting the source term and replacing $1/4\pi$ by $\gamma/4\pi$. These equations we shall call (19.5″), (19.17′), and (19.18′).

19.2.5. *The adjoint equations*

In order to apply matrix notation conveniently to the adjoint equations, the solutions of the latter should be represented as row vectors rather than column vectors. By the rules of matrix multiplication, a row vector should always be placed in front of the matrix operating on it, and the product of a row and a column vector is understood as a scalar product. From the discussion of §§ 3.5 and 3.6, the equations adjoint to (19.5″) and (19.17′) (just defined) are then readily seen to be

$$-\boldsymbol{\Omega}.\mathbf{grad}\,\psi^\dagger(\mathbf{r},\boldsymbol{\Omega})+\psi^\dagger(\mathbf{r},\boldsymbol{\Omega})\mathfrak{l}^{-1} = (\gamma/4\pi)\int\!\!\int \psi^\dagger(\mathbf{r},\boldsymbol{\Omega}')\,d\boldsymbol{\Omega}'\,\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1} \tag{19.19}$$

‡ Expressions of the form $(a+\mathfrak{b})$, a being a number and $\mathfrak{b}$ a matrix, should be understood as $a\mathfrak{I}+\mathfrak{b}$, $\mathfrak{I}$ being the unit matrix.

and

$$\boldsymbol{\rho}^{\dagger}(\mathbf{r}) = (\gamma/4\pi)\iiint \{dV'/|\mathbf{r}-\mathbf{r}'|^2\}\boldsymbol{\rho}^{\dagger}(\mathbf{r}')\exp[-t(\mathbf{r},\mathbf{r}')]\mathfrak{f}(\mathbf{r})\mathfrak{c}(\mathbf{r})\mathfrak{l}^{-1}(\mathbf{r}), \tag{19.20}$$

with a similar equation for $\boldsymbol{Q}^{\dagger}(\mathbf{r})$, where $\boldsymbol{\rho}^{\dagger}$, $\boldsymbol{Q}^{\dagger}$, and $\boldsymbol{\psi}^{\dagger}$ are row vectors. The orthogonality relation (3.29) becomes, with suitable normalization,

$$\iiint dV\,\boldsymbol{\rho}_j^{\dagger}(\mathbf{r})\boldsymbol{\rho}_i(\mathbf{r}) = \delta_{ij}, \tag{19.21}$$

where $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j^{\dagger}$ now denote eigenfunctions of (19.17′) and (19.20). On using (3.25) and (3.27) together with (3.28), we obtain the following relations between $\boldsymbol{\psi}^{\dagger}$ and $\boldsymbol{\rho}^{\dagger}$:

$$\boldsymbol{\psi}^{\dagger}(\mathbf{r},\Omega) = \frac{\gamma}{4\pi}\int_0^{\infty} ds\;\boldsymbol{\rho}^{\dagger}(\mathbf{r}+s\Omega)e^{-t(\mathbf{r},\mathbf{r}+s\Omega)} \tag{19.22}$$

and

$$\boldsymbol{\rho}^{\dagger}(\mathbf{r}) = \iint d\Omega\,\boldsymbol{\psi}^{\dagger}(\mathbf{r},\Omega)\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}. \tag{19.23}$$

19.3. Infinite and semi-infinite media

19.3.1. *An infinite source-free medium*

We now consider methods of solution of the equations (19.5′), etc. These equations are formally identical with the corresponding equations in the constant cross-section approximation. The only difference is that the coefficients are matrices and not ordinary numbers. It is therefore unnecessary to repeat the analysis given in earlier chapters; we shall merely indicate the differences arising from the coefficients' being matrices.

Let us first take problems for which an exact solution is available in the constant cross-section approximation, beginning with the case of an infinite source-free medium.

Proceeding as in § 5.1, we see that for an infinite source-free medium the most general solution of (19.5′) is

$$\boldsymbol{\psi}(\mathbf{r},\Omega) = \sum_k \iint G_k(\Omega_0)\left[\mathfrak{l}^{-1}+\frac{\Omega\,.\,\Omega_0}{L_k}\right]^{-1}\boldsymbol{B}_k\exp\left[\frac{\mathbf{r}\,.\,\Omega_0}{L_k}\right]d\Omega_0, \tag{19.24}$$

where $G_k(\Omega_0)$ are arbitrary scalar functions of Ω_0, the column vectors $\boldsymbol{B}_k$ are the suitably normalized solutions of the equations

$$\boldsymbol{B}_k = \mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\frac{L_k}{2}\log\frac{L_k+\mathfrak{l}}{L_k-\mathfrak{l}}\boldsymbol{B}_k, \tag{19.25}$$

and the L_k are the solutions of the determinantal equation

$$\left|1-\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\frac{L}{2}\log\frac{L+\mathfrak{l}}{L-\mathfrak{l}}\right| = 0, \tag{19.26}$$

which arises from the condition that (19.25) is soluble. The solutions of (19.26) appear, of course, in pairs, since that equation is unchanged when L is replaced by $-L$. Since the logarithm in (19.25) and (19.26) is the result of evaluating the integrals

$$\int_{-1}^{1}\left[\mathfrak{l}^{-1}+\frac{\mu}{L_k}\right]^{-1}d\mu,$$

the principal value of the logarithm should always be used, and only those roots of (19.26) are admissible which lie in the complex L-plane cut along the real axis from $-\max l_t$ to $+\max l_t$, $\max l_t$ being the largest of the l_t.

We first consider the determination of the roots of (19.26) which are large compared with all the l_t. In this case, it is convenient to rewrite (19.26) in the form

$$\left|\mathfrak{l}\left[\tfrac{1}{2}L\log\frac{L+\mathfrak{l}}{L-\mathfrak{l}}\right]^{-1}-\mathfrak{f}\mathfrak{c}\right| = 0.$$

If L is large compared with every l_t, we can expand each element of this determinant in powers of $L^{-1}\mathfrak{l}$, and the equation (19.26) then becomes

$$\left|1-\mathfrak{f}\mathfrak{c}-\frac{1}{3L^2}\mathfrak{l}^2-\frac{4}{45L^4}\mathfrak{l}^4-\ldots\right| = 0. \tag{19.26'}$$

It is seen immediately from an inspection of this equation that, if $\mathfrak{f}\mathfrak{c}$ is nearly a unit matrix, i.e. if

$$1-\mathfrak{f}\mathfrak{c} = O(\epsilon) \tag{19.27}$$

where ϵ is small, then the various l_t^2/L^2 will also be of the order of ϵ. Thus, retaining in (19.26') only the leading terms, we can replace that equation by

$$|1-\mathfrak{f}\mathfrak{c}-(1/3L^2)\mathfrak{l}^2| = 0. \tag{19.28}$$

The left side of (19.28) is a polynomial of order $2m$ in $1/L$, and therefore this equation has exactly as many pairs of roots $1/L = \pm 1/L_k$, say, as the number of groups used. Also, if (19.27) is well satisfied, all these $2m$ roots $L = \pm L_k$ will be large compared with any of the l_t, and therefore lie in the cut plane. Under the same conditions, there will be a root of (19.26') near each root of (19.28), and therefore a root of (19.26). If (19.27) holds, there will thus be at least as many pairs of roots of (19.26) as the number of groups used.

It can also be shown that under these conditions (19.26) cannot have more than m pairs of roots. For if it had any roots other than those associated with the roots of (19.28), the derivation of the latter equation would be inapplicable to these roots, and they would therefore have to be small, or comparable with some l_i. For such L, the terms which are small in (19.26) and can be neglected in the first approximation would, if (19.27) holds, be non-diagonal terms. That is, (19.26) could, so far as the L comparable with the l_i are concerned, be approximated by

$$\prod_{i=1}^{m}\left(1-f_{i\to i}c_i\frac{L}{2l_i}\log\frac{L+l_i}{L-l_i}\right)=0. \tag{19.29}$$

For $f_{i\to i}c_i = 1+O(\epsilon)$, each root of this equation is large compared with the corresponding l_i. That is, if (19.27) is satisfied, the roots of (19.29) are large compared with the l_i, and so (19.26) cannot have any roots in the cut plane which are small, or comparable with the l_i. This proves that, with the condition (19.27), any root of (19.26) is approximately given by the corresponding root of (19.28).

It should be noticed that the degree of smallness of $1-\mathfrak{f}\mathfrak{c}$ required for the above derivation depends on the spread in the values of the l_i. The greater this spread, the smaller must $1-\mathfrak{f}\mathfrak{c}$ be before it is certain that the roots of (19.28) are a good approximation to those of (19.26), or even that the two equations have the same number of roots.

It may also be remarked that the equation (19.28) represents a direct extension of the approximate formula (5.15 a) for the diffusion length in the constant cross-section approximation. This equation will be encountered again in § 19.5.3.

If (19.27) is not satisfied, the number of pairs of roots of (19.26) in the cut plane need no longer equal the number of groups. The change in the number of roots may occur as follows: suppose all $f_{j\to i}$ vanish for $j > i$, so that (19.26) reduces to (19.29). Let $\pm L_i$ be the roots of the ith factor in (19.29), and suppose that some of the $f_{i\to i}c_i$ are noticeably less than unity. For each such i, L_i will be comparable with l_i, and if the l_i differ appreciably among themselves, it may happen that some L_j are smaller than the largest l_i. These L_j therefore lie on the cut, and are not permissible in the solution of (19.26). Of course, the assumption that $f_{j\to i} = 0$ for $j > i$ is made for simplicity, and the reduction in the number of roots can occur when no $f_{j\to i}$ vanishes.

These arguments suggest that, if (19.26) has m' pairs of roots in the cut plane, then m' is bounded by

$$1 \leqslant m' \leqslant m, \tag{19.30}$$

where m is the number of groups used. The result (19.30) has been proved rigorously for two-group theory (Melvin, unpublished), by following the change in arg $F(L)$ as one goes round the cut, $F(L)$ being the left side of (19.26). The proof is not difficult, since it is sufficient to keep track of the changes of quadrant of $F(L)$, i.e. of the changes of sign of re $F(L)$ and im $F(L)$. The extension of this rigorous proof to a larger number of groups has not yet been attempted, however.

We have drawn attention to the possibility of situations where $m' < m$ (fewer pairs of roots than groups) because in these cases some difficulties are encountered in extending the diffusion approximation and the Serber–Wilson method (see § 19.4.3).

The following remarks may be made regarding the general properties of the solution in an infinite source-free medium. The neutron flux associated with (19.24) is given, as is seen by integrating (19.24) over all angles, by

$$\rho(\mathbf{r}) = \sum_k \tfrac{1}{2} L_k \log \frac{L_k+1}{L_k-1} B_k \phi_k(\mathbf{r}), \tag{19.31}$$

where $$\phi_k(\mathbf{r}) = 4\pi \iint G_k(\Omega_0) \exp[\mathbf{r}.\Omega_0/L_k]\, d\Omega_0$$

is an arbitrary solution of

$$(1-L_k^2\nabla^2)\phi_k(\mathbf{r}) = 0 \tag{19.32}$$

regular in all space.

The corresponding expression for the neutron current can be obtained as follows. Starting from the transport equation (19.5′) and proceeding as in the footnote to (8.14) we have

$$\iint_A \boldsymbol{j}(\mathbf{r}).\mathbf{dA} = [\mathfrak{f}c-1][\mathfrak{l}^{-1} \iiint_V \rho(\mathbf{r})\, dV.$$

Denoting the kth term in (19.31) by $\rho_k(\mathbf{r})$ and using (19.32), we can rewrite the last expression as

$$\iint_A \boldsymbol{j}(\mathbf{r}).\mathbf{dA} = [\mathfrak{f}c-1][\mathfrak{l}^{-1} \iiint_V \sum_k L_k^2 \nabla^2 \rho_k(\mathbf{r})\, dV,$$

and hence, by Gauss's theorem, since $\nabla^2 \equiv \mathrm{div}\,\mathbf{grad}$,

$$\boldsymbol{j}(\mathbf{r}) = (\mathfrak{f}c-1)\mathfrak{l}^{-1} \sum_k L_k^2\, \mathbf{grad}\, \rho_k(\mathbf{r}), \tag{19.33}$$

which represents the natural extension of (8.14).

These remarks also apply to the solution of the adjoint equation. Here the determinantal equation will differ from (19.26′) only by the fact that the matrix $\mathfrak{f}$ is replaced by its transpose. This obviously cannot affect the value of the determinant. Thus the values of L_k for the adjoint

equation are the same as for the original equation, though the corresponding row vectors $B_k^\dagger$ need not be simply related to the column vectors B_k. The same will be true of the determinantal equations which we shall encounter in § 19.5.

19.3.2. *'Solutions' which do not satisfy the conditions at infinity*

In the constant cross-section approximation, it was unnecessary to consider the infinite-medium solutions which do not satisfy the conditions at infinity. However, the discussion preceding (19.30) suggests that in multi-group theory this may be necessary. We therefore examine the solutions in question. Let us first recall the derivation of the permissible solutions given in the preceding section. For simplicity we consider only the plane case, i.e. $\psi(\mathbf{r}, \Omega) \equiv \psi(x, \mu)$. The first step in the derivation was to take $\psi(x, \mu)$ in the form

$$\psi(x, \mu) = A(\mu)\exp(x/L),$$

and to substitute this into (19.5'), giving

$$\left(\frac{\mu}{L}+\mathfrak{l}^{-1}\right)A(\mu) = \frac{1}{4\pi}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\int\int A(\mu')\,d\Omega'. \tag{19.34}$$

The next step was to solve (19.34) for $A(\mu)$, regarding $\iint A(\mu')\,d\Omega'$ as known. If the solutions must satisfy the conditions at infinity, L must lie in the cut plane, and then the only solution of (19.34) is

$$A(\mu) = \left(\frac{\mu}{L}+\mathfrak{l}^{-1}\right)^{-1}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\frac{1}{4\pi}\int\int A(\mu')\,d\Omega', \tag{19.35}$$

which, on integration over μ, leads immediately to (19.25). However, if the conditions at infinity can be violated, L need no longer lie in the cut plane, but may be on the cut itself. In this case

$$A(\mu) = \delta(\mu+L\mathfrak{l}^{-1})X+\text{the right side of (19.35)}, \tag{19.36}$$

where X is an arbitrary constant column vector, is also a solution. Integrating (19.36) over all angles and putting $\iint A(\mu)\,d\Omega = C$, we obtain

$$C = 2\pi\mathfrak{s}(L)X+\tfrac{1}{2}L\log\left(\left|\frac{L+\mathfrak{l}}{L-\mathfrak{l}}\right|\right)\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}C, \tag{19.37}$$

where $\mathfrak{s}(L)$ is a diagonal matrix whose ith element is unity if $-l_i \leqslant L \leqslant l_i$, and zero otherwise. The equations (19.37) are the direct counterpart of (19.25), as can be seen by multiplying them by $\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}$ and putting $B = \mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}C$. However, whereas (19.25) formed a system of homogeneous equations, (19.37) is a system of inhomogeneous equations with an arbitrary (but adjustable) free term, and therefore always has a solu-

tion.† That is, any value of L on the cut is permissible for solutions which can violate the conditions at infinity. The expression (19.37) can be written more explicitly as follows. Since X is arbitrary, we have by the definition of $s(L)$:

$$\left.\begin{aligned} &C_i \text{ is arbitrary} && \text{if } -l_i \leqslant L \leqslant l_i \\ &C_i = \tfrac{1}{2}L\log\frac{L+l_i}{L-l_i}\sum_j f_{j\to i}\frac{c_j}{l_j}C_j && \text{otherwise} \end{aligned}\right\}, \quad (19.38)$$

provided that the determinant of the latter set of equations does not vanish. If it does vanish, then a relation must be introduced to connect the C_i whose $l_i \geqslant |L|$, but some of the C_i whose $l_i < |L|$ will become arbitrary. We give, as an illustration, the form of the solution for the constant cross-section approximation. Replacing L by $-s$ to avoid confusion with the diffusion length, and choosing the arbitrary multiplier X in a convenient manner, we have

$$\psi_s(x,\mu) = \left[2\left(1-\frac{sc}{2l}\log\frac{l+s}{l-s}\right)\delta\left(\mu-\frac{s}{l}\right)+\frac{c}{1-l\mu/s}\right]e^{-x/s}, \quad (19.39)$$

where the subscript s indicates the particular value of s used.

The interpretation of the 'solution' (19.39) is not difficult. The appearance of the term $\delta(\mu-s/l)$ indicates that it corresponds to the presence of a collimated source which emits neutrons at an angle $\cos^{-1}(s/l)$ to the direction of the x-axis. For $s > 0$, (19.39) violates the conditions at $x = -\infty$, but this objection no longer holds if (19.39) is applied to the half-space $x > 0$ only, for instance, instead of all space. The flux associated with (19.39) is $e^{-x/s}$, and conversely, if the flux is $e^{-x/s}$, the angular distribution is of the form (19.39); this follows from the above derivation. Now in the plane case of the constant cross-section approximation $\rho_{tr}(x)$ for $x > 0$, due to a boundary or a plane source at $x = 0$, was given by an expression of the form

$$\rho_{tr}(x) = \int_0^l g(s)e^{-x/s}\,ds.$$

It follows that the transient part of the corresponding angular distribution is

$$\psi_{tr}(x,\mu) = \int_0^l g(s)\psi_s(x,\mu)\,ds. \quad (19.40)$$

The above arguments can be extended at once to other geometries and to multi-group theory. The infinite-medium solutions which violate

† Unless X is otherwise fixed, it need not be stipulated that L is not a root of (19.26). For if L is a root of (19.26) lying on the cut, the equations (19.37) can again be rendered soluble by putting $X = 0$, say.

the conditions at infinity are therefore important as giving, by superposition, the transient terms, although none of these solutions is physically permissible by itself.

The discussion of these 'solutions' will be made use of later; for the moment we shall continue with the problems for which an exact solution is known in the constant cross-section approximation.

19.3.3. *An infinite medium containing sources*

The formal analysis in the case of an infinite medium containing sources is the same as in the constant cross-section approximation. For instance, in the case of an isotropic point source of strength S (i.e. emitting S_i neutrons of the ith group per unit time per unit solid angle), we can repeat the arguments of § 5.3, taking care not to permute factors which do not commute, and obtain

$$\rho(r) = \frac{1}{i\pi r}\int_{-\infty}^{\infty} \mathfrak{l}\left(1-\frac{1}{p}\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}.\mathfrak{f}\mathfrak{c}\right)^{-1}\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}.Se^{ipr}\,dp. \tag{19.41}$$

The singularities of the integrand in the complex p-plane cut along the imaginary axis from $i/\max l_j$ to $i\infty$ and from $-i\infty$ to $-i/\max l_j$ are given by the roots of

$$\left|1-\frac{1}{p}\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}.\mathfrak{f}\mathfrak{c}\right| = 0, \tag{19.42}$$

and these are closely related to those of (19.26). In fact, if $\mathfrak{U}$ and $\mathfrak{B}$ are any two matrices, $|\mathfrak{U}\mathfrak{B}| = |\mathfrak{B}\mathfrak{U}|$ (this notation signifies the determinants), and hence $|1-\mathfrak{U}\mathfrak{B}| = |\mathfrak{U}(1-\mathfrak{B}\mathfrak{U})\mathfrak{U}^{-1}| = |1-\mathfrak{B}\mathfrak{U}|$. Using this and the substitution $L = i/p$, we immediately see the equivalence of (19.26) and (19.42).

If the number of pairs of roots of (19.26) in the cut plane is equal to the number of groups, it can be shown that, along the cut, the quantity

$$\left(1-\frac{1}{p}\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}.\mathfrak{f}\mathfrak{c}\right)^{-1}\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}$$

remains finite and varies smoothly, so that, on changing the path of integration and using the method of residues, we have

$$\rho(r) = \sum_k \frac{1}{r}e^{-r/L_k}\mathfrak{B}_k S + \int_{1/\max l_j}^{\infty} \mathfrak{W}(s)e^{-rs}\,ds.S, \tag{19.43}$$

where the $\mathfrak{B}_k$ are some constant matrices and $\mathfrak{W}(s)$ is a smoothly varying bounded matrix function of s. The first term in (19.43) can clearly be identified with $\rho_{as}(r)$, and the second with $\rho_{tr}(r)$.

If the number of pairs of roots of (19.26) in the cut plane is less than the number of groups, it may happen, as we have already seen, that (19.42) has further roots which lie on the cut itself, or that it has roots beyond the cut on other sheets of the Riemann surface, which are yet so near the cut as to affect the values of the integrand on the cut. In the former case (poles of the integrand on the cut) the contour must be indented at these poles in following the cut; alternatively, the principal value of the integral along each side of the cut can be taken and half the residue at each pole added for each side of the cut. The contribution of these added half-residues will clearly be the same in form as that due to the poles in the cut plane, i.e. of the form

$$(1/r)\exp(-r/L^*)\mathfrak{B}^*S, \tag{19.44}$$

and, in a certain range of r values, this may predominate over the principal value of the integral along the cut. In the other case (poles of the integrand near the cut on another sheet), the integration along the cut can be carried out without difficulty, but the value of $\mathfrak{B}(s)$ will be very large near these poles, and in a certain range of r values the contribution from this part of the path of integration may predominate over that from both smaller and larger s. The contribution from this maximum of $\mathfrak{B}(s)$ will again be of the form (19.44).

For very large r, the contribution from the smallest s will be dominant, and for very small r, that due to non-scattered neutrons. Therefore, if for example the number of pairs of roots of (19.26) in the cut plane is one less than the number of groups, the behaviour of $\rho_{tr}(r)$ will be of the form

$$\rho_{tr}(r) = \left\{\begin{array}{ll} O(1/r^2) & \text{for small } r, \\ O([1/r]\exp[-r/L^*]) & \text{for intermediate } r, \\ O(\exp[-r/\max l_j]) & \text{for very large } r. \end{array}\right\} \tag{19.45}$$

The appearance of a distinct intermediate range is connected with the disappearance of one of the terms in $\rho_{as}(r)$.

The same kind of analysis can also be carried out for other types of source. On repeating the arguments of § 5.5.2, we can see that, for plane sources and surface sources of large curvature, the importance of the intermediate r range in (19.45) is greater than for a point source.

19.3.4. *Semi-infinite media*

Having discussed the cases where solutions are available for an infinite homogeneous medium, we now turn to the case of an infinite half-space. Here the situation is in one respect markedly different from that in the constant cross-section approximation. In the latter

('one-group theory'), Milne's problem could be solved by means of the Wiener–Hopf method, but in multi-group theory this solution is no longer available except in some special cases. This difference arises in the following way. The essential step in the application of the Wiener–Hopf method to one-group theory was to represent

$$1-c\frac{\tan^{-1}pl}{pl} \tag{19.46}$$

as the product of two functions, one regular in the upper half-plane ($\operatorname{im} p > 0$), and the other regular in the lower half-plane ($\operatorname{im} p < 0$). This was achieved by taking the logarithm of (19.46) and representing it as the sum of two functions satisfying the prescribed regularity conditions. In multi-group theory, instead of (19.46), we have to consider the matrix function of p

$$1-\frac{1}{p}\,\mathfrak{l}^{-1}\tan^{-1}p\mathfrak{l}\,\mathfrak{f}\mathfrak{c}, \tag{19.46'}$$

and represent this as the product of two matrix functions satisfying the regularity conditions. However, the logarithm of the product of two matrices is not equal to the sum of their logarithms unless the matrices commute. Thus the essential step in the Wiener–Hopf method cannot be carried out by the same means as in one-group theory except in special cases, and no alternative procedure has yet been suggested.

One of these special cases is that where the various groups differ only as regards the mean number of secondaries per collision c_i, the mean free path being the same for all groups. In this case $\mathfrak{l}^{-1}(\tan^{-1}p\mathfrak{l})/p$ is not a matrix but a number, and the following procedure can be used. Let the matrix $\mathfrak{f}\mathfrak{c}$ be represented in diagonal form:

$$\mathfrak{f}\mathfrak{c} = \mathfrak{S}\mathfrak{g}\mathfrak{S}^{-1}, \tag{19.47}$$

where $\mathfrak{g}$ is diagonal. Then (19.46′) can be rewritten as

$$\mathfrak{S}\left(1-\mathfrak{g}\frac{\tan^{-1}pl}{pl}\right)\mathfrak{S}^{-1},$$

where $\mathfrak{S}$ is independent of p and $1-\mathfrak{g}(\tan^{-1}pl)/pl$ is diagonal. A diagonal matrix can always be represented as a product of the kind required by applying to each of the diagonal elements the same transformations as in one-group theory. This gives

$$1-\mathfrak{g}(\tan^{-1}pl)/pl = \mathfrak{h}^{+}(p)[\mathfrak{h}^{-}(p)]^{-1},$$

say, where $\mathfrak{h}^{+}(p)$ and $\mathfrak{h}^{-}(p)$ are also diagonal. The required representation of (19.46′) is then

$$\mathfrak{S}\mathfrak{h}^{+}(p)\mathfrak{S}^{-1}.\,\mathfrak{S}[\mathfrak{h}^{-}(p)]^{-1}\mathfrak{S}^{-1}.$$

It is to be noticed that the factors into which (19.46′) has been divided commute in this case, and this is why the required representation is possible.

The commutability of these factors could have been foreseen in the following manner. The matrices f and c enter only as fc, and so, if l is a number and not a matrix, the only matrices that can be encountered are fc and those constructed from it by means of analytical operations with numerical coefficients. It is not possible to arrive by this means at non-commuting matrices, starting from a single matrix.

19.4. The diffusion approximation and the Serber–Wilson method

19.4.1. *General remarks*

We now consider the available approximate methods. These we shall take in the same order as when discussing the constant cross-section approximation. However, it must be pointed out that the relative advantages of the various methods are no longer the same as in that approximation. If $\rho(\mathbf{r})$ is represented in the form

$$\rho(\mathbf{r}) = \rho_{\mathrm{diff}}(\mathbf{r}) + \rho_{\mathrm{tr}}(\mathbf{r}), \tag{19.48}$$

where $\rho_{\mathrm{diff}}(\mathbf{r})$ satisfies the same differential equation as the permissible infinite-medium solution, and $\rho_{\mathrm{tr}}(\mathbf{r})$ is of the order of $\exp[-d/\max l_i]$, the situation in multi-group theory differs markedly from that in the constant cross-section approximation. Firstly, it follows from the remarks in § 19.3.3 that less information is now available about $\rho_{\mathrm{tr}}(\mathbf{r})$ and its importance compared with the other term. Secondly, as can be expected from the discussion of (19.30) and (19.45), $\rho_{\mathrm{tr}}(\mathbf{r})$ may, for some values of f, c, and l, be of greater importance from the analytical point of view than it could be in the constant cross-section approximation.

In the spherical harmonics method and the discrete ordinates method, $\rho_{\mathrm{diff}}(\mathbf{r})$ and $\rho_{\mathrm{tr}}(\mathbf{r})$ are treated on the same footing. In the diffusion approximation and the Serber–Wilson method, however, $\rho_{\mathrm{tr}}(\mathbf{r})$ is neglected, and only an indirect allowance is made for it. This implies that the two former methods can be used as before, and the accuracy in each order of approxination will be about the same as in one-group theory. In the latter two methods, on the other hand, much greater uncertainties and limitations may be met with than in one-group theory. The discussion of these methods will therefore be of a somewhat negative character.

19.4.2. *The case where the infinite-medium equation has enough solutions*

In the diffusion approximation and the Serber–Wilson method, we try to approximate the solution for a finite medium by means of a suitable combination of solutions of the differential equation satisfied by $\rho(\mathbf{r})$ in an infinite medium. That is, we try to approximate $\rho(\mathbf{r})$ by an expression of the form (19.31), where the B_k are solutions of (19.25) and the $\phi_k(\mathbf{r})$ are solutions of (19.32) which are regular in the region considered, though not necessarily regular in all space. The possibility of doing so may depend on the number of these solutions, i.e. on whether the number of pairs of roots of (19.26) in the cut plane is equal to or less than the number of groups used.

We first consider the former case. Here the general solution of the infinite-medium differential equation contains exactly as many arbitrary parameters (or depends on exactly as many arbitrary functions) as are needed to satisfy two conditions for each group at each interface and one condition for each group at each free surface. That is, the same number of conditions can be satisfied for each group at each boundary as in the constant cross-section approximation, and this is clearly what is wanted. We shall therefore call this *the case where the infinite-medium equation has enough solutions.*

Let us now consider the actual conditions to be imposed. For the Serber–Wilson method these are self-evident. In one-group theory we exactly satisfied the integral equation at the centre of the system; we shall now do so for each group separately. Passing to Wilson's formulation of the method, we have

$$\psi_{SW}(r, -1) \text{ is continuous,} \tag{19.49}$$

and we should also wish to satisfy the condition that the total number of neutrons in each group leaving a medium is equal to the number entering the next medium, which gives

$$j_{SW}(r) \text{ is continuous.} \tag{19.50}$$

The diffusion approximation gives rise to greater difficulties, since the integral conditions of the Serber–Wilson method are to be replaced by purely local boundary conditions. In the absence of an exact solution for an infinite half-space and for adjacent semi-infinite media, there is no adequate guidance for this purpose.

The conditions most frequently imposed at an interface are that the flux and the normal component of the current are continuous in each group, i.e.

$$\rho_{\text{diff}} \text{ is continuous} \tag{19.51}$$

and

$$j_{\text{diff},n} \text{ is continuous,} \tag{19.52}$$

where $j_{\mathrm{diff},n}$ is the normal component of j_{diff}. The condition (19.52) is a natural extension of (8.13), and would be expected to be a fairly good approximation, like (8.13). The condition (19.51), on the other hand, corresponds to (8.15), which is comparatively poor, so that (19.51) cannot be very accurate either. The best condition on the flux in one-group theory was (8.12), and this, by (8.14) and (8.13), could be written

$$l(\mathbf{r})(\partial/\partial n)[\log \rho(\mathbf{r})] \textit{ is continuous}, \tag{19.53}$$

where the derivative is taken along the normal to the interface. If f is nearly diagonal for both the media at the interface, i.e. if

$$|f_{j\to i}| \ll 1 \quad \text{for } j \neq i, \tag{19.54}$$

the transfer of neutrons from one group to another within a few mean free paths of the interface can be neglected. Thus each group can be considered separately near an interface, and the boundary conditions are the same as in the constant cross-section approximation. Taking the latter in the form (19.53), we find

$$l_i(\mathbf{r})\frac{\partial}{\partial n}[\log \rho_i(\mathbf{r})] \textit{ is continuous for all } i. \tag{19.55}$$

However, unless (19.54) is satisfied, little reliance can be placed on formula (19.55).

For the free surface, we again assume (19.54), repeat the arguments leading to (19.55), and use (8.9) and (6.29); this leads to the condition that

$$\rho_i(\mathbf{r}) \textit{ vanishes at a distance } 0{\cdot}7104 l_i/c_i f_{i\to i} \textit{ beyond the surface.} \tag{19.56}$$

Alternatively, we can put:

$$j_{i,n}(\mathbf{r})/\rho_i(\mathbf{r}) \textit{ at the surface, for each } i, \textit{ equals its value in the one-group diffusion approximation with } c = c_i f_{i\to i}. \tag{19.57}$$

If (19.54) holds, both these methods should lead to very similar results, but little is known about them when (19.54) is not valid. We shall return to this question at the end of § 19.5.3.

19.4.3. *The case where the infinite-medium equation has not enough solutions*

(*a*) *Preliminary considerations.* The uncertainties involved in the method we are considering are still greater when the number of pairs of roots of (19.26) in the cut plane is less than the number of groups ($m' < m$). In this case the solution of the form (19.31) does not provide enough parameters (or arbitrary functions) to satisfy all the conditions

imposed. Yet it is unsafe to disregard any of the boundary conditions, despite the uncertainty in their derivation. We should therefore supplement (19.31) by further terms to give the additional parameters required. The form of these additional terms can be found as follows. It has been remarked in § 19.3.2 that the transient part of the exact solution can always be regarded as a superposition of infinite-medium solutions which do not satisfy the conditions at infinity. The additional terms to be introduced in the diffusion approximation and in the Serber–Wilson method should, of course, be as similar as possible to the terms which actually occur in the exact solution. We therefore take them in the form

$$C_L \phi_L(\mathbf{r}), \tag{19.58}$$

where L now lies on the cut, $\phi_L(\mathbf{r})$ satisfies $(\nabla^2 - 1/L^2)\phi_L(\mathbf{r}) = 0$, and the components of the column vector C_L satisfy (19.38). If the correct linear combination of the expressions (19.58) for all L on the cut is taken, the exact $\rho_{tr}(\mathbf{r})$ is obtained, and by adding an expression of the form (19.31) we find the exact solution. The approximation made consists in this case of taking only a few terms of the form (19.58) and not those for all L on the cut.

(*b*) *The Serber–Wilson method.* Let us examine to what extent the introduction of $m - m'$ terms of the form (19.58) will resolve our difficulties. We take first the Serber–Wilson method, which is of course restricted to spherically symmetrical systems. If the medium where $m' < m$ (number of infinite-medium solutions inadequate) does not extend to infinity, we must use the most general spherically symmetrical solution of $(\nabla^2 - 1/L^2)\phi_L(\mathbf{r}) = 0$, i.e.

$$\phi_L(r) = (1/r)\{A^- e^{-r/L} + A^+ e^{r/L}\} \quad (L > 0). \tag{19.59}$$

In this expression, $A^+ \exp(r/L)$ arises from the transient due to the outer boundary of the medium considered, and therefore does not in general vanish. For $A^+ \neq 0$, however, the contribution of (19.59) to $\psi_{SW}(r, -1)$ will contain a term proportional to

$$A^+ \int^{\infty} \frac{dr}{r} \exp\left[\frac{r}{L} - \frac{r}{\max l_i}\right],$$

and this diverges for any non-negative L on the cut. Thus Wilson's generalization (9.11) of Serber's condition (9.1) is no longer justifiable, while (9.1) itself does not provide enough information. Thus, if the medium for which $m' < m$ does not extend to infinity, we cannot supplement (19.31) by (19.58) in the Serber–Wilson method, which must consequently be abandoned.

However, if $m' < m$ only for the outermost medium of the system, and this extends to infinity, then A^+ in (19.59) vanishes by the conditions at infinity, and therefore $\psi_{SW}(r, -1)$ does not involve divergent integrals. The arguments underlying the Serber–Wilson method are thus applicable. The choice of L in (19.58) can be decided as follows. The additional terms (19.58) ought to simulate the behaviour of $\rho_{tr}(\mathbf{r})$. Since the medium extends to infinity, large distances are the most relevant, and at these distances $\rho_{tr}(\mathbf{r})$ is of the order of $\exp[-r/\max l_i]$. Thus, if $m-m' = 1$, i.e. only one pair of roots in the cut plane is lacking, and so only one term of the form (19.58) is needed, the most reasonable choice of L in this term is $\max l_i$. The total $\rho(r)$ in the outermost medium is therefore approximated (with a slight change of notation) by

$$\rho_{SW}(r) = \frac{1}{r}\left\{\sum_{k=1}^{m-1} C_k e^{-r/L_k} + C' e^{-r/\max l_i}\right\}, \tag{19.60}$$

where C_k is the solution of the equations

$$C_k = \tfrac{1}{2} L_k \log \frac{L_k + \mathfrak{l}}{L_k - \mathfrak{l}} \mathfrak{f} \mathfrak{l}^{-1} C_k$$

and the column vector C' is such that its component for the group with the longest mean free path is arbitrary, while the other components are the solutions of (19.38) with L replaced by $\max l_i$. This procedure has been reported to yield satisfactory results.

If more than one pair of roots of (19.26) in the cut plane is lacking ($m-m' \geqslant 2$), several terms of the form (19.58) must be used. One of these will again correspond to $L = \max l_i$, but the question of the best choice of L in other terms has not yet been investigated.

(c) *The diffusion approximation.* We now consider the applicability of this method of supplementing (19.31) by terms of the form (19.58) when the diffusion approximation is used. Here one uses purely local rather than quasi-integral boundary conditions, and so it does not matter whether a medium where $m' < m$ extends to infinity or not. The restriction on the applicability of (19.58) which was encountered in the Serber–Wilson method will thus not occur in the diffusion approximation. However, a certain limitation arises for a finite medium because of the uncertainty of choice of L in (19.58) Since (19.58) should simulate the behaviour of $\rho_{tr}(\mathbf{r})$, it follows from (19.45) that the best value of L in (19.58) is either $L = L^*$ or $L = \max l_i$, depending on the distances involved. The choice between these two alternatives can sometimes be made on general grounds without detailed supplementary calculations, but if this is not the case the diffusion approximation should be aban-

doned and the spherical harmonics method used (see the following section).

19.5. The spherical harmonics method and the discrete ordinates method

19.5.1. *The spherical harmonics method for plane geometries*

We now consider methods in which $\rho_{tr}(\mathbf{r})$ is treated on the same footing as $\rho_{diff}(\mathbf{r})$, and which are therefore free from the uncertainties and limitations encountered in the Serber–Wilson method and the diffusion approximation. These are the spherical harmonics and discrete ordinates methods. The general development of these two methods in multi-group theory is very similar to that in the constant cross-section approximation. However, we shall give it in some detail, mainly in order to show how the matrix notation introduced in § 19.2.4 can be used to simplify the algebraic manipulation. This work is due to Mandl (30).

Let us discuss first the spherical harmonics method, beginning with the case of plane symmetry and no sources. In this case, expanding (19.5′) in spherical harmonics, we obtain as in the derivation of (10.4)

$$(n+1)\psi'_{n+1}(x)+n\psi'_{n-1}(x)+(2n+1)\mathfrak{l}^{-1}\psi_n(x) = \mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\psi_0(x)\delta_{0n}. \quad (19.61)$$

The general solution of (19.61) can of course be obtained as the superposition of solutions varying exponentially with x, i.e. of solutions of the form

$$\psi_n(x) = A(n, L)e^{x/L}. \quad (19.62)$$

The simplest way of determining the permissible values of L and the corresponding $A(n, L)$ is as follows. We proceed as in § 10.2, introducing the auxiliary column vectors $G_n(L)$ defined by

$$G_n(L) = (-1)^n\{P_n(L\mathfrak{l}^{-1})-L[Q_0(L\mathfrak{l}^{-1})P_n(L\mathfrak{l}^{-1})-Q_n(L\mathfrak{l}^{-1})]\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}\}B, \quad (19.63)$$

where P_n are the Legendre polynomials, Q_n are Legendre functions of the second kind, and B is a column vector to be defined.

The first two $G_n(L)$ are clearly

$$\left.\begin{aligned} G_0(L) &= B \\ \text{and} \qquad G_1(L) &= [\mathfrak{f}\mathfrak{c}-1]L\mathfrak{l}^{-1}B \end{aligned}\right\}, \quad (19.64)$$

while the recurrence relations for P_n and Q_n show that

$$(n+1)G_{n+1}(L)+(2n+1)L\mathfrak{l}^{-1}G_n(L)+nG_{n-1}(L) = 0 \quad \text{for } n \geqslant 1. \quad (19.65)$$

Using these relations, we see at once that

$$\psi_n(x) = G_n(L)e^{x/L} \quad (19.62')$$

is the formal solution of (19.61).

In the P_N approximation $\psi_{N+1}(x)$ is put equal to zero, and therefore $G_{N+1}(L) = 0$. This gives for B the equation

$$L^{-N-1}P_{N+1}(L\mathfrak{l}^{-1})B = L^{-N}[Q_0(L\mathfrak{l}^{-1})P_{N+1}(L\mathfrak{l}^{-1}) - Q_{N+1}(L\mathfrak{l}^{-1})]\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}B, \tag{19.66}$$

where both sides have been multiplied by $1/L^{N+1}$ in order to make the coefficients polynomials in $1/L$. The matrix equation (19.66) is, of course, equivalent to a set of m ordinary equations for the m components of B, where m is the number of groups used. If these m equations have a non-trivial solution, L must satisfy the determinantal equation

$$|L^{-N-1}P_{N+1}(L\mathfrak{l}^{-1}) - L^{-N}[Q_0(L\mathfrak{l}^{-1})P_{N+1}(L\mathfrak{l}^{-1}) - Q_{N+1}(L\mathfrak{l}^{-1})]\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}| = 0. \tag{19.67}$$

In odd-order approximations, the left side of (19.67) is always a polynomial in $1/L^2$ of order $\frac{1}{2}m(N+1)$, i.e. it has exactly $m(N+1)$ finite roots for $1/L$, i.e. non-zero roots for L. Denoting these roots by $\pm L_k$ and the corresponding solutions of (19.66) by B_k, we find for the general expression for the neutron flux in the P_N approximation:

$$\psi_0(x) = \sum_{k=1}^{\frac{1}{2}m(N+1)} B_k\{A_k^+ e^{x/L_k} + A_k^- e^{-x/L_k}\}, \tag{19.68 a}$$

where A_k^+ and A_k^- are arbitrary numerical constants, and the B_k are normalized in some convenient manner, e.g. by stipulating that the largest component of each B_k is unity. The corresponding expression for the higher moments is

$$\psi_n(x) = \sum_{k=1}^{\frac{1}{2}m(N+1)} G_n(L_k)\{A_k^+ e^{x/L_k} + (-1)^n A_k^- e^{-x/L_k}\}, \tag{19.68 b}$$

it being understood that B in $G_n(L_k)$ is replaced by B_k.

If at least one pair of roots L_k of (19.67) is purely imaginary, then we have a multiplying medium. If no roots are finite and purely imaginary, but there is at least one pair of infinite roots ($L_k = \infty$, i.e. $1/L_k = 0$), capture and multiplication balance each other. If all the roots L_k of (19.67) are finite and not purely imaginary, the medium is capturing.

The constants A_k^+ and A_k^- in (19.68) are determined by the boundary conditions. These are evidently that the neutrons in each group satisfy the same boundary conditions as in the constant cross-section approximation. For example, at an interface all the relevant moments in each group should be continuous, i.e.

$$\psi_n(x) \textit{ is continuous for } n = 0, 1, \ldots, N,$$

while at a free surface the neutrons of each group satisfy either Marshak's or Mark's boundary conditions. Since there are exactly $m(N+1)$

arbitrary constants per medium, there are just enough conditions to determine all the constants.

19.5.2. *The spherical harmonics method for other geometries*

The above analysis can be immediately extended to other geometries. We first consider the neutron flux $\psi_0(\mathbf{r})$. In connexion with equation (12.13′) we remarked that the equation governing $\psi_0(\mathbf{r})$ can involve $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ only through ∇^2, ∇^4, etc., and consequently, to determine this equation, it suffices to take the corresponding equation in the plane case and replace $(d/dx)^n$ by ∇^n. The same is obviously true of multi-group theory. It is also clear that the equation governing $\psi_0(x)$ in the plane case, in the P_N approximation of multi-group theory, is obtained by replacing L and B in (19.66) by $(d/dx)^{-1}$ and $\psi_0(x)$ respectively. The corresponding equation for $\psi_0(\mathbf{r})$ in a general geometrical arrangement is therefore found by replacing L and B in (19.66) by $(\nabla^2)^{-\frac{1}{2}}$ and $\psi_0(\mathbf{r})$ respectively. Since each side of (19.66) is a polynomial in $1/L^2$, the equation so found will not contain negative or fractional powers of ∇^2. It can be verified that this equation is satisfied by $\boldsymbol{B}_k\phi_k(\mathbf{r})$, where $\boldsymbol{B}_k$ and L_k are defined as in the plane case and $\phi_k(\mathbf{r})$ is a solution of

$$(\nabla^2 - 1/L_k^2)\phi_k(\mathbf{r}) = 0 \tag{19.69}$$

which is regular in the region occupied by the medium considered. Superposing all such solutions, we obtain

$$\psi_0(\mathbf{r}) = \sum_{k=1}^{\frac{1}{2}m(N+1)} \boldsymbol{B}_k\phi_k(\mathbf{r}), \tag{19.70}$$

and this is the most general solution of the equation for $\psi_0(\mathbf{r})$, since it has the correct degree of arbitrariness.

If the symmetry properties of the system are such that the higher moments $\Psi_n(\mathbf{r})$ are determined uniquely by $\psi_0(\mathbf{r})$, then they are given by

$$\Psi_n(\mathbf{r}) = \sum_{k=1}^{\frac{1}{2}m(N+1)} G_n(L_k)[\phi_k(\mathbf{r})]_n, \tag{19.71}$$

where $G_n(L_k)$ is the same as in the plane case and $[\phi_k(\mathbf{r})]_n$ is constructed from $\phi_k(\mathbf{r})$ in the same way as in the constant cross-section approximation. If these symmetry properties are absent, so that within each group $\Psi_n(\mathbf{r})$ contains more than one component, the appropriate complementary functions have to be added to (19.71). In the constant cross-section approximation, the equations governing these complementary functions did not involve $\mathbf{c}$. The corresponding equations in multi-group theory therefore do not involve the matrices $\mathfrak{f}$ and $\mathfrak{c}$. The only matrices that can appear in these latter equations are therefore powers of the

diagonal matrix $\mathfrak{l}$. This means that the complementary functions concerned can be determined separately for each group, and within each group they are governed by the same equations as in the constant cross-section approximation.

19.5.3. *The P_1 approximation and a comparison of it with the diffusion approximation*

It may be of interest to examine more closely the lowest approximation in the spherical harmonics method, namely the P_1 approximation, and to compare it with the diffusion approximation. The relations (19.64) and (19.65) give at once

$$G_2(L) = [\tfrac{3}{2}L\mathfrak{l}^{-1}(1-\mathfrak{f}\mathfrak{c})L\mathfrak{l}^{-1}-\tfrac{1}{2}]B,$$

and hence the equation (19.66) becomes in the P_1 approximation

$$(\tfrac{3}{2}\mathfrak{l}^{-2}-\tfrac{1}{2}L^{-2})B = \tfrac{3}{2}\mathfrak{l}^{-1}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1}B,$$

which can also be written

$$(1-\{1/3L^2\}\mathfrak{l}^2-\mathfrak{f}\mathfrak{c})\mathfrak{l}^{-1}B = 0.$$

Thus the equation (19.67) in the P_1 approximation is

$$|1-(1/3L^2)\mathfrak{l}^2-\mathfrak{f}\mathfrak{c}| = 0, \tag{19.72}$$

which is the same as (19.28). This shows that, if (19.27) is satisfied, the values of L_k used in the P_1 approximation are very close to those used in the diffusion approximation, and, as in one-group theory, the P_1 approximation may be regarded as a variant of the diffusion approximation, the more so since the P_1 approximation contains no provision for taking the transient terms into account.

In discussing the constant cross-section approximation, we remarked that the P_1 approximation is a poor version of the diffusion approximation. The same would be true in multi-group theory if the diffusion approximation could be developed as far as it can in one-group theory. However, we have seen in § 19.4 that this is not so, and by multi-group diffusion theory one usually means a cross between the P_1 approximation and an ideal diffusion approximation resembling that of one-group theory. Thus, in using the diffusion approximation boundary conditions ((19.51) and (19.52)), the values of L_k are taken from the ideal diffusion approximation, whilst these boundary conditions themselves come from the P_1 approximation. We have remarked in § 10.5 that such a procedure is often less satisfactory than the consistent use of the P_1 approximation, since the distortion of the diffusion length tends to compensate for the neglect of the transient flux. The same is clearly true in multi-group theory.

Thus, if (19.27) holds, the P_1 approximation is usually more satisfactory than the available multi-group diffusion approximation. If (19.27) does not hold, but (19.26) still has as many pairs of roots in the cut L-plane as there are groups, the multi-group diffusion approximation, even in its present unsatisfactory form, may be the more reliable. If the number of pairs of roots of (19.26) in the cut L-plane is less than the number of groups, reliable results can be obtained only by passing to an approximation, such as the P_3 approximation, which contains a provision for taking account of the transient terms.

At the end of § 19.4.2 we mentioned that it might be possible to prescribe the surface value of the ratio $j(\mathbf{r})/\rho(\mathbf{r})$ as a free-surface boundary condition in the diffusion approximation. This idea, like (19.51), is taken from the P_1 approximation. If this is done consistently, the ratio in question should be given not by (19.57), but by

$$j(\mathbf{r})/\rho(\mathbf{r}) = \tfrac{2}{3} \text{ at the free surface,} \tag{19.73}$$

as required by Marshak's boundary conditions, which are more satisfactory than Mark's in low orders of approximation. The condition (19.73) is simpler than (19.57), but its application to the diffusion approximation is open to the same criticisms as that of (19.51).

This concludes our examination of the multi-group diffusion approximation in the light of our knowledge of the multi-group spherical harmonics method.

19.5.4. *The discrete ordinates method*

We ought now to discuss the discrete ordinates method. However, the development of this method in multi-group theory is the same as in the constant cross-section approximation. The equation (19.5′) for the plane case is now approximated by

$$\mu_i \psi'(x, \mu_i) + \mathfrak{l}^{-1}\psi(x, \mu_i) = \tfrac{1}{2}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1} \sum a_j \psi(x, \mu_j), \tag{19.74}$$

and if

$$\psi(x, \mu_i) = \sum_k A_{i,k}\, e^{x/L_k}, \tag{19.75}$$

the equations for the $A_{i,k}$ are

$$A_{i,k} = \tfrac{1}{2}\mathfrak{l}[1+\mu_i\, \mathfrak{l}/L_k]^{-1}\mathfrak{f}\mathfrak{c}\mathfrak{l}^{-1} \sum_j a_j\, A_{j,k}.$$

Hence, if we form the equation for

$$B_k = \mathfrak{l}^{-1} \sum_j a_j\, A_{j,k}$$

and stipulate that it should have non-trivial solutions, we obtain

$$\left| 1 - \tfrac{1}{2} \sum_i a_i [1+\mu_i\, \mathfrak{l}/L_k]^{-1}\mathfrak{f}\mathfrak{c} \right| = 0, \tag{19.76}$$

which defines the permissible values of L_k. The nature of the boundary conditions is evident; they are fixed for each group separately. In discussing such points as the choice of summation formula, the arguments given for the constant cross-section approximation continue to be valid.

19.6. The perturbation method

The extension of the perturbation method, and in particular of the statistical weight theorem, to multi-group theory has been carried out by Tait (48). The method is the same as in the constant cross-section approximation. Applying a variation to the equation (19.17′) for $\rho_i(\mathbf{r})$ (the ith eigenfunction of that equation), multiplying through by $\rho_i^\dagger(\mathbf{r})$ (the eigenfunction of (19.20) for the same eigenvalue) and integrating over all $\mathbf{r}$, we obtain similarly to (14.6):

$$\begin{aligned}\frac{\delta\gamma_i}{\gamma_i}\iiint \rho_i^\dagger(\mathbf{r})\rho_i(\mathbf{r})\,dV &= -\frac{\gamma_i}{4\pi}\iiint dV \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\rho_i^\dagger(\mathbf{r})\times \\ &\quad \times\exp[-t(\mathbf{r},\mathbf{r}')]\delta[\mathfrak{f}(\mathbf{r}')\mathfrak{c}(\mathbf{r}')\mathfrak{l}^{-1}(\mathbf{r}')]\rho_i(\mathbf{r}')+ \\ &\quad +\frac{\gamma_i}{4\pi}\iiint dV \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\rho_i^\dagger(\mathbf{r})\times \\ &\quad \times\exp[-t(\mathbf{r},\mathbf{r}')]\int_0^{|\mathbf{r}-\mathbf{r}'|} ds\,\delta\mathfrak{l}^{-1}(\mathbf{r}'')\mathfrak{f}(\mathbf{r}')\mathfrak{c}(\mathbf{r}')\mathfrak{l}^{-1}(\mathbf{r}')\rho_i(\mathbf{r}'), \qquad (19.77)\end{aligned}$$

where $$\mathbf{r}'' = \mathbf{r}+s(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|,$$

and the factor $$\exp[-t(\mathbf{r},\mathbf{r}')]\int_0^{|\mathbf{r}-\mathbf{r}'|} ds\,\delta\mathfrak{l}^{-1}(\mathbf{r}'')$$

represents simply $-\delta\exp[-t(\mathbf{r},\mathbf{r}')]$. The terms in $\delta\rho_i(\mathbf{r})$ disappear on integration by virtue of the equation for $\rho_i^\dagger(\mathbf{r})$. Using this equation, the first term on the right of (19.77) can be rewritten as

$$-\iiint dV'\,\rho_i^\dagger(\mathbf{r}')\mathfrak{l}(\mathbf{r}')\mathfrak{c}^{-1}(\mathbf{r}')\mathfrak{f}^{-1}(\mathbf{r}')\delta[\mathfrak{f}(\mathbf{r}')\mathfrak{c}(\mathbf{r}')\mathfrak{l}^{-1}(\mathbf{r}')]\rho_i(\mathbf{r}').$$

On changing the variables of integration in the same manner as in (14.10), we can rewrite the second term on the right of (19.77) as

$$\begin{aligned}\frac{\gamma_i}{4\pi}\iiint dV''\iint d\Omega\int_0^\infty ds\int_0^\infty ds'\,\rho_i^\dagger(\mathbf{r}''+s\Omega)\times \\ \times\exp[-t(\mathbf{r}'',\mathbf{r}''+s\Omega)-t(\mathbf{r}'',\mathbf{r}''-s'\Omega)]\times \\ \times\delta\mathfrak{l}^{-1}(\mathbf{r}'')\mathfrak{f}(\mathbf{r}''-s'\Omega)\mathfrak{c}(\mathbf{r}''-s'\Omega)\mathfrak{l}^{-1}(\mathbf{r}''-s'\Omega)\rho_i(\mathbf{r}''-s'\Omega).\end{aligned}$$

The matrices $\delta\mathfrak{l}^{-1}(\mathbf{r}'')$ and $\exp[-\mathfrak{t}(\mathbf{r}'', \mathbf{r}''-s'\Omega)]$ are both diagonal, so that the order of these two factors can be interchanged. Then $\gamma_i/4\pi$ times the integral over s' can be at once identified as $\psi_i(\mathbf{r}'', \Omega)$, while according to (19.22) the integral over s is $(4\pi/\gamma_i)\psi_i^\dagger(\mathbf{r}'', \Omega)$. The second term on the right of (19.77) thus becomes finally

$$(4\pi/\gamma_i)\iiint dV'' \iint d\Omega\, \psi_i^\dagger(\mathbf{r}'', \Omega)\delta\mathfrak{l}^{-1}(\mathbf{r}'')\psi_i(\mathbf{r}'', \Omega).$$

Collecting the above results, and assuming that ρ_i and $\rho_i^\dagger$ are normalized in accordance with (19.21), we obtain

$$\delta\gamma_i = -\gamma_i \iiint dV\, \rho_i^\dagger(\mathbf{r})\mathfrak{l}(\mathbf{r})\mathfrak{c}^{-1}(\mathbf{r})\mathfrak{f}^{-1}(\mathbf{r})\delta[\mathfrak{f}(\mathbf{r})\mathfrak{c}(\mathbf{r})\mathfrak{l}^{-1}(\mathbf{r})]\rho_i(\mathbf{r}) + \\ + 4\pi \iiint dV \iint d\Omega\, \psi_i^\dagger(\mathbf{r}, \Omega)\delta\mathfrak{l}^{-1}(\mathbf{r})\psi_i(\mathbf{r}, \Omega). \quad (19.78)$$

This is the multi-group form of the statistical weight theorem.

To find the perturbation in the lowest eigenvalue of a critical system ($\gamma_i = \gamma_0 = 1$) it is sometimes convenient to rewrite (19.78) in a slightly different form. If $\psi_0(\mathbf{r}, \Omega)$ and $\psi_0^\dagger(\mathbf{r}, \Omega)$ are expanded in spherical harmonics, we have

$$\psi_{00}(\mathbf{r}) = \iint \psi_0(\mathbf{r}, \Omega)\, d\Omega = \rho_0(\mathbf{r}),$$

$$\psi_{00}^\dagger(\mathbf{r}) = \iint \psi_0^\dagger(\mathbf{r}, \Omega)\, d\Omega = \rho_0^\dagger(\mathbf{r})\mathfrak{l}(\mathbf{r})\mathfrak{c}^{-1}(\mathbf{r})\mathfrak{f}^{-1}(\mathbf{r});$$

the last of these relations comes from equation (19.23). The formula (19.78) can then be rewritten, if (19.27) holds, as

$$\delta\gamma_0 \cong -\iiint dV\, \rho_0^\dagger(\mathbf{r})\mathfrak{l}(\mathbf{r})\mathfrak{c}^{-1}(\mathbf{r})\mathfrak{f}^{-1}(\mathbf{r})\delta[\mathfrak{f}(\mathbf{r})\mathfrak{c}(\mathbf{r})]\mathfrak{l}^{-1}(\mathbf{r})\rho_0(\mathbf{r}) + \\ + 4\pi \iiint dV \iint d\Omega\, [\psi_0^\dagger(\mathbf{r}, \Omega) - (1/4\pi)\psi_{00}^\dagger(\mathbf{r})]\delta\mathfrak{l}^{-1}(\mathbf{r}) \times \\ \times [\psi_0(\mathbf{r}, \Omega) - (1/4\pi)\psi_{00}(\mathbf{r})]. \quad (19.78')$$

Under the conditions of the diffusion approximation,

$$\psi_0(\mathbf{r}, \Omega) - (1/4\pi)\psi_{00}(\mathbf{r})$$

is approximately equal to $(3/4\pi)j_0(\mathbf{r})$, while $\psi_0^\dagger(\mathbf{r}, \Omega) - (1/4\pi)\psi_{00}^\dagger(\mathbf{r})$ is, if (19.27) holds, approximately equal to $(3/4\pi)j_0^\dagger(\mathbf{r})\mathfrak{l}(\mathbf{r})$, and this leads to the diffusion approximation form of the statistical weight theorem in multi-group theory.

19.7. The Monte Carlo method and anisotropic scattering

The variational method can be extended to multi-group theory in the same manner as the perturbation method, and the extension of the iteration and Monte Carlo methods to multi-group theory is evident.

It is, however, of interest to remark that, whereas in all other methods the amount of numerical work involved increases very rapidly with the number of groups used, in the Monte Carlo method the amount of work increases only slightly with the number of groups, if at all, once the constant cross-section approximation is abandoned. There is, in fact, no need to replace the exact equations of Chapter II by the approximate ones derived in § 19.2. This ability of the Monte Carlo method to deal with the actual form of the energy dependence of cross-sections is one of its main attractions.

We have hitherto discussed only problems where the scattering is isotropic in the L system. All the methods of solution described above can, of course, be at once extended to the case where the $f_{j\to i}$ in (19.5) are not constants, but are polynomials of comparatively low order in $\Omega.\Omega'$. However, caution is necessary regarding the validity of the equations obtained in this way from (19.5). We have seen in Chapter I that anisotropic scattering in the L system may arise either from scattering by heavy nuclei, because the scattering in the C system is anisotropic, or from scattering by light nuclei, because of the difference between the C and the L systems. In the former case, the replacement of $f_{j\to i}$ in (19.5) by $f_{j\to i}(\Omega'\to\Omega)$ is often possible and justifiable. The latter case (light nuclei) is, however, much more often the important one, and here, as we saw in Chapter II, the energy loss per collision depends on the scattering angle in the L system. This kind of situation cannot be described by simply replacing $f_{j\to i}$ in (19.5) by $f_{j\to i}(\Omega'\to\Omega)$, unless the number of groups is extremely large.

Of course, the Monte Carlo method is equally applicable whether the scattering angle is random or a function of the energy loss. However, if an analytical solution is required for the case where the anisotropy (in the L system) of scattering by light nuclei is important, then one of the procedures described in the following chapters should be used.

Nevertheless, this does not exclude the possibility of a semi-empirical multi-group treatment of media containing light nuclei. We shall not, however, discuss such a treatment of the problem.

XX

THE METHOD OF POLYNOMIAL APPROXIMATIONS

20.1. The type of problem for which the method is intended

In using multi-group theory, we were concerned with situations which were, in general, characterized by the following three features:

(1) the system was spectrum-regenerating;

(2) the variation of the relevant cross-sections with energy could be approximated by means of a step-function with relatively few discontinuities;

(3) the anisotropy in the L system of the scattering by light nuclei could be neglected.

The third of these assumptions implies, of course, that (except in semi-empirical methods) scattering by light nuclei is unimportant, the main mechanisms of neutron energy loss being inelastic scattering, or elastic scattering by moderately heavy nuclei, or both.

We now consider problems in which one or more of the above assumptions are seriously incorrect. In the present chapter we shall still assume that the system is spectrum-regenerating, but the other two assumptions will be completely reversed: we shall assume that the relevant cross-sections (or rather these cross-sections multiplied by the velocity) can be approximated not by step-functions, but by polynomials in the velocity, of a relatively low order, and that scattering by light nuclei is not at all negligible, but rather the *only* important mechanism whereby a neutron in the system can lose energy. That is, we neglect the inelastic scattering, and regard all nuclei, apart from the few lightest, as being infinitely heavy.

It should be noted that such a combination of assumptions introduces a certain limitation on the composition of the systems that can be considered. Firstly, if the system is to be really spectrum-regenerating, as opposed to not degrading the spectrum indefinitely (see § 18.2), it must contain fissile material. Secondly, there is a restriction on the relative numbers of light and fissile nuclei. If the number of light nuclei in every medium in the system is very small compared with the number of fissile nuclei, the spread in energy among the neutrons in the system is small, and the constant cross-section approximation can be applied. If, conversely, the number of light nuclei in every medium is very large compared with the number of fissile nuclei, or, more accurately, if

$N_l\sigma_l \gg N_f\sigma_f$ (N = number of nuclei per unit volume, σ_l = fast neutron scattering cross-section for light nuclei, σ_f = fast neutron fission cross-section), then the fissions caused by non-thermal neutrons can be neglected, and it is more profitable to use the method given in § 23.5. The method described in the present chapter is applied in practice to two cases. Davison (13) originally suggested it for use when $N_l\sigma_l$ is comparable with $N_f\sigma_f$ in the most important region of the system, while Elliott (18) has extended the method to the case where in some regions of the system $N_f\sigma_f \gg N_l\sigma_l$, while in other regions the inequality is reversed.

We shall call this method the *polynomial approximations method*. The real reason for this name is that the functions G_n, introduced in § 20.3, are approximated by polynomials in v.

20.2. The transformation of the Boltzmann equation

20.2.1. *Simplifications made in the presentation of the method*

For the sake of brevity, we shall assume in our presentation of the polynomial approximations method that the light nuclei in the system are those of hydrogen. The modifications necessary if other light nuclei are present will be evident. We shall not restrict ourselves to the actual form of the variation of the hydrogen cross-sections with energy, but take them as arbitrary polynomials in the velocity. The medium used will thus be a 'pseudo-hydrogen', with unit nuclear mass but varying cross-sections which are not necessarily those of hydrogen.

We also assume for simplicity that the scattering by the heavy nuclei is isotropic, while the fission spectrum can be approximated by a delta function. The results when the former of these assumptions is no longer made will be evident, whilst we have shown in § 1.2.4 that the spread of the fission spectrum in the logarithmic energy scale is small. The second assumption is thus reasonable, since (2.16) shows that it is more natural to use the logarithmic scale. The scattering in pseudo-hydrogen is taken as isotropic in the C system, which is correct for all light nuclei (see § 1.2.2). With these simplifications we can use (2.8′), (2.16), and (2.18) to rewrite (2.7) in the stationary case as

$$\frac{c(v')}{l_{\mathrm{tot}}(v')} f(v'\Omega' \to v\Omega)$$

$$= \frac{\nu}{4\pi l_f(v')}\delta(v-v_0) + \frac{1}{4\pi l_e(v')}\delta(v'-v) + \frac{v}{\pi v'^2 l_h(v')}\delta\left(\Omega\,.\,\Omega' - \frac{v}{v'}\right), \quad (20.1)$$

where v_0 is the speed of neutrons released in fission, $l_h(v')$ is the scattering

mean free path in pseudo-hydrogen, $l_e(v')$ is the scattering mean free path for heavy nuclei, and the other symbols mean the same as in Chapter II. Substituting (20.1) into (2.4′) and dividing by v, we obtain

$$\Omega \cdot \mathbf{grad}\, N(\mathbf{r}, v\Omega) + \{1/l_{tot}(v)\} N(\mathbf{r}, v\Omega)$$
$$= \frac{1}{4\pi l_e(v)} \iint N(\mathbf{r}, v\Omega')\, d\Omega' + \frac{v\delta(v-v_0)}{4\pi v_0} \iiint \frac{N(\mathbf{r}, v'\Omega')}{l_f(v')} v'\, dv' d\Omega' +$$
$$+ \frac{1}{\pi} \iiint \frac{N(\mathbf{r}, v'\Omega')}{v' l_h(v')} \delta\left(\frac{v}{v'} - \Omega \cdot \Omega'\right) dv'\, d\Omega'. \quad (20.2)$$

20.2.2. *The application of the spherical harmonics method*

In multi-group theory, our first step was to reduce the determination of $N(\mathbf{r}, v\Omega)$ to that of only a finite number of functions of $\mathbf{r}$ and Ω. In the present chapter, the order of elimination is different, and we start by reducing the determination of $N(\mathbf{r}, v\Omega)$ to that of a finite number of functions of $\mathbf{r}$ and v. This limits us to the spherical harmonics method, since we wish to determine the angular dependence for any geometrical arrangement to any degree of accuracy. We expand (20.2) in spherical harmonics, using the identity

$$\delta(\mu - \mu_0) = \tfrac{1}{2} \sum_{n=0}^{\infty} (2n+1) P_n(\mu) P_n(\mu_0)$$

and taking the plane case for simplicity, and thus obtain

$$(n+1) \frac{\partial N_{n+1}(x,v)}{\partial x} + n \frac{\partial N_{n-1}(x,v)}{\partial x} + (2n+1) \frac{N_n(x,v)}{l_{tot}(v)}$$
$$= \left\{ \frac{N_0(x,v)}{l_e(v)} + \frac{v\delta(v-v_0)}{v_0} \int_0^{v_0} \frac{N_0(x,v')v'\, dv'}{l_f(v')} \right\} \delta_{0n} +$$
$$+ 2(2n+1) \int_v^{v_0} \frac{N_n(x,v')\, dv'}{v' l_h(v')} P_n\left(\frac{v}{v'}\right), \quad (20.3)$$

where $N_n(x,v)$ is defined, similarly to (10.3), by

$$N_n(x,v) = \iint N(x,v,\mu) P_n(\mu)\, d\Omega,$$

μ being the x-component of Ω.

The modifications necessary in (20.3) for other geometries are evident. For instance, in the spherical case, the left side of (20.3) becomes

$$(n+1)\left(\frac{\partial}{\partial r} + \frac{n+2}{r}\right) N_{n+1}(r,v) + n\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right) N_{n-1}(r,v) + (2n+1) \frac{N_n(r,v)}{l_{tot}(v)},$$

and for more general geometries we proceed as in Chapter XII.

In the P_N approximation we neglect the term in $N_{N+1}(x, v)$ and the equations (20.3) for $n > N$. (N, the order of the approximation, should not be confused with $N(\mathbf{r}, v\Omega)$ or with $N_n(r, v)$.) The infinite system of integro-differential equations (20.3) is thereby reduced to a finite system. In what follows we shall understand by (20.3) this finite system, unless otherwise stated.

The boundary conditions on the solution of (20.3) in an odd-order approximation follow at once from the discussion in Chapter X. They are that at the interfaces all the $N_n(x, v)$ are continuous for all v, and at the free surfaces either Mark's or Marshak's conditions are satisfied for all v.

20.3. The method of polynomial approximations

20.3.1. *The basic idea of the method*

The system (20.3) is not easily solved exactly except in special cases. We therefore attempt to devise an approximate method, using the following considerations. Firstly, for any $v < v_0$, we can reduce the system (20.3) to a system of partial differential equations in x and v, since the $P_n(v/v')$ are polynomials in v. The general theory of partial differential equations then shows that the solutions of this system, i.e. the $N_n(x, v)$, are regular functions of v except at the singularities of $1/l_{\text{tot}}(v)$, $1/l_s(v)$, and $1/l_h(v)$. Since we have assumed (§ 20.1) that $v/l_{\text{tot}}(v)$, $v/l_s(v)$, and $v/l_h(v)$ are smooth functions of v which can be approximated by low-order polynomials, it follows that, for $v < v_0$, the various $N_n(x, v)$ can have singularities only for $v \to 0$, and with suitable restrictions (see § 20.4.1) on the cross-sections as $v \to 0$ the $N_n(x, v)$ will be regular for $0 \leqslant v < v_0$. At $v = v_0$, the $N_n(x, v)$ each contain a term in $\delta(v - v_0)$, but if we separate the delta-function terms by putting, say,

$$N_n(x, v) = g_n(x)\delta(v - v_0) + G_n(x, v), \tag{20.4}$$

where the $G_n(x, v)$ do not involve $\delta(v - v_0)$, the above arguments show that the G_n are regular up to and including v_0. That is, with suitable restrictions on the cross-sections as $v \to 0$, the G_n are smooth functions of v in the range $0 \leqslant v \leqslant v_0$, and can be well approximated by a few terms of an expansion in orthogonal polynomials. Since the polynomials orthogonal in $[0, v_0]$ are $P_s(2v/v_0 - 1)$, where P_s are the Legendre polynomials, we have, denoting the order of the highest polynomial retained by S:

$$G_n(x, v) = \frac{1}{v_0} \sum_{s=0}^{S} P_s\left(2\frac{v}{v_0} - 1\right) \int_0^{v_0} P_s\left(2\frac{v'}{v_0} - 1\right) G_n(x, v')\, dv', \tag{20.5}$$

and this is a fairly good approximation, even for relatively small S.

By means of (20.4) and (20.5), (20.3) can always be reduced to a system of ordinary differential equations in x alone, and these can be solved by standard methods.

The approximation in which the expansion of $N(x, v, \mu)$ in spherical harmonics in μ is terminated at $P_N(\mu)$ and (20.5) is terminated at $P_S(2v/v_0-1)$ is called the $P_N(\Omega)P_S(v)$ approximation.

We shall now make some general remarks concerning formulae (20.4) and (20.5). Firstly, we have assumed in deriving (20.5) that $l_{\text{tot}}(v)$, $l_e(v)$, and $l_h(v)$ are such that $G_n(x, v)$ is regular as $v \to 0$. This assumption is, however, unnecessarily restrictive. It can easily be verified that the series (20.5) converges even when the G_n are singular at $v = 0$, provided that they do not increase too rapidly as $v \to 0$. The modifications necessary when this condition is not met will be discussed in § 20.4.

Secondly, we have derived (20.5) by converting (20.3) into partial differential equations. This conversion is possible when the light nuclei involved are hydrogen or pseudo-hydrogen, but not for nuclei with mass numbers greater than unity. However, we have used these differential equations only in order to obtain a qualitative result, which is equally valid, if less familiar, for integral equations. The expansion (20.5) is thus equally satisfactory for all light nuclei.

The representation of $N_n(x, v)$ in the form (20.4) has a simple physical interpretation. It corresponds to a classification of the neutrons into those which have had no collisions with light nuclei and those which have had at least one such collision. If the fission spectrum is not approximated by a delta function, but the other assumptions leading to (20.2) are retained, then (20.4) must be replaced by

$$N_n(x, v) = g_n(x)F_f(v)+G_n(x, v), \tag{20.4'}$$

where $F_f(v)$ represents the fission spectrum.

20.3.2. *The application of Mellin transforms*

We now consider in more detail the application of the ideas of the preceding section. Substituting (20.4) into (20.3) and equating to zero separately the terms in $\delta(v-v_0)$ and the remaining terms, we have

$$(n+1)g'_{n+1}(x)+ng'_{n-1}(x)+(2n+1)g_n(x)/l_{\text{tot}}(v_0)$$

$$= \left\{\left(\frac{1}{l_e(v_0)}+\frac{\nu}{l_f(v_0)}\right)g_0(x)+\frac{\nu}{v_0}\int_0^{v_0}\frac{G_0(x, v')v'\,dv'}{l_f(v')}\right\}\delta_{0n} \tag{20.6}$$

and

$$(n+1)\frac{\partial G_{n+1}(x,v)}{\partial x}+n\frac{\partial G_{n-1}(x,v)}{\partial x}+\frac{(2n+1)}{l_{\mathrm{tot}}(v)}G_n(x,v)$$
$$=\frac{G_0(x,v)}{l_e(v)}\delta_{0n}+2(2n+1)\int_v^{v_0}\frac{G_n(x,v')\,dv'}{v'l_h(v')}P_n\left(\frac{v}{v'}\right)+$$
$$+2(2n+1)g_n(x)\frac{1}{v_0\,l_h(v_0)}P_n\left(\frac{v}{v_0}\right). \quad (20.7)$$

As with (20.3), the equations (20.6) and (20.7) are used only for $n \leqslant N$, and $g'_{N+1}(x)$ and $\partial G_{N+1}(x,v)/\partial x$ are neglected.

In order to obtain equations for the quantities on the right of (20.5), we might multiply the equations (20.7) by $P_s(2v/v_0-1)$ and integrate, expressing the resulting integrals in terms of those appearing in (20.5). However, this would be rather laborious, and an equivalent system of equations can be much more simply obtained as follows. Each term on the right of (20.5) is a linear combination of the integrals

$$\int_0^{v_0}\left(\frac{v}{v_0}\right)^s G_n(x,v)\frac{dv}{v_0}=G_{n,s}(x) \text{ say,} \quad (20.8)$$

i.e. of the Mellin transforms of $G_n(x,v)$ with respect to v. Multiplying (20.7) by v^s/v_0^{s+1} and integrating over all v, putting

$$\frac{v}{l_{\mathrm{tot}}(v)}=v_0\sum_j \alpha_{\mathrm{tot},j}\left(\frac{v}{v_0}\right)^j \quad (20.9)$$

say, with a similar definition of $\alpha_{e,j}$, etc., we find

$$(n+1)G'_{n+1,s}(x)+nG'_{n-1,s}(x)+(2n+1)\sum_j \alpha_{\mathrm{tot},j}\,G_{n,j+s-1}(x)$$
$$=\sum_j \alpha_{e,j}G_{0,j+s-1}(x)\,\delta_{0n}+2(2n+1)k_{n,s}\sum_j \alpha_{h,j}G_{n,j+s-1}(x)+$$
$$+2(2n+1)k_{n,s}\frac{g_n(x)}{v_0\,l_h(v_0)}, \quad (20.10)$$

where

$$k_{n,s}=\int_0^1 u^s P_n(u)\,du=\begin{cases}\dfrac{s(s-2)\ldots(s-n+2)}{(s+1)(s+3)\ldots(s+n+1)} & \text{if } n \text{ is even,}\\[2ex] \dfrac{(s-1)(s-3)\ldots(s-n+2)}{(s+2)(s+4)\ldots(s+n+1)} & \text{if } n \text{ is odd.}\end{cases} \quad (20.11)$$

Similarly, (20.6) becomes

$$(n+1)g'_{n+1}(x)+ng'_{n-1}(x)+(2n+1)g_n(x)/l_{\mathrm{tot}}(v_0)$$
$$=\left\{\left(\frac{1}{l_e(v_0)}+\frac{\nu}{l_f(v_0)}\right)g_0(x)+\nu v_0\sum_j \alpha_{f,j}\,G_{0,j}(x)\right\}\delta_{0n}. \quad (20.6')$$

The system of equations (20.10) and (20.6′) involves more unknowns than there are equations in the system, and none of the unknown transforms can be neglected. However, we have not yet made use of the assumption (20.5). In view of the orthogonality properties of Legendre polynomials, this assumption is equivalent to the assumption that $G_n(x, v)$ is orthogonal to all $P_{S+1+s'}(2v/v_0-1)$ with $s' \geqslant 0$. Thus, if we put

$$P_s(2v/v_0-1) = \sum_j p_{s,j}(v/v_0)^j,$$

the assumption (20.5) is equivalent to

$$\sum_j p_{S+1+s',j} G_{n,j}(x) = 0 \quad (s' = 0, 1, 2, \ldots). \tag{20.12}$$

By supplementing (20.10) and (20.6′) by a suitable number of the equations (20.12), we therefore obtain a system of equations which is strictly equivalent to the one obtained by multiplying (20.7) by $P_s(2v/v_0-1)$ and integrating. We call (20.12) *the supplementary equations*.

The number of these equations needed for each n depends, of course, on the order of the polynomials used to approximate to $v/l_{\text{tot}}(v)$, $v/l_c(v)$, etc. In the case where all the cross-sections follow the $1/v$ law, the equations (20.12) need be used only for $s' = 0$, i.e. we use only

$$\sum_j p_{S+1,j} G_{n,j}(x) = 0.$$

In the $P_N(\Omega)P_S(v)$ approximation, we eliminate all $G_{n,S+1+s'}(x)$ with $s' \geqslant 0$ by means of (20.12), obtaining a system of $(N+1)(S+2)$ equations in the same number of unknowns $g_n(x)$ and $G_{n,s}(x)$, with $0 \leqslant n \leqslant N$, $0 \leqslant s \leqslant S$. These equations can then be solved in the standard manner, putting, say,

$$\left.\begin{aligned} g_n(x) &= \sum_j b_{n,j}[A_j^+ \exp(x/L_j) + (-1)^n A_j^- \exp(-x/L_j)] \\ G_{n,s}(x) &= \sum_j B_{n,s,j}[A_j^+ \exp(x/L_j) + (-1)^n A_j^- \exp(-x/L_j)] \end{aligned}\right\}, \tag{20.13}$$

substituting into the equations, eliminating $b_{n,j}$ and $B_{n,s,j}$, and thus obtaining a determinantal equation for the permissible values of L_j. If L_j is a root of this equation, so is $-L_j$.

For other geometries, the expression $[A_j^+ \exp(x/L_j) + A_j^- \exp(-x/L_j)]$ in $g_0(x)$ and $G_{0,s}(x)$ must be replaced by $\phi_j(\mathbf{r})$, where $(\nabla^2 - 1/L_j^2)\phi_j(\mathbf{r}) = 0$, with corresponding changes for $n \neq 0$, while the L_j are the same as for the plane case. The boundary conditions must be satisfied by the g_n separately and by the $G_{n,s}$ for each $s \leqslant S$ separately. This follows from the fact that the boundary conditions for the $N_n(x, v)$ must be satisfied for each v separately.

Since the order of the determinantal equation for the L_j increases rapidly as the order of approximation increases,† we are limited in practice to fairly low orders of approximation, particularly since the determinantal equation has complex roots in many cases, and the labour needed to find a complex root is more than twice that needed to find a real root. Fairly satisfactory results, however, are obtained even in low-order approximations. For instance, it has been found that, if $v/l_{tot}(v)$, $v/l_c(v)$, etc., are linear in v, and $G_0(x, v)$ does not increase too fast as $v \to 0$, the difference between the $P_N(\Omega)P_1(v)$ and $P_N(\Omega)P_2(v)$ approximations is quite small, which indicates good convergence. It has also been confirmed by direct comparison with the exact solutions of (20.3) (where available) that the $P_N(\Omega)P_2(v)$ approximation, or even the $P_N(\Omega)P_1(v)$, gives satisfactory results (Davison (13)).

20.3.3. *The determinantal equation*

In the above discussion we have referred to the determinant giving L_j as being of the order $(N+1)(S+2)$. However, the fact that the L_j appear in pairs of opposite sign suggests that it should be possible to reduce this determinant by relatively simple means to one of half the order, i.e. to one of order $\frac{1}{2}(N+1)(S+2)$. This reduction is important, because of the rate at which the numerical work necessary increases with the order of the determinant. The reduction may be performed as follows. Let ζ_j be the assembly of all $b_{n,j}$ and $B_{n,s,j}$ for even n and given j, and let η_j be the corresponding assembly for odd n. The equations which determine the $b_{n,j}$ and the $B_{n,s,j}$ (i.e. those obtained by substituting (20.13) into (20.10), etc.) can be symbolically written

$$\left.\begin{aligned}(1/L_j)\mathfrak{P}\zeta_j + \mathfrak{Q}\eta_j &= 0\\ (1/L_j)\mathfrak{S}\eta_j + \mathfrak{T}\zeta_j &= 0\end{aligned}\right\}. \tag{20.14}$$

This follows since differentiation with respect to x in (20.10) is always associated with a change in parity of the n-subscript. Multiplying the first of (20.14) by $\mathfrak{P}^{-1}$ and the second by $\mathfrak{S}^{-1}$, and eliminating η_j, say, we have

$$(1/L_j^2 - \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{S}^{-1}\mathfrak{T})\zeta_j = 0,$$

† The situation here appears less favourable than in multi-group theory, since the order of the determinant giving the values of L_j depends on both S and N, whereas in the multi-group theory spherical harmonics method it depended only on the number of groups used, while the order of the approximation affected only the order of the elements in the determinant (§ 19.5). In the latter case, however, scattering was taken as isotropic in the L system. If the anisotropy is taken into account, the order of the determinant in multi-group theory is equal to the number of groups multiplied by the number of terms retained in the spherical harmonics expansion of $f(\Omega' \to \Omega)$.

and the equation for the L_j^s thus becomes

$$|1/L_j^s - \mathfrak{P}^{-1}\mathfrak{Q}\mathfrak{S}^{-1}\mathfrak{T}| = 0. \tag{20.15}$$

Since the matrices $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{S}$, and $\mathfrak{T}$ in (20.14) are clearly all of the order $\frac{1}{2}(N+1)(S+2)$, we have to deal only with matrices of this order, both in obtaining (20.15) and in solving it. It may, of course, be necessary to deal with matrices of order $(N+1)(S+2)$, or even higher orders in more general geometries, when we come to satisfy the boundary conditions.

20.4. Extensions and modifications of the method

20.4.1. *The behaviour of the exact solution for small energies*

Our description of the polynomial approximations method has so far been limited to the case where

(1) it is a good approximation to assume that $v/l_{\text{tot}}(v)$, etc., are low-order polynomials in v;

(2) the behaviour of $l_{\text{tot}}(v)$, $l_e(v)$, etc., as $v \to 0$ is such as not to make the $G_n(\mathbf{r}, v)$ increase too fast as $v \to 0$.

Before discussing the modifications which are necessary when these statements are not both valid, we shall find the behaviour of the $G_n(\mathbf{r}, v)$ as $v \to 0$, which depends on that of $l_{\text{tot}}(v)$, $l_e(v)$, etc., in this region.

Since $N(\mathbf{r}, v, \mu)$ is not negative, none of the $|G_n(\mathbf{r}, v)|$ can exceed $G_0(\mathbf{r}, v)$ for the same arguments, and it is therefore sufficient to examine the behaviour of the latter quantity. Further, in problems with spectrum regeneration we are chiefly interested in regions of space where the neutron population is appreciable, and such regions will on the whole lose by migration more neutrons than they gain. That is, $G_0(\mathbf{r}, v)$ cannot increase, as v decreases, faster than it would in the absence of migration, i.e. faster than the corresponding solution in an infinite medium in which ν is adjusted to make the infinite medium critical and thus give a solution independent of position. Thus, if the solution for this infinite medium is $X_0(v)$, we have, for all relevant regions of a finite spectrum-regenerating system, the inequality

$$G_0(\mathbf{r}, v) < X_0(v)\phi(\mathbf{r}), \tag{20.16}$$

where $\phi(\mathbf{r})$ is some bounded function of position. The inequality (20.16), which we have derived by semi-intuitive means, can also be proved rigorously; the proof, though too lengthy to be given here, is straightforward. For a single homogeneous spectrum-regenerating body, the proof consists essentially in beginning the solution of the undistorted system (20.3) (i.e. no G_n neglected), and noting that any solution where a $|G_n(\mathbf{r}, v)|$ exceeds $G_0(\mathbf{r}, v)$ is inadmissible. For systems in which a

spectrum-regenerating core is surrounded by slowing-down media, it must be remembered that $\phi(\mathbf{r})$ in (20.16) may decrease more slowly, as $|\mathbf{r}|$ increases, than $G_0(\mathbf{r}, v)$ for any v.

The determination of $X_0(v)$ in (20.16) is simplified by the following considerations. Since $X_0(v)$ is a solution, independent of position, in an infinite medium, the corresponding angular distribution should be isotropic, i.e. all the spherical harmonics above the first should be absent. The equation for $X_0(v)$, corresponding to (20.7), thus becomes

$$\left(\frac{1}{l_{tot}(v)}-\frac{1}{l_s(v)}\right)X_0(v) = 2\int_v^{v_0}\frac{X_0(v')\,dv'}{v'l_h(v')}+\text{constant}. \qquad (20.17)$$

It has this form, of course, only under the conditions of § 20.2.1, i.e. for non-thermal neutrons in a mixture of (pseudo-) hydrogen and infinitely heavy nuclei. We shall discuss in Chapter XXII the case where other light nuclei are present; for the moment we keep to the conditions of § 20.2.1, so that (20.17) is valid. Converting the latter into a differential equation and solving, we have

$$X_0(v) = \frac{\text{constant}}{1/l_{tot}(v)-1/l_s(v)}\exp\left[2\int_v^{v_0}\frac{dv'}{v'}\,\frac{1/l_h(v')}{1/l_{tot}(v')-1/l_s(v')}\right] \quad (v<v_0). \qquad (20.18)$$

This formula is, of course, valid only for $v < v_0$; at $v = v_0$, X_0 is discontinuous, and for $v > v_0$ it vanishes.

In considering the behaviour of $X_0(v)$ for small v, we do not need to distinguish between captured neutrons and those which cause fission. Accordingly, we put

$$1/l_f(v)+1/l_c(v) = 1/l_a(v)$$

and use $$1/l_{tot}(v) = 1/l_s(v)+1/l_h(v)+1/l_a(v),$$

which follows from (1.7) and the notation of (20.2). We can then rewrite (20.18) in the form

$$X_0(v) = \frac{\text{constant}}{1/l_h(v)+1/l_a(v)}\exp\left[2\int_v^{v_0}\frac{dv'}{v'}\,\frac{1/l_h(v')}{1/l_h(v')+1/l_a(v')}\right] \quad (v<v_0). \qquad (20.18')$$

This shows that, if the absorption cross-section varies as $1/v$, and $1/l_h(v) \leqslant 1/l_a(v)$ (i.e. the probability of absorption is never less than that of scattering at (pseudo-) hydrogen), then $X_0(v)$ is bounded as $v \to 0$. If, however, $1/l_a(v)$ tends to zero as $v \to 0$, while $1/l_h(v)$ tends to a finite limit, then the expression (20.18′) clearly behaves as $1/v^2$. These are the two extreme cases which are likely to be encountered in solving (20.17) with $v/l_a(v)$ and $v/l_h(v)$ approximated by polynomials in v.

Of course, since we are concerned with a numerical method, the above results are applicable only if all the coefficients in the polynomials approximating $v/l_a(v)$ and $v/l_h(v)$ are of the same order of magnitude. If some of the coefficients are very small compared with others, the presence of the corresponding terms can have little effect in low-order approximations, and the initial convergence is similar to what it would be if those terms were absent. For instance, if $1/l_h = \text{constant}$, $1/l_a = \epsilon/v$, where $\epsilon \ll v_0/l_h$, we have $X_0(0) = 0$, but the behaviour of the low-order approximations is much the same as if $X_0(v) \sim 1/v^2$ as $v \to 0$.

20.4.2. *The extension of the method to cases where low-energy neutrons predominate*†

Having examined the possible behaviour of $X_0(v)$, and thus of $G_n(\mathbf{r}, v)$, as $v \to 0$, we return to the polynomial approximations method. In § 20.3 we considered only cases where $X_0(v)$ could be approximated by a low-order polynomial in v, and this implied certain restrictions on its behaviour as $v \to 0$. For example, X_0 had to be integrable, and this will not always be the case. However, it follows from formula (20.18′) that there is always a relatively small integer k such that $v^k X_0$ is integrable and can be approximated by a polynomial of low order. Then (20.16) implies that the $v^k G_n(\mathbf{r}, v)$ also can be approximated by such polynomials, and it is reasonable to suppose that this approximation is uniform with respect to $\mathbf{r}$ and n.

If $v^k G_n(\mathbf{r}, v)$, with $k \neq 0$, is approximated by a low-order polynomial in v, this simply means that the supplementary conditions (20.12) are to be replaced by

$$\sum_t p_{S+1+s',t} G_{n,t+k}(\mathbf{r}) = 0, \tag{20.19}$$

and no other alteration is necessary in the method described in § 20.3.

If $v^k X_0(v)$ can be approximated by a polynomial of order S in v, then $v^{k+1} X_0(v)$ can be approximated by one of order $S+1$, to at least the same accuracy. The value of k in (20.19) can therefore always be increased without affecting the validity of the method. However, we clearly wish to take the smallest permissible value of k. We have indicated above that the smallest value of k, such that $v^k X_0(v)$ is integrable, is probably permissible. It may be argued that a smaller value still is permissible for k, since in a finite system a rapid accumulation of slow neutrons at any point will be prevented by migration, and that $G_0(\mathbf{r}, v)$ will therefore increase more slowly than $X_0(v)$ when $\mathbf{r}$ is fixed and v tends to zero; consequently, $v^k G_0(\mathbf{r}, v)$ may be integrable for smaller k than $v^k X_0(v)$ is.

† The results contained in §§ 20.4.2 and 20.4.3 are due to J. P. Elliott.

However, it must be recalled that in the polynomial approximations method the $G_{n,s}(\mathbf{r})$ are obtained as the solutions of certain differential equations. If these equations form a reasonable approximation at all, they should do so for an infinite medium. In an infinite medium, on the other hand, there are regions, for any v, where $G_0(\mathbf{r}, v)$ increases as fast as $X_0(v)$. It is thus the behaviour of $X_0(v)$ as $v \to 0$, rather than that of $G_0(\mathbf{r}, v)$ for some $\mathbf{r}$, which determines the lowest permissible value of k in (20.19).

The above arguments are somewhat tentative, but the conclusions have been completely confirmed by means of numerical examples. For a medium where $X_0(v) = 1/v^2$, for example, the polynomial approximations method with $k = 2$ in (20.19) gave satisfactory results and rapid convergence. The use of $k = 1$ (too low a value, according to the above arguments) led to impossible results; although the medium was not multiplying, the use of (20.19) with $k = 1$ led to periodic solutions.

The fact that the supplementary conditions (20.19) involve the Mellin transforms $G_{n,s}(\mathbf{r})$ only for $s \geqslant k$ implies that in the $P_S(v)$ approximation for each medium we should normally work in terms of the $G_{n,s}(\mathbf{r})$ with $k \leqslant s \leqslant k+S$. Those with $0 \leqslant s \leqslant k-1$ will not be involved in calculating the permissible L_j (see (20.13)) and the form of $g_n(\mathbf{r})$, $G_{n,k}(\mathbf{r}), \ldots, G_{n,k+S}(\mathbf{r})$. However, it may happen that, when these latter have been determined, it is possible, by using (20.10), to find some of the earlier $G_{n,s}$, and in particular $G_{n,k-1}(\mathbf{r})$, for all n.

Although it is not always possible to determine all the earlier transforms, it is always possible to find those needed to evaluate the integral

$$\int_0^{v_0} \frac{G_0(\mathbf{r}, v')v' \, dv'}{l_f(v')} \tag{20.20}$$

which appears in (20.6). This may be seen by considering that $0 \leqslant 1/l_f(v) \leqslant 1/l_a(v)$, while

$$\int_0^{v_0} \frac{X_0(v)v \, dv}{l_a(v)} \tag{20.21}$$

is the number of neutrons absorbed per unit time and volume when the number of neutrons produced per unit time and volume is finite. Thus (20.21) is finite, so that $vX_0(v)/l_f(v)$ should be integrable over $[0, v_0]$. Since k is the lowest non-negative integer for which $v^k X_0(v)$ is integrable over this range, and $v/l_f(v)$ is approximated by a polynomial in v, the index of the lowest power of v in this polynomial cannot be less than k,

and the integral (20.20) is thus expressible in terms of $G_{0,k}(\mathbf{r})$, $G_{0,k+1}(\mathbf{r})$, etc., which are the transforms that can always be determined.

We now consider the boundary conditions. The boundary conditions are evident (see § 20.3.2) for free surfaces and for interfaces between media where the value of k in (20.19) is the same. It is therefore necessary to discuss only interfaces between media where the value of k and the transforms involved are different on the two sides. As an example, let us consider the plane case, with the interface at $x = 0$, and let the values of k on the two sides of this plane be 0 and 2. That is, for $x > 0$ we work in terms of $G_{n,0}(x)$, $G_{n,1}(x)$,..., $G_{n,S}(x)$, and for $x < 0$ in terms of $G_{n,2}(x)$, $G_{n,3}(x)$,..., $G_{n,S+2}(x)$. If the equations (20.10) for $x < 0$ do not lead to a finite $G_{n,1}(x)$ for all n, the normal procedure would be to find $G_{n,S+1}(0+)$ and $G_{n,S+2}(0+)$ in terms of $G_{n,0}(0+)$, $G_{n,1}(0+)$,..., $G_{n,S}(0+)$ by means of the supplementary equations (20.12) for $x > 0$, and then to equate $G_{n,s}(0+)$ to $G_{n,s}(0-)$ for $s = 2, 3,..., S+2$. If, however, the equations (20.10) for $x < 0$ allow us to determine $G_{n,1}(x)$, a better approximation is obtained by determining $G_{n,1}(0-)$ from these equations and equating $G_{n,s}(0-)$ and $G_{n,s}(0+)$ for $s = 1, 2,..., S+1$, disregarding the available value of $G_{n,S+2}(0-)$, which is now superfluous. This example makes clear the procedure in other cases.

In the above formulation of the boundary conditions, we have treated all n in the same way, by using the same set of Mellin transforms for each spherical harmonic moment $G_n(\mathbf{r}, v)$. This is done, however, simply for convenience, and is not related to any considerations of convergence.

20.4.3. *The modification of the method to take account of thermal neutrons*

We have so far been concerned with the solution of equation (20.2), but this equation itself involves certain approximations. In particular, we have used the scattering law (20.1) for all energies in the derivation of (20.2), whereas it is in fact a fair approximation only for non-thermal neutrons. For thermal neutrons, the scattering law is appreciably altered by the effects of the thermal motion and of the molecular binding. The term corresponding to scattering by light nuclei can thus no longer be expected to have the same form. When thermal neutrons are of importance, equation (20.2) has therefore to be modified, and in the present section we shall extend the polynomial approximations method to deal with equation (20.2) so modified. In doing so, however, we shall not attempt to represent the scattering of thermal neutrons with great accuracy, but simply use a modification which is at least in the right

direction. This consists in introducing a velocity v^* called the *upper limit of the spread of thermal neutrons*, and assuming that for $v' > v^*$ (20.1) is still valid, whilst for $v' < v^*$ the light nuclei scatter in the same way as the heavy nuclei, i.e. elastically and isotropically in the L system. Physically, this corresponds to the assumption that for $v' < v^*$ the neutrons are scattered not by individual nuclei but by molecules which are regarded as rigid and infinitely heavy, while for $v' > v^*$ the neutrons are scattered by free nuclei. Since the scattering cross-section of a fully bound nucleus is different from that of the same nucleus when free, we treat the scattering mean free path for light nuclei as changing discontinuously at $v = v^*$. We therefore adopt different notations for the mean free path according as $v > v^*$ or not. Again considering the case of pseudo-hydrogen, we call the mean free path l_d for $v < v^*$, retaining the notation $l_h(v)$ for $v > v^*$.

The only modification needed in (20.2) in this approximation is to replace the last term by

$$\frac{1}{\pi}\iiint_{v'>v^*} \frac{N(\mathbf{r}, v'\Omega')}{v' l_h(v')}\,\delta\left(\frac{v}{v'} - \Omega \cdot \Omega'\right) dv'\,d\Omega' + \frac{S(v^*-v)}{4\pi l_d(v)} \iint N(\mathbf{r}, v\Omega')\,d\Omega',$$

where $$S(v^*-v) = 1 \text{ if } v^* > v, \quad 0 \text{ if } v^* < v.$$

We call the modified form of equation (20.2) '(20.2′)'.

Transforming this equation in the same way as we did (20.2), but replacing (20.4) by

$$N_n(\mathbf{r}, v) = g_n(\mathbf{r})\delta(v-v_0) + G_n(\mathbf{r}, v) + H_n(\mathbf{r}, v) \tag{20.22}$$

say, where

$$G_n(\mathbf{r}, v) = 0 \text{ for } v < v^*, \qquad H_n(\mathbf{r}, v) = 0 \text{ for } v > v^*,$$

and again taking the plane case for simplicity, we see that $g_n(x)$ and $G_n(x, v)$ satisfy the same equations (20.6) and (20.7) as before, except that we replace

$$\int_0^{v_0} \frac{G_0(x, v')v'\,dv'}{l_f(v')}$$

in (20.6) by

$$\int_{v^*}^{v_0} \frac{G_0(x, v')v'\,dv'}{l_f(v')} + \int_0^{v^*} \frac{H_0(x, v')v'\,dv'}{l_f(v')}. \tag{20.23}$$

Also, equations (20.7) are valid only for $v^* < v < v_0$, while the equations for $H_n(x, v)$ are

$$(n+1)\frac{\partial H_{n+1}(x,v)}{\partial x}+n\frac{\partial H_{n-1}(x,v)}{\partial x}+\frac{(2n+1)}{l_{\mathrm{tot}}(v)}H_n(x,v)$$
$$=\left[\frac{1}{l_e(v)}+\frac{1}{l_d(v)}\right]H_0(x,v)\,\delta_{0n}+\frac{2(2n+1)}{l_h(v_0)}g_n(x)P_n\left(\frac{v}{v_0}\right)+$$
$$+2(2n+1)\int_{v^*}^{v_0}\frac{G_n(x,v')}{v'l_h(v')}P_n\left(\frac{v}{v'}\right)dv' \quad (v<v^*). \tag{20.24}$$

It must be noticed that the thermal scattering law assumed in (20.2′) is too crude to give the actual spectrum of thermal neutrons. We can hope to obtain with fair accuracy only the distribution in space and angle of the thermal neutrons. This means that we are not interested in the solution of equations (20.24) as they stand, but only in that of a reduced system from which the energy dependence has been eliminated. The spectrum of thermal neutrons is roughly a Maxwellian. It is thus independent of position, and we may assume it to be independent of direction also. This is equivalent to putting

$$H_n(x,v)\cong h_n(x)F(v) \tag{20.25}$$

say, where $F(v)$ is the same for all n. Normalizing $F(v)$ by the condition

$$\left.\begin{aligned}\int_0^{v_0} vF(v)\,dv &= v_0^2\\ \text{and putting as in § 4.4}\qquad\qquad&\\ v_0^2/l^*_{\mathrm{tot}} &= \int_0^{v_0} v\,dv\,F(v)/l_{\mathrm{tot}}(v),\ \text{etc.},\end{aligned}\right\} \tag{20.26}$$

we can clearly regard l^*_{tot}, l^*_f, etc., as the thermal mean free paths for the corresponding processes. Multiplying (20.24) by v, integrating from $v = 0$ to $v = v^*$ and using (20.25) and (20.26), we obtain

$$(n+1)h'_{n+1}(x)+nh'_{n-1}(x)+(2n+1)h_n(x)/l^*_{\mathrm{tot}}$$
$$=(1/l^*_d+1/l^*_e)h_0(x)\,\delta_{0n}+$$
$$+2(2n+1)\left\{p_n(v^*/v_0)g_n(x)/l_h(v_0)+\int_{v^*}^{v_0}\frac{vG_n(x,v)}{l_h(v)}p_n\left(\frac{v^*}{v}\right)dv\right\}, \tag{20.27}$$

where $p_n(\alpha)=\int_0^{\alpha}P_n(y)y\,dy$, $P_n(y)$ being a Legendre polynomial, and the second term in (20.23) is given by

$$v_0^2h_0(x)/l^*_f.$$

Thus, instead of solving the system of integro-differential equations (20.24), we need only solve the system of ordinary differential equations (20.27). It remains to reduce the equations for $G_n(x, v)$ to a system of ordinary differential equations in x. To do so, we can again apply polynomial approximations. Since the equations for the $G_n(x, v)$ are the same as in the preceding section, the behaviour of $G_n(x, v)$ for small v/v_0 is also the same. Since $v^* \ll v_0$, the conditions for the applicability of low-order polynomials in the range $v^* \leqslant v \leqslant v_0$ should be much the same as in the range $0 \leqslant v \leqslant v_0$. Thus it is again $v^k G_n(x, v)$ rather than $G_n(x, v)$ which can be approximated by a low-order polynomial, and the value of k in each medium is the same as in § 20.4.2. In the $P_S(v)$ approximation, we should therefore put

$$G_n(x, v) = \frac{1}{v^k} \sum_{i=0}^{S} a_{n,i}(x) v^i. \tag{20.28}$$

The similarity to the preceding section suggests that, now that (20.7) holds only for $v^* \leqslant v \leqslant v_0$, the best procedure is to use the *incomplete Mellin transforms* defined by

$$G_{n,s}(x|v^*/v_0) = \int_{v^*}^{v_0} (v/v_0)^s G_n(x, v)\, dv/v_0. \tag{20.29}$$

The basic differential equations governing these incomplete transforms are obtained in the same manner as (20.10), i.e. by multiplying (20.7) by $(v/v_0)^s/v_0$, integrating over v from v^* to v_0, and expressing the resulting integrals in terms of the quantities (20.29). However, to express

$$\int_{v^*}^{v_0} \left(\frac{v}{v_0}\right)^s \frac{dv}{v_0} \int_{v}^{v_0} \frac{G_n(x, v')\, dv'}{v' l_h(v')} P_n\left(\frac{v}{v'}\right)$$

in terms of the quantities (20.29) requires more of the latter than when $v^* = 0$, and in particular some $G_{n,s}(x|v^*/v_0)$ for negative s. If we write the supplementary equations obtained from (20.28) in the form

$$\sum_i p_{S+1+s',i}\left(\frac{v^*}{v_0}\right) G_{n,i+k}(x|v^*/v_0) = 0 \quad (s' = 0, 1, \ldots), \tag{20.30}$$

which is strictly analogous to (20.19), with $p_{s,i}(v^*/v_0)$ the coefficients of the expansion of $P_s[(2v - v^* - v_0)/(v_0 - v^*)]$ in powers of v/v_0, the conditions (20.30) will not be sufficient to eliminate the incomplete Mellin transforms for $s < k$. However, for $v^* > 0$ the relations (20.30) are not the only ones which can be derived from (20.28). On the contrary, any number can be derived, since for $v^* \neq 0$ any power of v can itself be

expanded in a convergent series of polynomials orthogonal in $[v^*, v_0]$. The general procedure is to eliminate the $S+1$ coefficients $a_{n,s}(x)$ between the equations obtained by substituting (20.28) into (20.29), and to express the $G_{n,s}(x|v^*/v_0)$ for $s < 0$ and $s > S$ in terms of those for $0 \leqslant s \leqslant S$. For $k = 0$ and $s > S$, we obtain (20.30), but for $s < 0$ we get the further supplementary conditions required.

Combining these with the differential equations, discussed above, for the incomplete Mellin transforms, (20.27), and (20.6) modified as shown in (20.23), we have in the $P_N(\Omega)P_S(v)$ approximation a system of $(N+1)(S+3)$ equations in the same number of unknowns, and these can be solved in the ordinary way.

Whereas, in the method of § 20.4.2, different sets of Mellin transforms were used in different media, here we use the same set of incomplete transforms in each medium. The difference is due to the fact that in § 20.4.2 it was necessary to ensure the convergence of the transforms by eliminating all quantities sensitive to the behaviour of the slow neutrons, whereas the incomplete Mellin transforms converge for all s. The quantities previously eliminated could thus be retained, and indeed our purpose was to take more accurate account of the slow neutrons.

The modification given in this section can also be used to determine the spectrum of non-thermal neutrons. Once the $g_n(x)$ have been determined, we have no further interest in the $H_n(x, v)$, and are consequently no longer restricted to a particular interpretation of v^*. We can first solve the problem with v^* equal to the upper limit of the spread of thermal neutrons and find the $g_n(x)$. Then, treating these $g_n(x)$ as previously given, we can vary v^* so as to lie anywhere in the non-thermal region. Differentiating the resulting expressions for $G_{n,0}(x|v^*/v_0)$ with respect to v^* and using (20.29), we find the solution for $G_n(x, v)$. This procedure has been carried out and, despite the additional error due to the numerical differentiation, the results are quite satisfactory, even in approximations of fairly low order.

XXI

FEYNMAN'S METHOD

21.1. A single homogeneous body with isotropic scattering

21.1.1. *General outline of the method*

In the preceding two chapters we have discussed methods which were based on simplifying assumptions about the energy variation of the cross-sections present in the system. The present chapter is concerned with an approach whereby no restriction need be placed on the cross-sections in one of the media, although at the cost of rather drastic restrictions on the cross-sections in the other media, if any. Of course, although no restriction is necessary in the case of the selected medium, it is customary to approximate the cross-sections in some way, in order to reduce the amount of numerical work involved. The method in question is due to Feynman (unpublished).

We start with a single homogeneous body, which may be re-entrant. The basis of Feynman's method is the integral equation rather than the integro-differential equation, and the integral equation for the neutron emission density $Q(\mathbf{r}, v)$ given by (2.31), rather than that for the neutron distribution $N(\mathbf{r}, v\boldsymbol{\Omega})$. Since the integral equation in question is much simpler when the scattering is isotropic in the L system, we start by considering this case. The equation to be solved for the neutron emission density $Q(\mathbf{r}, v)$ is then

$$Q(\mathbf{r}, v) = \frac{\gamma}{4\pi} \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} \int dv' \frac{c(v')}{l_{\text{tot}}(v')} f(v' \to v) \exp[-\tau_{v'}(\mathbf{r}, \mathbf{r}')] Q(\mathbf{r}', v') \tag{21.1}$$

(see (2.32)), where $l(v)$, $c(v)$, and $f(v' \to v)$ are the same throughout the system, and we have written $\exp[-\tau_{v'}(\mathbf{r}, \mathbf{r}')]$ rather than $\exp[-|\mathbf{r}-\mathbf{r}'|/l_{\text{tot}}(v')]$ only because the body may be re-entrant. The parameter γ is introduced so that we shall be able to consider the problem as an eigenvalue problem under variable composition (see § 3.5).

The basic idea of Feynman's method is as follows. Suppose that for every v we have solved the one-group theory problem, for the body of the specified shape and dimensions, in which $l = l(v)$ (i.e. the assumed energy-independent mean free path in the one-group problem is equal to the actual mean free path for the velocity v), while c is a variable parameter. The auxiliary one-group problem is thus also an eigenvalue

problem under variable composition. Let $c_j(v)$ be the eigenvalues of this auxiliary problem and $\rho_j(\mathbf{r}|v)$ the corresponding eigenfunctions, i.e.

$$\rho_j(\mathbf{r}|v) = \frac{c_j(v)}{4\pi l_{\text{tot}}(v)} \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} \exp[-\tau_v(\mathbf{r},\mathbf{r}')]\rho_j(\mathbf{r}'|v). \quad (21.2)$$

For any fixed v, the eigenfunctions $\rho_j(\mathbf{r}|v)$ are mutually orthogonal (see (4.30)), and $Q(\mathbf{r}, v)$ can thus be expanded in terms of these eigenfunctions for each v. The coefficients in the expansion depend, of course, on v, so that the expansion has the form

$$Q(\mathbf{r}, v) = \sum_j p_j(v)\rho_j(\mathbf{r}|v). \quad (21.3)$$

Substituting into (21.1) and using (21.2), we easily obtain

$$\begin{aligned}\sum_j p_j(v)\rho_j(\mathbf{r}|v) &= \sum_{j'} \int dv' p_{j'}(v') f(v' \to v) \frac{\gamma c(v')}{4\pi l_{\text{tot}}(v')} \iiint \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2} \rho_{j'}(\mathbf{r}'|v') \times \\ &\qquad \times \exp[-\tau_{v'}(\mathbf{r},\mathbf{r}')] \\ &= \gamma \sum_{j'} \int dv' p_{j'}(v') f(v' \to v) \frac{c(v')}{c_{j'}(v')} \rho_{j'}(\mathbf{r}|v'); \quad (21.4)\end{aligned}$$

multiplying (21.4) by $\rho_j(\mathbf{r}|v)$ and integrating over the volume of the body, and assuming that the $\rho_j(\mathbf{r}|v)$ have been normalized by the condition

$$\iiint \rho_j(\mathbf{r}|v)\rho_{j'}(\mathbf{r}|v)\, dV = \delta_{jj'}, \quad (21.5)$$

we have

$$p_j(v) = \gamma \sum_{j'} \int dv' p_{j'}(v') f(v' \to v) \frac{c(v')}{c_{j'}(v')} A_{j' \to j}(v' \to v), \quad (21.6)$$

where we have put for brevity

$$A_{j' \to j}(v' \to v) = \iiint \rho_j(\mathbf{r}|v)\rho_{j'}(\mathbf{r}|v')\, dV. \quad (21.7)$$

The solution of the integral equation (21.1) in velocity and position has thus been reduced to that of a system of integral equations in the velocity alone, whilst the shape and dimensions of the body appear in this system only through the coefficients $A_{j' \to j}(v' \to v)$ and $c_j(v)$.

It may be noted in passing that, in deriving the equation (21.4), we have used the fact that the kernel of (21.1) contains $\tau_{v'}(\mathbf{r}, \mathbf{r}')$ rather than $\tau_v(\mathbf{r}, \mathbf{r}')$. This is why we had to start from the integral equation for $Q(\mathbf{r}, v)$ rather than that for $n(\mathbf{r}, v)$. To use the latter would have led to a much more involved system of equations than (21.6).

21.1.2. *The coefficients* $A_{j'\to j}(v'\to v)$ *in* (21.6)

Before proceeding with the solution of the equations (21.6), we shall examine the way in which the coefficients $A_{j'\to j}(v'\to v)$ depend on j, j', v, and v'. For simplicity, we first assume that the body is non-re-entrant and very large compared with the total mean free paths involved, i.e.

$$a \gg \max_v l_{\text{tot}}(v), \tag{21.8}$$

where a is a characteristic dimension of the body. If (21.8) holds, we may use the diffusion approximation to determine $\rho_j(\mathbf{r}|v)$ for any j and v, i.e. $\rho_j(\mathbf{r}|v)$ may be taken as the solution of the differential equation

$$(\nabla^2 - 1/L_j^2(v))\rho_j(\mathbf{r}|v) = 0 \tag{21.9}$$

which vanishes at a distance

$$z_0[c_j(v), l_{\text{tot}}(v)] \cong 0{\cdot}71 l_{\text{tot}}(v)/c_j(v)$$

beyond the surface; see (8.11), recalling that the body is for the moment assumed non-re-entrant, so that the surface is free, and see also (6.29).

In (21.9), $L_j(v)$ is, of course, a variable parameter, because $c_j(v)$ is one in (21.2).

If (21.8) is satisfied, then so is

$$a \gg \max_{j,v} z_0[c_j(v), l_{\text{tot}}(v)],$$

and we can neglect z_0 in comparison with the dimensions of the body, regarding $\rho_j(\mathbf{r}|v)$ as vanishing at the surface of the body. The boundary conditions for the solution of (21.9) then become independent of the velocity, and $L_j(v)$ will also be independent of the velocity, since it is a variable parameter which is required to be an eigenvalue. In this approximation, therefore, $\rho_j(\mathbf{r}|v)$ will itself be independent of the velocity, and, according to (21.5), (21.7) then gives

$$A_{j'\to j}(v'\to v) = \delta_{j'j}. \tag{21.10}$$

The restriction that the body is non-re-entrant was introduced only to simplify the boundary conditions, and the result is true independently of this assumption. Thus (21.10) holds whenever (21.8) is satisfied. In actual applications, (21.8) will not in general be satisfied, so that z_0 cannot be neglected in comparison with the dimensions of the body, and the body may be so small that even the use of the diffusion approximation is doubtful. However, it is fairly evident that, even when a is comparable with the values of $l_{\text{tot}}(v)$ involved, $\rho_j(\mathbf{r}|v)$ for a given j will not be very sensitive to the value of v. Thus, although (21.10) will not generally be true, we can put

$$A_{j'\to j}(v'\to v) = \delta_{j'j} + \epsilon \tilde{A}_{j'\to j}(v'\to v) \tag{21.11}$$

say, where $\epsilon\tilde{A}_{j'\to j}(v'\to v)$ is a small perturbation.

21.1.3. *The energy-dependent factors* $p_j(v)$

By the use of the usual technique of perturbation, with (21.11) and

$$p_j(v) = \sum_{s=0}^{\infty} \epsilon^s p_{js}(v), \qquad \gamma = \sum_{s=0}^{\infty} \epsilon^s \gamma_s, \tag{21.12}$$

we can easily reduce the solution of the system (21.6) to the solution of one equation at a time. Substituting (21.11) and (21.12) into (21.6) and collecting first the terms independent of ϵ, we obtain

$$p_{j0}(v) = \gamma_0 \int p_{j0}(v') \frac{c(v')}{c_j(v')} f(v' \to v)\, dv'. \tag{21.13}$$

Let $\gamma_{0,jk}$ be the kth eigenvalue of the jth equation (21.13), and let $p_{j0,k}(v)$ be the corresponding eigenfunction. It will be recalled that our real interest is in the lowest eigenvalue of γ. According to the definition of $c_j(v)$, the smallest $c_j(v)$ for a given v is $c_0(v)$. We thus see that $\gamma_{0,00}$ is the smallest $\gamma_{0,jk}$, so that $\gamma_0 = \gamma_{0,00}$. That is, γ_0 is the lowest eigenvalue of

$$p_{00}(v) = \gamma_0 \int p_{00}(v')\{c(v')/c_0(v')\} f(v' \to v)\, dv', \tag{21.14}$$

and p_{00} is the corresponding eigenfunction. Since this γ_0 is not an eigenvalue of any of the equations (21.13) with $j \geqslant 1$, we clearly have

$$p_{j0}(v) = 0 \quad \text{for } j \geqslant 1. \tag{21.15}$$

Collecting now the terms of the first order in ϵ in (21.6), we obtain

$$\begin{aligned} p_{01}(v) = {} & \gamma_0 \int p_{01}(v')\{c(v')/c_0(v')\} f(v' \to v)\, dv' + \\ & + \gamma_1 \int p_{00}(v')\{c(v')/c_0(v')\} f(v' \to v)\, dv' + \\ & + \gamma_0 \int p_{00}(v')\{c(v')/c_0(v')\} f(v' \to v) \tilde{A}_{0\to 0}(v' \to v)\, dv' \end{aligned} \tag{21.16}$$

and

$$\begin{aligned} p_{j1}(v) = {} & \gamma_0 \int p_{j1}(v')\{c(v')/c_j(v')\} f(v' \to v)\, dv' + \\ & + \gamma_0 \int p_{00}(v')\{c(v')/c_0(v')\} \tilde{A}_{0\to j}(v' \to v) f(v' \to v)\, dv' \quad (j \geqslant 1). \end{aligned} \tag{21.17}$$

Since γ_0 is the eigenvalue of the homogeneous equation corresponding to (21.16), the free term of the latter equation should be orthogonal to the solution $p^{\dagger}_{00}(v)$ of the adjoint homogeneous equation

$$p^{\dagger}_{00}(v) = \{\gamma_0 c(v)/c_0(v)\} \int p^{\dagger}_{00}(v') f(v \to v')\, dv',$$

which, using (21.14), gives

$$\gamma_1 = -\gamma_0^2\Big[\iint p_{00}(v')\{c(v')/c_0(v')\}f(v' \to v)\bar{A}_{0\to 0}(v' \to v)p_{00}^{\dagger}(v)\,dv'dv\Big] \div \div\Big[\int p_{00}(v)p_{00}^{\dagger}(v)\,dv\Big]. \quad (21.18)$$

Using this value of γ_1, we can solve equation (21.16) by Neumann series. γ_0 is not an eigenvalue of any of the equations (21.17), and therefore each of them can be solved by Neumann series. When $p_{01}(v)$ and $p_{j1}(v)$ have been determined, the terms of order ϵ^2 in (21.6) can be collected, and so on.

The method of solving (21.14), (21.16), (21.17), etc., which is adopted in practical applications depends largely on the actual form of $f(v' \to v)$, $c(v)$, and $c_j(v)$. When the assumptions of multi-group theory are satisfied, the integral equations (21.14), (21.16), etc., reduce to linear algebraic equations. It has been pointed out by Woodcock (57) that, since $\epsilon\bar{A}_{j'\to j}(v' \to v)$ in (21.11) is usually very small, it is often advantageous, in cases where both the multi-group theory and Feynman's method are applicable, to start as in the latter method and use the assumptions of multi-group theory only for the solution of (21.14), (21.16), etc. This remark is particularly important when the number of groups is relatively large (of the order of ten or more), since the direct application of multi-group theory as described in Chapter XIX is then prohibitively laborious.

Where the assumptions of multi-group theory are inapplicable, other simplifications of equations (21.14), etc., are often possible. We shall not describe these here, but their nature will be fairly apparent from our discussion of a closely related integral equation in § 22.2.

To conclude our examination of equation (21.6), we must mention an evident but important corollary. If l_{tot} is independent of v, then all the $\rho_j(\mathbf{r}|v)$ defined by (21.2) are independent of v also. The term $\epsilon\bar{A}_{j'\to j}(v' \to v)$ therefore vanishes, and (21.15) shows that then all the $p_j(v)$ vanish for $j \neq 0$. The series (21.3) thus reduces to

$$Q(\mathbf{r}, v) = p_0(v)\rho_0(\mathbf{r}). \quad (21.3')$$

Comparing this with (2.38) for l_{tot} independent of v, we see that

$$n(\mathbf{r}, v) = \bar{p}_0(v)\rho_0(\mathbf{r}). \quad (21.3'')$$

Since $\rho_0(\mathbf{r})$ is determined from the constant cross-section approximation with a suitable value of c, we see that *the spatial distribution of neutrons can be determined by the constant cross-section approximation, whether $c(v)$ is constant or not, for bare bodies (i.e. single media) where $l_{\text{tot}}(v)$ is constant.* If $c(v)$ is not a constant, a suitably chosen weighted average $\bar{c}$ is used. This is found from (21.14) if $f(v' \to v)$ is known.

21.1.4. *The critical-size problem*

Hitherto we have considered the problem of the eigenvalues of (21.1) under variable composition. In practice, we are much more frequently interested in the problem of critical size. The preceding analysis, however shows at once how to apply Feynman's method to the latter problem. To determine the critical value of the characteristic dimension a for a body of fixed shape and composition, we take a set of trial values of a and find the corresponding $\gamma(a)$ as above. The critical value of a is then the solution of

$$\gamma(a) = 1.$$

21.1.5. *A remark on the solution of the auxiliary one-group problem*

Before extending Feynman's method to more complicated systems, we shall add some comments on the practical application of the method. It has been assumed in the analysis of § 21.1.1 that the eigenvalues and eigenfunctions of (21.2) are found exactly. This will not be so in practice, and approximate eigenfunctions must be used for the $\rho_j(\mathbf{r}|v)$. Whereas the exact eigenfunctions of (21.2) for a given v are mutually orthogonal, we know that only some of the approximate methods preserve this property, while others lead to approximate eigenfunctions which are not mutually orthogonal (see § 8.8, for example). The derivation of (21.6), however, depends on the orthogonality of the $\rho_j(\mathbf{r}|v)$ for a given v. Thus, without altering the entire method to take account of deviations from orthogonality, we can use only those approximate solutions of (21.2) which preserve this property, even though they may be inferior in other respects. For instance, if we wish to use the diffusion approximation with the boundary condition (8.11), we must take the same value

$$z_0 = 0{\cdot}7104 l_{\mathrm{tot}}(v) \tag{21.19}$$

for z_0, whatever the value of $c_j(v)$, since the more accurate form (6.29) would make the resulting eigenfunctions no longer orthogonal; furthermore, even with (21.19), the eigenfunctions are orthogonal only when integrated over the extrapolated volume of the body, and this must be taken into account in evaluating the normalization factor in $\rho_j(\mathbf{r}|v)$ and the integrals $A_{j'\to j}(v' \to v)$. The Serber–Wilson method is inapplicable for our present purpose, for the same reason. Thus the more accurate of the simple approximations cannot be used to solve (21.2), and it is necessary either to use the less accurate versions (e.g. replacing (6.29) by (21.19)) or to pass to the more powerful but more laborious methods, such as the spherical harmonics method. (We have proved in § 10.6 that the spherical harmonics method preserves the orthogonality of the approximate eigenfunctions.)

21.2. A reactive core in an elastically scattering reflector

21.2.1. *Formulation of the problem*

Having described Feynman's method for the case of a single homogeneous body, we now turn to problems where the system consists of two or more media. In the present section, we take the case where neutron collisions can result in a change of energy in only one of the media, whilst collisions in the other media can result only in either capture, or scattering elastic in the L system. We call the former medium the *core* and the other media the *reflector*. The case considered in this section is thus that of a reflector which scatters only elastically.† For the moment we shall continue to assume that all scattering is isotropic in the L system, so that

$$\left.\begin{aligned} f_c(v'\Omega' \to v\Omega) &= (1/4\pi) f(v' \to v) \\ f_r(v'\Omega' \to v\Omega) &= (1/4\pi)\delta(v'-v) \end{aligned}\right\}, \tag{21.20}$$

and the second formula is valid throughout the reflector. The form of $c(v)$ and of $l(v)$ will, in general, be different for core and reflector, and for different parts of the reflector if it contains more than one medium.

In order to extend to this case the method developed in the preceding section of this chapter, it is necessary to introduce an auxiliary function $K(\mathbf{r}, \mathbf{r}'|v)$, which gives the probability per unit volume that a neutron starting at the point $\mathbf{r}'$ in the core, with velocity v, will have its next collision, in the core, at the point $\mathbf{r}$. That is, $K(\mathbf{r}, \mathbf{r}'|v)$ is the neutron flux at $\mathbf{r}$ due to an isotropic point source of unit strength at $\mathbf{r}'$, in a fictitious system where the properties of the reflector are the same as for the actual system, whilst in the core $c(v)$ becomes zero and $l(v)$ is unchanged; the points $\mathbf{r}$ and $\mathbf{r}'$ are both in the core. Using this interpretation of $K(\mathbf{r}, \mathbf{r}'|v)$ and the optical reciprocity theorem (see (4.26)), we see that $K(\mathbf{r}, \mathbf{r}'|v)$ is symmetrical in $\mathbf{r}$ and $\mathbf{r}'$, i.e.

$$K(\mathbf{r}, \mathbf{r}'|v) = K(\mathbf{r}', \mathbf{r}|v). \tag{21.21}$$

The auxiliary function $K(\mathbf{r}, \mathbf{r}'|v)$ is needed only for the analysis; it does not have to be evaluated explicitly. In the absence of the reflector it reduces to

$$\{1/l(v)|\mathbf{r}-\mathbf{r}'|^2\}\exp[-\tau_v(\mathbf{r}, \mathbf{r}')].$$

The equation for the neutron emission density $Q(\mathbf{r}, v)$ can be rewritten, in terms of the function $K(\mathbf{r}, \mathbf{r}'|v)$, so as to involve integration over the core only. If $Q(\mathbf{r}, v)$ is the neutron emission density, i.e. $Q(\mathbf{r}', v')\,dV'\,dv'$ is the number of neutrons generated (in fission or scattering) per unit

† Here, and throughout Chapter XXI, *elastic scattering* means scattering elastic in the L system, that is, the medium concerned is composed of heavy nuclei only.

time in the volume element dV' of the core and the velocity interval from v' to $v'+dv'$, then the number of neutrons of velocity v' to $v'+dv'$ which undergo collisions in the volume element dV of the core is

$$dV\,dv' \iiint_{\text{core}} K(\mathbf{r}, \mathbf{r}'|v')Q(\mathbf{r}', v')\,dV'.$$

The number of neutrons generated (in fission or scattering) per unit time in the volume element dV of the core and the velocity interval from v to $v+dv$, i.e. $Q(\mathbf{r}, v)\,dV\,dv$, is thus

$$dV\,dv\,\frac{1}{4\pi}\int c(v')f(v'\to v)\,dv' \iiint_{\text{core}} K(\mathbf{r}, \mathbf{r}'|v')Q(\mathbf{r}', v')\,dV'.$$

Cancelling the differentials, we have

$$Q(\mathbf{r}, v) = \frac{1}{4\pi}\int c(v')f(v'\to v)\,dv' \iiint_{\text{core}} K(\mathbf{r}, \mathbf{r}'|v')Q(\mathbf{r}', v')\,dV' \quad \text{for } \mathbf{r} \text{ in the core.} \tag{21.22}$$

Thus equation (21.22) is strictly equivalent to our previous equation

$$Q(\mathbf{r}, v) = \frac{1}{4\pi}\int \frac{c_{\mathbf{r}}(v')f_{\mathbf{r}}(v'\to v)\,dv'}{l_{\text{tot},\mathbf{r}}(v')} \iiint_{\text{system}} \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}\exp[-\tau_{v'}(\mathbf{r}, \mathbf{r}')]Q(\mathbf{r}', v') \tag{21.23}$$

(see (2.32′)). The subscript $\mathbf{r}$ in $c_{\mathbf{r}}(v')$, etc. (not to be confused with r for reflector), is here introduced to distinguish quantities belonging to different media. It was unnecessary in (21.22), since it was obvious that only quantities belonging to the core could appear there.

In what follows, we shall introduce the factor γ into equation (21.22), as we did for equation (21.1), so as to be able to deal with the problem of eigenvalues under variable composition:

$$Q(\mathbf{r}, v) = \frac{\gamma}{4\pi}\int c(v')f(v'\to v)\,dv \iiint_{\text{core}} K(\mathbf{r}, \mathbf{r}'|v')Q(\mathbf{r}', v')\,dV'. \tag{21.22′}$$

The corresponding change in (21.23) is to introduce the factor $\gamma(\mathbf{r})$, where

$$\gamma(\mathbf{r}) = \begin{cases} \gamma & \text{for } \mathbf{r} \text{ in the core,} \\ 1 & \text{for } \mathbf{r} \text{ in the reflector.} \end{cases}$$

21.2.2. *The application of the eigenfunction expansion*

We now proceed as in § 21.1.1. That is, we consider a fictitious one-group problem in which c and l_{tot} in the reflector, and l_{tot} in the core, have the same values as in the actual system for neutrons of velocity v, while the value of c in the core is a variable parameter which is to be an eigenvalue. Let $c_j(v)$ be the jth eigenvalue and $\rho_j(\mathbf{r}|v)$ the corresponding

eigenfunction for the neutron flux. Like the equation for $Q(\mathbf{r}, v)$, the equation for $\rho_j(\mathbf{r}|v)$ can be written in two alternative forms, namely

$$\rho_j(\mathbf{r}|v) = \frac{c_j(v)}{4\pi} \iiint_{\text{core}} K(\mathbf{r}, \mathbf{r}'|v)\rho_j(\mathbf{r}'|v)\, dV' \quad \text{for } \mathbf{r} \text{ in the core} \tag{21.24}$$

and

$$\rho_j(\mathbf{r}|v) = \frac{1}{4\pi} \iiint_{\text{system}} \frac{c_{j,\mathbf{r}'}(v)}{l_{\text{tot},\mathbf{r}'}(v)} \rho_j(\mathbf{r}'|v) \exp[-\tau_v(\mathbf{r}, \mathbf{r}')] \frac{dV'}{|\mathbf{r}-\mathbf{r}'|^2}, \tag{21.25}$$

where

$$c_{j,\mathbf{r}'}(v) = \begin{cases} c_j(v) & \text{for } \mathbf{r}' \text{ in the core,} \\ c_{\mathbf{r}'}(v) & \text{for } \mathbf{r}' \text{ in the reflector.} \end{cases}$$

The equivalence of the equations (21.24) and (21.25) can easily be established as for (21.22) and (21.23).

Since the values of c in the reflector are fixed for a given v, only the value of c in the core being variable, the orthogonality relation (4.30) gives

$$[c_j(v) - c_{j'}(v)] \iiint_{\text{core}} \rho_j(\mathbf{r}|v)\rho_{j'}(\mathbf{r}|v)\, dV = 0,$$

or, with suitable normalization,

$$\iiint_{\text{core}} \rho_j(\mathbf{r}|v)\rho_{j'}(\mathbf{r}|v)\, dV = \delta_{jj'}. \tag{21.26}$$

Then, for $\mathbf{r}$ in the core, $Q(\mathbf{r}, v)$ can be expanded in terms of these $\rho_j(\mathbf{r}|v)$ in a series of the same form (21.3) as before. Substituting this series into (21.22′), we obtain the same equation (21.6) for $p_j(v)$ as before, except that in the definition (21.7) of $A_{j'\to j}(v' \to v)$ the integration extends over the core only.

Since the equations (21.25) are strictly equivalent to (21.24), the quantities $c_j(v)$, $\rho_j(\mathbf{r}|v)$, and consequently $A_{j'\to j}(v' \to v)$, can be found by solving (21.25), without using (21.24). It is therefore unnecessary to know $K(\mathbf{r}, \mathbf{r}'|v)$ explicitly, as has been remarked above. The arguments of § 21.1.2 can then be applied, leading to the same conclusion (21.11), and the equations (21.6) can thus be solved as before (see § 21.1.3).

21.3. An inelastically scattering reflector

Whereas the application of Feynman's method to the systems considered in §§ 21.1 and 21.2 is fairly simple, and its convenience and accuracy have been well confirmed for these cases, there is as yet no completely satisfactory extension of this method to systems where changes in the neutron energy can occur in more than one medium.

There have, however, been several attempts to make such an extension and we shall describe below the method proposed by Fuchs (unpublished), giving also its limitations.

We assume that there are only two media in the system, which we call the *core* and the *reflector*, in either of which a neutron may change its energy. Both media are assumed to be of finite extent, with scattering isotropic in the L system.

The procedure is at first similar to that in § 21.2. As a generalization of the auxiliary function $K(\mathbf{r}, \mathbf{r}'|v)$, we introduce $K(\mathbf{r}' \to \mathbf{r}|v' \to v)$, defined as the probability, per unit volume and velocity interval, that a neutron generated at a point $\mathbf{r}'$ in the core with velocity v' has its first collision, in the core, at $\mathbf{r}$ with velocity v. That is, $K(\mathbf{r}' \to \mathbf{r}|v' \to v)$ is the density at $\mathbf{r}$ of neutrons with velocity v due to a point source of unit strength at $\mathbf{r}'$ which emits neutrons of velocity v' only, in a fictitious system where the reflector properties are the same as in the actual system, while in the core $l_{\text{tot}}(v)$ is unchanged and $c(v)$ is zero; the points $\mathbf{r}$ and $\mathbf{r}'$ are both in the core.

If the reflector cannot scatter inelastically, i.e. if we return to the case of § 21.2, the function $K(\mathbf{r}' \to \mathbf{r}|v' \to v)$ clearly reduces to

$$K(\mathbf{r}, \mathbf{r}'|v)\delta(v'-v).$$

The equation governing the neutron emission density can now be rewritten as

$$Q(\mathbf{r}, v) = \frac{1}{4\pi} \int c(v'')f(v'' \to v)\, dv'' \int dv' \iiint_{\text{core}} K(\mathbf{r}' \to \mathbf{r}|v' \to v'') \times \times Q(\mathbf{r}', v')\, dV' \quad (\mathbf{r} \text{ in the core}), \quad (21.27)$$

the proof being similar to that of (21.22).

We also introduce the corresponding auxiliary function for the reflector, $\bar{K}(\mathbf{r}' \to \mathbf{r}|v' \to v)$, defined in the same way except that the core and the reflector are interchanged. That is, $\bar{K}(\mathbf{r}' \to \mathbf{r}|v' \to v)$ is the density at $\mathbf{r}$ of neutrons of velocity v, etc., in a fictitious system where the core properties are the same as in the actual system, while in the reflector $l_{\text{tot}}(v)$ is unchanged and $c(v)$ is zero; the points $\mathbf{r}$ and $\mathbf{r}'$ are both in the reflector. The equation governing the neutron emission density can be rewritten as

$$Q(\mathbf{r}, v) = \frac{1}{4\pi} \int c(v'')f(v'' \to v)\, dv'' \int dv' \iiint_{\text{refl}} \bar{K}(\mathbf{r}' \to \mathbf{r}|v' \to v'') \times \times Q(\mathbf{r}', v')\, dV' \quad (\mathbf{r} \text{ in the reflector}), \quad (21.28)$$

where the reflector values of $c(v'')$ and of $f(v'' \to v)$ are used, instead of the core values as in (21.27). In what follows, we shall introduce the factor γ into equation (21.28), as with (21.1) and (21.22'):

$$Q(\mathbf{r}, v)$$

$$= \frac{\gamma}{4\pi} \int c(v'')f(v'' \to v)\, dv'' \int dv' \iiint_{\text{refl}} \bar{K}(\mathbf{r}' \to \mathbf{r} | v' \to v'')Q(\mathbf{r}', v')\, dV'. \tag{21.28'}$$

Let us suppose that we have solved an auxiliary problem where the core properties are the same as in the actual case, whilst all scattering in the reflector is regarded as elastic in the L system, and the number of secondaries per collision in the reflector is $\chi c(v)$, $c(v)$ being the actual number of secondaries and χ a variable parameter. Let χ_j be the jth eigenvalue of this parameter, and $\phi_j(\mathbf{r}, v)$ the corresponding solution for the neutron emission density. The equation satisfied by $\phi_j(\mathbf{r}, v)$ can be written down as follows in terms of $\bar{K}(\mathbf{r}' \to \mathbf{r} | v' \to v)$:

$$\phi_j(\mathbf{r}, v) = \frac{\chi_j c(v)}{4\pi} \int dv' \iiint_{\text{refl}} \bar{K}(\mathbf{r}' \to \mathbf{r} | v' \to v)\phi_j(\mathbf{r}', v')\, dV'$$

($\mathbf{r}$ in the reflector). (21.29)

Since both media are finite, the χ_j form a discrete set. We also suppose solved the corresponding adjoint equation

$$\phi_j^\dagger(\mathbf{r}, v) = \frac{\chi_j}{4\pi} \int c(v')\, dv' \iiint_{\text{refl}} \bar{K}(\mathbf{r} \to \mathbf{r}' | v \to v')\phi_j^\dagger(\mathbf{r}', v')\, dV'$$

($\mathbf{r}$ in the reflector). (21.30)

Since $\phi_j(\mathbf{r}, v)$ refers to a problem in which the reflector does not scatter inelastically, the determination of χ_j and of $\phi_j(\mathbf{r}, v)$ in the core can be carried out by the method of § 21.2, except that all the eigenvalues are now needed, and not only the lowest one. When $\phi_j(\mathbf{r}, v)$ has been determined in the core, its values in the reflector for each j can be obtained by the solution of a one-group problem. The solution of an adjoint equation can always be obtained in essentially the same way as that of the original equation. The solution of (21.30) therefore offers no further difficulties.

It can easily be proved in the usual manner that the $\phi_j(\mathbf{r}, v)$ and the $\phi_j^\dagger(\mathbf{r}, v)$ form a biorthogonal set, and we shall normalize them so that

$$\int dv \iiint_{\text{refl}} \phi_j(\mathbf{r}, v)\phi_{j'}^\dagger(\mathbf{r}, v)\, dV = \delta_{jj'}. \tag{21.31}$$

We now assume that the $\phi_j(\mathbf{r}, v)$ form a complete set in the reflector

region of the four-dimensional $(\mathbf{r}, v)$ space, and expand $Q(\mathbf{r}, v)$ in terms of the $\phi_j(\mathbf{r}, v)$, putting

$$Q(\mathbf{r}, v) = \sum_j g_j \phi_j(\mathbf{r}, v), \tag{21.32}$$

where the g_j are constant, unlike the p_j of (21.3).

Substituting (21.32) into (21.28′) and using (21.29), we obtain

$$\sum_j g_j \phi_j(\mathbf{r}, v) = \gamma \sum_{j'} \frac{g_{j'}}{\chi_{j'}} \int f(v'' \to v) \phi_{j'}(\mathbf{r}, v'')\, dv'' \quad (\mathbf{r} \text{ in the reflector}),$$

and hence, using the orthogonality relations (21.31), we find the system of equations

$$g_j = \gamma \sum_{j'} \frac{g_{j'}}{\chi_{j'}} \int\int dv dv'' f(v'' \to v) \iiint \phi_{j'}(\mathbf{r}, v'') \phi_j^{\dagger}(\mathbf{r}, v)\, dV \tag{21.33}$$

for the coefficients g_j. The value of γ is determined by the condition that this system of equations has a solution which is non-trivial. The adaptation of this procedure to the case where the critical size is required is evident.

This method remains only an attempt to extend Feynman's, on account of a number of weaknesses in it. Mathematically, its weakness lies in the fact that there is as yet little evidence that the $\phi_j(\mathbf{r}, v)$ form a complete set. In practice, a serious defect of the method is that, even if the series (21.32) converges fairly rapidly, it is still necessary to solve a large number of auxiliary problems before the result is obtained, and this means that the numerical work involved will be considerable. Furthermore, if the $\phi_j(\mathbf{r}, v)$ do form a complete set for the reflector region of the four-dimensional space, the eigenvalues χ_j should increase very slowly with j, and thus it would be expected that quite a large number of terms would be needed in (21.32) in order to obtain reasonable accuracy. For these reasons, the above extension of Feynman's method has seldom, if at all, been applied, although it was suggested some years ago. The generalization of Fuchs's method to systems of more than two media is evident. It is plain that the addition of media where the neutrons can change their energy will significantly increase the amount of numerical work required.

The analysis in the present section is based on the equation (21.28), so that in the auxiliary problem of determining $\phi_j(\mathbf{r}, v)$ the core properties were unchanged, while the reflector was replaced by one in which the neutrons cannot change their energy. It might seem at first sight that we might alternatively have based the analysis on equation (21.27), and determined $\phi_j(\mathbf{r}, v)$ from an auxiliary problem in which the reflector properties are unchanged, while the core is replaced by one in which the

neutrons cannot change their energy. However, it is generally the case that the reflector contains no fissile material, so that the neutrons can lose, but not gain, energy there. If, therefore, the reflector is left unchanged, while the core becomes elastically scattering only, the result is a slowing down problem, and not one with spectrum regeneration. In such a system, the eigenvalue problem defining $\phi_j(\mathbf{r}, v)$ has no solution; the set of the $\phi_j(\mathbf{r}, v)$ will be empty, and therefore not complete. This is why we must start from (21.28) and not (21.27). If the reflector is spectrum-regenerating and the neutrons cannot lose energy in the core, then we should, of course, start from (21.27).† If both media are spectrum-regenerating, either equation can be used, while if neutrons cannot lose energy in the core and cannot gain it in the reflector, neither procedure is possible and Feynman's method in its present form is inapplicable. It follows from these remarks that, even if the functions $\phi_j(\mathbf{r}, v)$ form a complete set for some system, they will not always do so; indeed, a system can be found for which the set of the $\phi_j(\mathbf{r}, v)$ is not complete, but empty.

21.4. Feynman's method for anisotropic scattering

21.4.1. *The case of a bare slab. Transformation of the equations*

In the preceding analysis, we have assumed the scattering to be isotropic in the L system. Davison (11) has suggested an extension of the method to the case where the scattering is anisotropic. We begin by describing this extension for a bare infinite plane slab, i.e. a single homogeneous body having plane symmetry. In § 21.4.4 we consider the cases of single homogeneous bodies of other shapes, and of bodies in reflectors which do not scatter inelastically.

When the scattering is anisotropic, the neutron emission is also anisotropic, and the neutron emission density $Q(\mathbf{r}, v)$ must now be replaced by the neutron emission angular distribution $Q(\mathbf{r}, v\boldsymbol{\Omega})$. The integral equation satisfied by the latter quantity has been mentioned, though not written down, in § 2.7. It may be easily obtained by means of the arguments of §§ 2.4 and 2.5, or, where the body is not single, homogeneous, and non-re-entrant, of § 2.6.

The neutron emission angular distribution is clearly the right side of the Boltzmann equation (2.4′), so that (2.31) must now be replaced by

$$Q(\mathbf{r}, v\boldsymbol{\Omega}) = \iiint dv' d\boldsymbol{\Omega}' \{v'c(v')/l_{\mathrm{tot}}(v')\} N(\mathbf{r}, v'\boldsymbol{\Omega}') f(v'\boldsymbol{\Omega}' \to v\boldsymbol{\Omega}). \quad (21.34)$$

† If neutrons cannot lose energy in some medium, then the eigenfunctions of (18.3), if any, are proportional to $\delta(v - v_{\max})$, and these clearly cannot form a complete set.

The source term is omitted, since we are at present considering systems containing no independent sources. When the neutron emission angular distribution is anisotropic, equation (2.36) is seen to become

$$N(\mathbf{r}, v\Omega) = \frac{1}{v}\int_0^{R_0} Q(\mathbf{r}-R\Omega, v\Omega)e^{-R/l_{\text{tot}}(v)}\,dR, \tag{21.35}$$

as can be found by suitably modifying the arguments leading to (2.36). In the plane case, the position and angle dependence of $N(\mathbf{r}, v\Omega)$ and $Q(\mathbf{r}, v\Omega)$ reduces to a dependence on one Cartesian coordinate (x, say) and one angular coordinate ($\Omega_x = \mu$, say), while $f(v'\Omega' \to v\Omega)$ can depend on the angle only through $\Omega'.\Omega$. We expand $N(x, v, \mu)$, $Q(x, v, \mu)$ and $f(v'\Omega' \to v\Omega) \equiv f(v' \to v, \Omega'.\Omega)$ in spherical harmonics of μ and of $\Omega'.\Omega$, putting

$$\left.\begin{aligned} f(v'\Omega' \to v\Omega) &= \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)f_n(v' \to v)P_n(\Omega'.\Omega), \\ \left.\begin{matrix} N(x, v, \mu) \\ Q(x, v, \mu) \end{matrix}\right\} &= \frac{1}{4\pi}\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\begin{cases} N_n(x, v), \\ Q_n(x, v). \end{cases} \end{aligned}\right\} \tag{21.36}$$

Substituting these into (21.34) and (21.35) and collecting the coefficients of $P_n(\mu)$, we have

$$Q_n(x, v) = \int dv'\, c(v')f_n(v' \to v)v'N_n(x, v')/l_{\text{tot}}(v') \tag{21.37}$$

and

$$N_n(x, v) = \frac{1}{4\pi v}\int\int P_n(\mu)\,d\Omega \int_0^{R_0} e^{-R/l_{\text{tot}}(v)}\,dR \sum_{n'=0}^{\infty}(2n'+1)Q_{n'}(x-R\mu, v)P_{n'}(\mu). \tag{21.38}$$

Taking $x' = x-R\mu$ as a new variable of integration instead of R, and changing the order of integration, we can rewrite (21.38) in the more convenient form

$$N_n(x, v) = \frac{1}{v}\int_{-a}^{a}\sum_{n'=0}^{\infty} Q_{n'}(x', v)K_{n'\to n}(x' \to x|v)\,dx', \tag{21.39}$$

where $x = \pm a$ correspond to the two faces of the slab, and the $K_{n'\to n}(x' \to x|v)$ are certain integrals over Ω, whose integrands contain the product

$$P_n(\mu)P_{n'}(\mu)\exp[-|x-x'|/\mu l_{\text{tot}}].$$

The exact form of these integrals is not of immediate interest, but we may note a simple interpretation of them. Anisotropic sources can always be regarded as a superposition of isotropic sources, dipole

sources, quadrupole sources, etc.; and for mathematical convenience we may consider a pure dipole, quadrupole, etc., source. Let a $2^{n'}$-pole source of neutrons of velocity v be situated at the plane $x = x'$ in a medium similar to the slab considered, except that $c \equiv 0$. Then $K_{n'\to n}(x' \to x|v)$ gives the nth spherical harmonic moment of the angular distribution at the plane $x = x$. This interpretation will be utilized in §§ 21.4.2 and 21.4.4.

The systems of equations (21.37) and (21.39) can, of course, be regarded as a direct generalization, to the case of anisotropic scattering, of the plane-symmetrical forms of (2.31) and (2.38). Eliminating successively $Q_n(x, v)$ and $N_n(x, v)$ between (21.37) and (21.39), we obtain

$$N_n(x,v) = \frac{1}{v}\int_{-a}^{a}\sum_{n'=0}^{\infty} K_{n'\to n}(x' \to x|v)\,dx' \int \frac{c(v')f_{n'}(v' \to v)}{l_{\text{tot}}(v')}\,v'N_{n'}(x', v')\,dv' \tag{21.40}$$

and

$$Q_n(x,v) = \int \frac{c(v')f_n(v' \to v)}{l_{\text{tot}}(v')}\,dv' \int_{-a}^{a}\sum_{n'=0}^{\infty} K_{n'\to n}(x' \to x|v')Q_{n'}(x', v')\,dx', \tag{21.41}$$

and these represent the corresponding generalization of (2.30) and (2.32). As with the two latter equations, the kernels of (21.40) and (21.41) differ only slightly, in that $f_{n'}(v' \to v)$ and $K_{n'\to n}(x' \to x|v)$ in (21.40) are replaced by $f_n(v' \to v)$ and $K_{n'\to n}(x' \to x|v')$ respectively.

21.4.2. *The application of the eigenfunction expansion*

Let us now consider the solution of the equations (21.41). We proceed as in § 21.1.1, by supposing solved the equations

$$\psi_{n\to n}(x|v) = \frac{c}{l(v)}\int_{-a}^{a}\psi_{n\to n}(x'|v)K_{n\to n}(x' \to x|v)\,dx', \tag{21.42}$$

where c is treated as a variable parameter. Let c_n^j be the jth eigenvalue† of this parameter, and $\psi^j_{n\to n}(x|v)$ the corresponding eigenfunction of (21.42). We shall also use the abbreviation

$$\psi^j_{n'\to n}(x|v) \equiv \frac{c^j_{n'}(v)}{l(v)}\int_{-a}^{a}\psi^j_{n'\to n'}(x'|v)K_{n'\to n}(x' \to x|v)\,dx'. \tag{21.43}$$

† From now on we shall place the affix j relating to the eigenvalue in the superscript position, the subscript position being used for the affixes relating to the spherical harmonic moments.

Using the interpretation of $K_{n\to n}(x' \to x|v)$ as the nth spherical harmonic moment at x of the distribution due to a 2^n-pole of unit strength at x' in a certain fictitious system, and applying the optical reciprocity theorem, we readily see that $K_{n\to n}(x' \to x|v)$ is symmetrical in x and x'. Consequently, for fixed n and v, the $\psi^j_{n\to n}(x|v)$ must form an orthogonal set in the interval $-a \leqslant x \leqslant a$. We assume that for any fixed n and v they form a complete set, and expand $Q_n(x, v)$ in terms of these orthogonal functions; the coefficients in the expansion depend, of course, on n and v. Thus

$$Q_n(x, v) = \sum_j p^j_n(v)\psi^j_{n\to n}(x|v). \tag{21.44}$$

Substituting into (21.41) and using (21.42) and (21.43), we obtain

$$\sum_j p^j_n(v)\psi^j_{n\to n}(x|v) = \sum_{n'}\sum_{j'}\int \frac{c(v')}{c^{j'}_{n'}(v')} f_n(v' \to v)p^{j'}_{n'}(v')\psi^{j'}_{n'\to n}(x, v')\,dv'. \tag{21.45}$$

Normalizing the $\psi^j_{n\to n}(x|v)$ by the condition

$$\int_{-a}^{a} \psi^j_{n\to n}(x|v)\psi^{j'}_{n\to n}(x|v)\,dx = \delta_{jj'},$$

introducing the abbreviations

$$A^{j'\to j}_{n'\to n}(v' \to v) = \int_{-a}^{a} \psi^j_{n\to n}(x|v)\psi^{j'}_{n'\to n}(x|v')\,dx,$$

and replacing $c(v')$ by $\gamma c(v')$ in order to deal with the eigenvalue problem under varying composition, we reduce (21.45) to

$$p^j_n(v) = \gamma \sum_{n'}\sum_{j'}\int \frac{c(v')}{c^{j'}_{n'}(v')} A^{j'\to j}_{n'\to n}(v' \to v)f_n(v' \to v)p^{j'}_{n'}(v')\,dv'. \tag{21.46}$$

If $f_n(v' \to v)$ decreases sufficiently rapidly as n increases, (21.46) can be solved in a similar manner to (21.6). The case where $f_n(v' \to v)$ decreases only slowly as n increases would occur in practice only in the presence of an appreciable quantity of hydrogen, and here the method of Chapter XX is much more suitable than Feynman's method.

21.4.3. *The solution of the auxiliary one-group problem*

In the preceding section we have assumed that the solutions of (21.42) and the values of (21.43) are known. We shall now show how to obtain these in practice. On comparing (21.42) and (21.43) with (21.40), it is seen that, for fixed n' and j, $\psi^j_{n'\to n}(x|v)$ may be regarded as the nth spherical harmonic moment in a fictitious one-group problem where the scattering law is

$$f(v'\Omega' \to v\Omega) = \{(2n'+1)/4\pi\}P_{n'}(\Omega'.\Omega)\delta(v'-v) \tag{21.47}$$

and $c(v) = c^j_{n'}(v)$. We shall call the corresponding angular distribution $\psi^j_{n'\to}(x, \mu|v)$. The system to which this refers is fictitious, of course, not only in that (21.47) differs from the actual scattering law, but also in that the law (21.47) cannot be realized in practice. This, however, does not affect our ability to determine $\psi^j_{n'\to}(x, \mu|v)$ by the ordinary methods of the constant cross-section approximation. Since we need the spherical harmonic moments $\psi^j_{n'\to n}(x|v)$ of all sufficiently low orders, it is natural to use the spherical harmonics method to determine $\psi^j_{n'\to}(x, \mu|v)$, thus obtaining $\psi_{n'\to n}$ as well as $\psi_{n'\to n'}$. The spherical harmonics method, too, is particularly suitable for the treatment of anisotropic scattering. In applying the spherical harmonics method with the scattering law (21.47), we must replace the functions $G_n(\nu)$ introduced in Chapter X by the functions $G_{n'\to n}(\nu, c)$, defined by the recurrence relations

$$(n+1)G_{n'\to n+1}(\nu, c)+nG_{n'\to n-1}(\nu, c)+\{(2n+1)/\nu\}(1-c\delta_{nn'})G_{n'\to n}(\nu, c) = 0$$

and

$$G_{n'\to 0}(\nu, c) = 1.$$

This clearly gives

$$G_{n'\to n}(\nu, c) = (-1)^n P_n(1/\nu) \qquad (n \leqslant n')$$

and

$$G_{n'\to n}(\nu, c) = (-1)^n\left\{P_n\left(\frac{1}{\nu}\right) - \frac{2n'+1}{\nu}cP_{n'}\left(\frac{1}{\nu}\right)\left[\frac{P_n(1/\nu)Q_{n'}(1/\nu)-P_{n'}(1/\nu)Q_n(1/\nu)}{P_{n'+1}(1/\nu)Q_{n'}(1/\nu)-P_{n'}(1/\nu)Q_{n'+1}(1/\nu)}\right]\right\} \qquad (n \geqslant n'), \quad (21.48)$$

where the Q_n are Legendre functions of the second kind.

We then find as in Chapter X

$$\psi^j_{n'\to n}(x|v) = \sum_k a^j_{n',k}(v)G_{n'\to n}(\nu^j_{n',k}(v), c^j_{n'}(v))\exp[\nu^j_{n',k}(v)x/l(v)], \quad (21.49)$$

where, in the P_N approximation, the $\nu^j_{n',k}$ are the roots of

$$G_{n'\to N+1}(\nu, c^j_{n'}(v)) = 0,$$

while the coefficients $a^j_{n',k}$ and the eigenvalues $c^j_{n'}(v)$ are determined in the usual manner from the standard boundary conditions.

21.4.4. *Bodies of other shapes and those surrounded by a reflector which scatters only elastically*

Our discussion of the extension of Feynman's method has hitherto been limited to the case of a bare slab. We now consider bodies of other shapes. For a bare sphere, the procedure is very similar to the above, although μ is now, of course, defined as in Chapter XI, i.e. $\mu = (\mathbf{r}\,.\,\Omega)/r$,

and the spherical harmonic moments are calculated relative to this μ. Let $K_{n'\to n}(r'\to r|v)$ be the nth spherical harmonic moment at a distance r from the centre of the sphere, due to a concentric spherically symmetric $2^{n'}$-pole shell source of radius r' and unit strength per unit area, in a fictitious medium where the mean free path is equal to the actual mean free path for neutrons of velocity v, and the number of secondaries per collision is zero. The equations governing the spherical harmonic moments $Q_n(r,v)$ of the neutron emission angular distribution $Q(r,v,\mu)$ can be written

$$Q_n(r,v) = 4\pi \int \frac{c(v')f_n(v'\to v)}{l_{\text{tot}}(v')}dv' \int_0^a r'^2 \sum_{n'=0}^{\infty} K_{n'\to n}(r'\to r|v)Q_{n'}(r',v')\,dr', \tag{21.41'}$$

where a is the radius of the sphere. By the optical reciprocity theorem, $K_{n\to n}(r'\to r|v)$ is symmetrical in r and r', and consequently, if $\psi^j_{n\to n}(r|v)$ is the jth eigenfunction of

$$\psi_{n\to n}(r|v) = \frac{4\pi c}{l_{\text{tot}}(v)} \int_0^a r'^2 \psi_{n\to n}(r'|v)K_{n\to n}(r'\to r|v)\,dr', \tag{21.42'}$$

the $\psi^j_{n\to n}(r|v)$ are orthogonal, for fixed n and v, with the weight factor r^2, i.e. the $r\psi^j_{n\to n}(r|v)$ are orthogonal for fixed n and v. Defining $\psi^j_{n'\to n}(r|v)$ for $n'\neq n$ in the same way as before (this definition will be called '(21.43')'), we see that for fixed j and n' and various n they represent the nth spherical harmonic moment of the neutron angular distribution in a bare sphere of radius a composed of the fictitious medium whose scattering law is given by (21.47). It follows that these $\psi^j_{n'\to n}(r|v)$ can be determined without using (21.42') and (21.43') directly. The nth spherical harmonic moment in a spherically symmetric system of radius r should behave as r^n for small r. Thus the $r\psi^j_{n\to n}(r|v)$ for fixed n and v and various j will not, except for $n=0$, form a complete set of orthogonal functions quadratically integrable in the interval $0\leqslant r\leqslant a$, but otherwise unrestricted. However, it is reasonable to suppose that they will form a complete set of orthogonal functions behaving as r^{n+1} for small r. Since $Q_n(r,v)$ is itself the nth spherical harmonic moment in a spherically symmetric system, $rQ_n(r,v)$ should behave as r^{n+1} for small r. It should therefore be possible to expand $rQ_n(r,v)$ in terms of the orthogonal functions $r\psi^j_{n\to n}(r|v)$, and the analysis then proceeds as before.

The modifications required when the slab or sphere is not bare, but is surrounded by a reflector which does not scatter inelastically, are evident. The quantities $K_{n'\to n}(x'\to x|v)$ and $K_{n'\to n}(r'\to r|v)$ will be

defined as the nth spherical harmonic moment of the angular distribution due to the appropriate $2^{n'}$-pole source in a fictitious system where $c(v)$ is zero in the core and the properties of the reflector are unaltered. The integration in (21.41) or (21.41′) extends over the core only. Further, $\psi^j_{n'\to}(x, \mu|v)$ and $\psi^j_{n'\to}(r, \mu|v)$ are the angular distributions in the fictitious system when the scattering law in the core becomes (21.47), remaining unchanged in the reflector. The analysis then proceeds as before.

No work has yet been done on the application of this extension of Feynman's method to bodies having neither plane nor spherical symmetry. Though no difficulties would be expected, it is not possible to discuss these questions in detail; in particular, we have seen that anisotropic scattering renders the calculations quite laborious, even in the constant cross-section approximation.

PART IV

SLOWING-DOWN PROBLEMS

XXII

A GENERAL SURVEY OF SLOWING-DOWN PROBLEMS. THE SPATIAL MOMENTS OF THE NEUTRON DISTRIBUTION AS FUNCTIONS OF ENERGY

22.1. General survey

22.1.1. *The main types of slowing-down problem*

After dealing with the spectrum-regeneration problems, we now turn to slowing-down problems, i.e. those where equation (18.3) has no eigenfunctions in the medium which is chiefly of interest. This causes two important differences between spectrum-regeneration problems and slowing-down problems. Firstly, all the methods we have discussed hitherto have depended in some way on the existence of such eigenfunctions. These methods are therefore no longer applicable, and for this reason the progress that has been made in the solution of slowing-down problems has been much more limited and more uneven than in the case of spectrum-regeneration problems. Secondly, if (18.3) has no eigenfunctions, there must be sources present in order for there to be any neutrons in the medium at all, and the distance from these sources is certain to be an important parameter. In fact, slowing-down problems can be classified according to the magnitude of this distance. The sources may, of course, include a supply of neutrons from other media.

The greater the distance from the source, the more collisions a neutron must undergo, on the average, before reaching this distance, and thus the lower must be its energy. It is natural to introduce a *slowing-down length* (defined at the end of § 22.3.1) which is, roughly speaking, the average crow-flight distance a neutron travels from the source before reaching a given energy. For neutrons of a given energy, the situation will be noticeably different according as the distance from the source is comparable with or much greater than the slowing-down length. The former case corresponds to typical neutrons, and the latter to neutrons

which have travelled exceptionally far in reaching the energy in question. These two cases represent the two main types of slowing-down problem. In the first type, i.e. the problem of the neutron distribution at moderate distances from the source, some information may be obtained by considering the spatial moments of the neutron distribution, i.e. expressions of the form

$$\int\int d\Omega \int\int\int_{\text{all space}} r^s N(\mathbf{r}, v\Omega)\, dV.$$

The moment of order zero, i.e. the neutron energy spectrum integrated over all space, is sometimes of direct interest. We have already met with such situations (see, for example, the discussion of $f_{j\to i}$ in § 19.2.2, and that of $X_0(v)$ in § 20.4.1). The determination of the neutron energy spectrum integrated over all space may be regarded as a third and simplest type of slowing-down problem; it implies, of course, that the medium is infinite and homogeneous, since the spatial distribution of the neutrons would otherwise have to be determined first.

In comparing slowing-down problems and spectrum-regeneration problems, it should be remarked that, besides the effect of the intrinsic difference between the problems, our treatment will differ in another way. In previous chapters, we have taken account of the energy dependence only in so far as was necessary to determine the critical size and the spatial distribution, but henceforward the neutron energy spectrum itself will be one of the objects of our study.† This is the main reason why we regard the neutron energy spectrum integrated over all space as leading to a separate type of problem.

In all three types of slowing-down problem, we shall restrict ourselves to the case where the scattering is elastic and isotropic in the C system, and the sources are isotropic and monochromatic, i.e. all neutrons are emitted with the same energy. The solution for polychromatic sources can always be obtained by superposition.

Most of the methods which we shall describe in this and the following chapters can be extended without great difficulty to the case of scattering anisotropic in the C system. However, in view of the complexity of the formulae concerned, and of the fact that anisotropic scattering is seldom of importance in slowing-down problems (see § 1.2.2) and is liable to vary

† This does not imply that the neutron energy spectrum is never of interest in spectrum-regeneration problems, but in such cases the problem can always be divided into two parts, first finding the spatial distribution of fissions by the methods of the preceding chapters, and then solving a slowing-down problem with these fissions as given sources.

considerably from case to case, there is little reason to discuss its effects in general terms. The same applies to anisotropy of the sources.

On the other hand, the anisotropy of scattering in the L system, arising from the difference between the L and C systems, can no longer be neglected. In fact, we shall see that it is very important in dealing with the distribution of slowed-down neutrons at very large distances from the source.

The following remark may be made concerning the assumption that the scattering is elastic in the C system. If the inelastic scattering cross-sections and the spectrum of inelastically scattered neutrons after one collision (i.e. $f_{is}(v' \to v)$) are known, then the inelastic scattering can always be taken into account by adding the inelastic scattering cross-section to the capture cross-section, and introducing additional sources of lower-energy neutrons to represent the inelastically scattered neutrons.†

22.1.2. *The form of the Boltzmann equation in slowing-down problems.* Before discussing any of the three types of slowing-down problem, we write down the Boltzmann equation for slowing down, under the assumption of § 22.1.1, i.e. no fission, scattering isotropic and elastic in the C system, and sources isotropic and monochromatic. We can take the case of a point source without loss of generality. Using the definitions in § 1.3.1 (see (1.5) and (1.7)), denoting by $l_{s,k}(v)$ the scattering mean free path for nuclei of the kth kind and by $l_c(v)$ the capture mean free path for all nuclei, combining (2.4′) with (2.16) and dividing through by v, we obtain

$$\Omega \cdot \mathbf{grad}\, N(\mathbf{r}, v\Omega) + \left[\frac{1}{l_c(v)} + \sum_k \frac{1}{l_{s,k}(v)}\right] N(\mathbf{r}, v\Omega)$$

$$= \sum_k \frac{(M_k+1)^2}{4\pi M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{dv'}{v' l_{s,k}(v')} \iint d\Omega' N(\mathbf{r}, v'\Omega') \times$$

$$\times \delta\left(\Omega \cdot \Omega' - \frac{(M_k+1)v^2 - (M_k-1)v'^2}{2vv'}\right) + \frac{S}{4\pi v_0} \delta(v - v_0)\delta_3(\mathbf{r}), \tag{22.1}$$

† This procedure would, of course, be convenient only if the nuclear excitation levels involved in the inelastic scattering in the C system are fairly high, so that the neutron loses a considerable amount of energy in the process; this is usually the case in slowing-down problems. If, however, there are numerous low excitation levels, the above procedure is no longer suitable, but an alternative procedure is available, which will be outlined in § 23.4.3.

where M_k is the mass of a nucleus of the kth kind, v_0 is the initial velocity of the neutrons, S is the number of neutrons emitted by the source per unit time and per unit solid angle, and $\delta_3(\mathbf{r})$ is the three-dimensional delta function defined by

$$\delta_3(\mathbf{r}) = \delta(x)\delta(y)\delta(z).$$

We have not assumed that $N(\mathbf{r}, v\Omega)$ is spherically symmetrical, since the geometry itself may not be.

It is often convenient to expand the solution of equation (22.1) in spherical harmonics. For a plane source in a plane-symmetric system, putting as in Chapter X

$$\mu = \Omega_x, \qquad N(\mathbf{r}, v\Omega) = N(x, v, \mu) = (1/4\pi)\sum_n (2n+1)N_n(x, v)P_n(\mu),$$

and transforming the left side as in the derivation of (10.4) and the right side as in § 17.1, we obtain

$$(n+1)\frac{\partial}{\partial x}N_{n+1}(x,v)+n\frac{\partial}{\partial x}N_{n-1}(x,v)+(2n+1)\left[\frac{1}{l_c(v)}+\sum_k \frac{1}{l_{s,k}(v)}\right]N_n(x,v)$$

$$= (2n+1)\sum_k \frac{(M_k+1)^2}{2M_k}\int_{\frac{M_k+1}{M_k-1}v}^{\infty} \frac{dv'}{v' l_{s,k}(v')} N_n(x,v')\times$$

$$\times P_n\left[\frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'}\right]+\frac{S}{v_0}\delta(v-v_0)\,\delta(x)\,\delta_{n0}. \quad (22.2)$$

If only $N_0(\mathbf{r}, v)$ is required, its value for a point source in a spherically symmetric system can be obtained from the solution of (22.2) by the relation

$$N_{0,pl}(r,v) = -\frac{1}{2\pi r}\left[\frac{\partial}{\partial x}N_{0,pl}(x,v)\right]_{x=r};$$

see the footnote to (5.44).

To deal directly with the case of a point source in a spherically symmetric system, we put now $\mu = (\mathbf{r}\,.\,\Omega)/r$ and proceed as in the derivation of (11.5). It is then clear that the first two terms in (22.2) should be replaced by

$$(n+1)\left(\frac{\partial}{\partial r}+\frac{n+2}{r}\right)N_{n+1}(r,v)+n\left(\frac{\partial}{\partial r}-\frac{n-1}{r}\right)N_{n-1}(r,v),$$

whilst the only other alterations needed in (22.2) are the replacement of

$N_n(x, v)$, $N_n(x, v')$, and $\delta(x)$ by $N_n(r, v)$, $N_n(r, v')$, and $\delta_3(\mathbf{r}) = (1/4\pi r^2)\delta(r)$ respectively. We call this modified form of (22.2) '(22.2')'.

If the system is not spherically symmetric, we expand $N(\mathbf{r}, v\Omega)$ in spherical harmonics and use the procedure of Chapter XII. That is, similarly to (12.5), we represent $N(\mathbf{r}, v\Omega)$ in the form

$$N(\mathbf{r}, v\Omega) = \frac{1}{4\pi} \sum_n (2n+1)[N_n(\mathbf{r}, v, \mathrm{W})]_{W=1},$$

where $\mathrm{W} = W\Omega$ is a vector of arbitrary magnitude W and direction Ω, corresponding to $\mathrm{U} = U\Omega$ in Chapter XII, and $(1/4\pi)(2n+1)N_n(\mathbf{r}, v, \mathrm{W})$ represents W^n times the combined contribution to $N(\mathbf{r}, v\Omega)$ from all the spherical harmonics of order n. Proceeding as in the derivation of (12.11), we find that the first two terms on the left side of (22.2') are replaced by

$$\mathrm{div}_{\mathrm{W}} \mathbf{grad}_{\mathbf{r}} N_{n+1}(\mathbf{r}, v, \mathrm{W}) + [(2n-1)\mathrm{W}. - W^2 \mathrm{div}_{\mathrm{W}}] \mathbf{grad}_{\mathbf{r}} N_{n-1}(\mathbf{r}, v, \mathrm{W}),$$

whilst the only other alterations needed in (22.2') are the replacement of $N_n(r, v)$ and $N_n(r, v')$ by $N_n(\mathbf{r}, v, \mathrm{W})$ and $N_n(\mathbf{r}, v', \mathrm{W})$ respectively:

$$\mathrm{div}_{\mathrm{W}} \mathbf{grad}_{\mathbf{r}} N_{n+1}(\mathbf{r}, v, \mathrm{W}) + [(2n-1)\mathrm{W}. - W^2 \mathrm{div}_{\mathrm{W}}] \mathbf{grad}_{\mathbf{r}} N_{n-1}(\mathbf{r}, v, \mathrm{W}) +$$

$$+ (2n+1)\left[\frac{1}{l_c(v)} + \sum_k \frac{1}{l_{s,k}(v)}\right] N_n(\mathbf{r}, v, \mathrm{W})$$

$$= (2n+1) \sum_k \frac{(M_k+1)^2}{2M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{dv'}{v' l_{s,k}(v')} N_n(\mathbf{r}, v', \mathrm{W}) \times$$

$$\times P_n\left[\frac{(M_k+1)v^2 - (M_k-1)v'^2}{2vv'}\right] + \frac{S}{v_0} \delta(v - v_0)\delta_3(\mathbf{r})\delta_{n0}. \quad (22.2'')$$

The first two equations (22.2'') will be of particular interest later, and we shall rewrite them in a somewhat simpler form. It is clear from the definition of $N_n(\mathbf{r}, v, \mathrm{W})$ that $N_0(\mathbf{r}, v, \mathrm{W})$ is independent of W, and is identical with what we have previously called $n(\mathbf{r}, v)$ (see (2.26)). $N_1(\mathbf{r}, v, \mathrm{W})$ represents the scalar product of W and a vector which characterizes the current of neutrons of velocity v and may be called $\mathbf{j}(\mathbf{r}, v)$; this latter vector is, of course, independent of W. We eliminate W from the second of (22.2'') by operating on it with $\mathbf{grad}_{\mathrm{W}}$, and, since

$\mathbf{div}_{\mathbf{W}}\mathbf{grad}_{\mathbf{r}} = \mathbf{div}_{\mathbf{r}}\mathbf{grad}_{\mathbf{W}}$, the first two equations (22.2″) become

$$\left.\begin{aligned}
&\mathbf{div}_{\mathbf{r}}\mathbf{j}(\mathbf{r},v)+\left[\frac{1}{l_c(v)}+\sum_k\frac{1}{l_{s,k}(v)}\right]n(\mathbf{r},v)\\
&\qquad=\sum_k\frac{(M_k+1)^2}{2M_k}\int_v^{\frac{M_k+1}{M_k-1}v}\frac{n(\mathbf{r},v')}{v'l_{s,k}(v')}dv'+\frac{S}{v_0}\delta(v-v_0)\delta_3(\mathbf{r})\\
&\text{and}\\
&\mathbf{grad}_{\mathbf{W}}\mathbf{div}_{\mathbf{r}}\mathbf{grad}_{\mathbf{W}}N_3(\mathbf{r},v,\mathbf{W})+\mathbf{grad}_{\mathbf{r}}n(\mathbf{r},v)+\\
&\qquad+3\left[\frac{1}{l_c(v)}+\sum_k\frac{1}{l_{s,k}(v)}\right]\mathbf{j}(\mathbf{r},v)\\
&\quad=3\sum_k\frac{(M_k+1)^2}{2M_k}\int_v^{\frac{M_k+1}{M_k-1}v}\frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'}\frac{\mathbf{j}(\mathbf{r},v')}{v'l_{s,k}(v')}dv'
\end{aligned}\right\}. \quad (22.3)$$

22.1.3. *The lethargy and the collision interval*

In all the above equations, except in the arguments of $l_c(v)$ and $l_{s,k}(v)$, it is the ratios of velocities which are involved, rather than the velocities themselves. It is therefore often convenient to replace v by a new independent variable

$$u = 2\log(v_0/v) = \log(E_0/E). \quad (22.4)$$

This quantity is called the *lethargy*. If it is taken as an independent variable, however, some caution is necessary. Since $N(\mathbf{r},v\boldsymbol{\Omega})$, $n(\mathbf{r},v)$, etc., have been defined as the numbers of neutrons per unit velocity interval, they must be replaced by new functions $\bar{N}(\mathbf{r},u,\boldsymbol{\Omega})$, $\bar{n}(\mathbf{r},u)$, etc., given by the relations

$$\bar{N}(\mathbf{r},u,\boldsymbol{\Omega})|du| = N(\mathbf{r},v\boldsymbol{\Omega})|dv|;$$

$$\bar{n}(\mathbf{r},u)|du| = n(\mathbf{r},v)|dv|.$$

That is, $\bar{N}(\mathbf{r},u,\boldsymbol{\Omega})$, $\bar{n}(\mathbf{r},u)$, etc., are the numbers of neutrons per unit lethargy interval. The delta function, too, transforms according to the formula

$$\delta(v-v_0)|dv| = \delta(u)|du|.$$

Substituting these relations in (22.1), determining $|dv/du|$ from (22.4),

and multiplying the resulting expression by $\frac{1}{2}v$, we obtain

$$\Omega \cdot \operatorname{grad} \bar{N}(\mathbf{r}, u, \Omega) + \left[\frac{1}{l_c(u)} + \sum_k \frac{1}{l_{s,k}(u)}\right] \bar{N}(\mathbf{r}, u, \Omega)$$

$$= \sum_k \frac{(M_k+1)^2}{8\pi M_k} \int_{u-q_k}^{u} e^{-k(u-u')} \frac{du'}{l_{s,k}(u')} \iint \bar{N}(\mathbf{r}, u', \Omega') \times$$

$$\times \delta[\Omega \cdot \Omega' - \tfrac{1}{2}(M_k+1)e^{-k(u-u')} + \tfrac{1}{2}(M_k-1)e^{k(u-u')}]\, d\Omega' +$$

$$+ \frac{S}{4\pi v_0} \delta(u)\, \delta_3(\mathbf{r}), \quad (22.1\,l)$$

where $l_c(u)$ and $l_{s,k}(u)$ are the appropriate mean free paths expressed as functions of the lethargy, and

$$q_k = 2 \log \frac{M_k+1}{M_k-1}; \quad (22.5)$$

the letter l in the number of the equation stands for 'lethargy'. The equation (22.1 l) differs from that obtained by replacing $N(\mathbf{r}, v\Omega)$ and $\delta(v-v_0)$ in (22.1) by $N(\mathbf{r}, u, \Omega)$ and $\delta(u)$, because the factor $\exp \frac{1}{2}(u'-u)$ appears in the integrand.

The same modifications are necessary in equations (22.2) to (22.3) if the lethargy is taken as the independent variable instead of the velocity. We call these modified equations (22.2 l), (22.2′ l), (22.2″ l), and (22.3 l). They will not, however, be used immediately, although the concept of lethargy will be employed in our discussion, and in certain series expansions.

The largest value of q_k which appears in equations (22.1 l), (22.2 l), etc., i.e. the largest increase in lethargy that a neutron can undergo in one collision in the medium concerned, is called the *collision interval*. More generally, if the lethargy can increase from u_1 to u_2 in one collision, u_1 and u_2 are said to belong to the same collision interval. The same terminology can be used when working in terms of velocity, but the collision interval is not then independent of the velocity.

It has been assumed in the above definition of the collision interval that all the kinds of nucleus present in the medium are present in comparable amounts. If, however, there are so few nuclei of some kind that their presence can be regarded as a small perturbation, it is customary to disregard this kind of nucleus in defining the collision interval.

22.1.4. *The slowing-down density*

Another quantity which is important in dealing with slowing-down problems is the *slowing-down density* $\chi(\mathbf{r}, v)$. This is defined as the

number of neutrons per unit time per unit volume which are scattered near r from a velocity greater than v to one less than v. Since the number of neutrons in the velocity interval from v' to $v'+dv'$ which have collisions near r is

$$v'n(\mathbf{r}, v')/l_{\text{tot}}(v')$$

per unit time per unit volume (see § 2.1), and the fraction of these that are scattered and not captured is $l_{\text{tot}}(v')/l_s(v')$ (see § 1.3.1), we clearly have

$$\chi(\mathbf{r}, v) = \int_v^\infty dv' \frac{v'n(\mathbf{r}, v')}{l_s(v')} \int_0^v f(v' \to v'')\, dv'', \tag{22.6}$$

where

$$f(v' \to v'') = \iint f(v'\Omega' \to v''\Omega'')\, d\Omega''.$$

Since $f(v' \to v'')$ has the dimensions of reciprocal velocity, and the number of neutron collisions per unit time, volume, and lethargy interval is

$$\frac{v\bar{n}(\mathbf{r}, u)}{l(u)} = \frac{v^2}{2} \frac{n(\mathbf{r}, v)}{l(v)},$$

it follows from (22.6) that the slowing-down density is dimensionally equal to the neutron collision density per unit lethargy interval.

If the scattering is elastic and isotropic in the C system, combining (22.6) with (2.16) and adopting the notation of (22.1), we have

$$\chi(\mathbf{r}, v) = \sum_k \frac{(M_k+1)^2}{2M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{n(\mathbf{r}, v')\, dv'}{v' l_{s,k}(v')} \int_{\frac{M_k-1}{M_k+1}v'}^v v''\, dv''$$

or, what is the same thing,

$$\chi(\mathbf{r}, v) = \sum_k \left[v^2 \frac{(M_k+1)^2}{4M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{n(\mathbf{r}, v')\, dv'}{v' l_{s,k}(v')} - \frac{(M_k-1)^2}{4M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{v'n(\mathbf{r}, v')\, dv'}{l_{s,k}(v')} \right]. \tag{22.7}$$

The most obvious application of the concept of the slowing-down density is as follows. Consider the region of the four-dimensional $(\mathbf{r}, v)$ space defined by the conditions that v lies in an interval $[v_1, v_2]$, while r lies in a volume V bounded by a surface A. In a stationary problem, the number of neutrons in such a region will be conserved, i.e. the number entering it per unit time should equal the number leaving it. This condition can be written in the same way as (6.34), except that, in addition to the sources, capture, and migration in r-space, we must also

consider migration along the v-axis, i.e. slowing down. We then obtain in a notation which is evident from (6.34),

$$\iiint_V [\chi(\mathbf{r}, v_1) - \chi(\mathbf{r}, v_2)]\, dV + \int_{v_1}^{v_2} dv \iint_A \mathbf{j}(\mathbf{r}, v).\mathbf{dA}$$

$$= \int_{v_1}^{v_2} dv \iiint_V dV \left[s(\mathbf{r}, v) - \frac{vn(\mathbf{r}, v)}{l_c(v)}\right], \quad (22.8)$$

so that the slowing-down density may be regarded as the component of the current along the velocity (or lethargy) axis. Another useful relation is obtained by differentiating (22.8) with respect to v_2. If the sources are distributed uniformly in space, $N(\mathbf{r}, v\Omega)$ is independent of $\mathbf{r}$, and therefore of Ω, so that $\mathbf{j}(\mathbf{r}, v)$ is zero, and formula (22.8) reduces to

$$\chi(v_1) = \chi(v_2) + \int_{v_1}^{v_2} \left[s(v) - \frac{vn(v)}{l_c(v)}\right] dv. \quad (22.9)$$

If the sources occupy only a finite region of $\mathbf{r}$-space, we obtain the same formula (22.9) on taking $s(v)$, $n(v)$, and $\chi(v)$ as the integrals of $s(\mathbf{r}, v)$, $n(\mathbf{r}, v)$, and $\chi(\mathbf{r}, v)$ over all space.

22.2. The integral of the neutron energy spectrum over all space

22.2.1. *A single element with no capture*

We now turn to a consideration of the various types of slowing-down problem, and take first the simplest type, namely the determination of the integral of the neutron energy spectrum over all space. This is, of course, equivalent to the determination of the neutron spectrum at any point when the sources are distributed uniformly in all space. In the latter case, all points in space are equivalent, and the neutron angular distribution should therefore be isotropic. That is, all terms in the spherical harmonics expansion of $N(\mathbf{r}, v\Omega)$ except $N_0(\mathbf{r}, v) = n(\mathbf{r}, v)$ must vanish, and $n(\mathbf{r}, v)$ is independent of $\mathbf{r}$. The system of equations (22.2), for example, thus reduces to

$$\left[\frac{1}{l_c(v)} + \sum_k \frac{1}{l_{s,k}(v)}\right] n(v) = \sum_k \frac{(M_k+1)^2}{2M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{n(v')}{v' l_{s\,k}(v')}\, dv' + \frac{S}{v_0}\delta(v - v_0). \quad (22.10)$$

We first examine the simplest case, that of a single element with no capture. The equation (22.10) then becomes

$$\frac{n(v)}{l(v)} = \frac{(M+1)^2}{2M} \int_v^{\frac{M+1}{M-1}v} \frac{n(v')}{l(v')} \frac{dv'}{v'} + \frac{S}{v_0} \delta(v-v_0). \tag{22.11}$$

For $v > v_0$, $n(v)$ must vanish, since there cannot be any neutrons in the system with greater energy than those emitted by the source. Thus, if v lies in the first collision interval $[(M-1)/(M+1)]v_0 < v < v_0$, the upper limit of integration can be replaced by v_0, and equation (22.11) then becomes an ordinary linear differential equation for

$$\int_v^{v_0} \frac{n(v')}{l(v')} \frac{dv'}{v'}.$$

This equation can, of course, be solved at once by the standard methods. For the second collision interval

$$[(M-1)/(M+1)]^2 v_0 < v < [(M-1)/(M+1)]v_0,$$

the integral involved can be written

$$\int_v^{v_0} \frac{n(v')}{l(v')} \frac{dv'}{v'} - \int_{\frac{M+1}{M-1}v}^{v_0} \frac{n(v')}{l(v')} \frac{dv'}{v'}. \tag{22.12}$$

The second term in (22.12) is already known from the solution for the first collision interval, and we thus have a differential equation for

$$\int_v^{v_0} \frac{n(v')}{l(v')} \frac{dv'}{v'},$$

of the same form as before but with a different free term. The value of $n(v)$ for any v could in principle be determined in this manner by proceeding from one collision interval to the next. This procedure becomes prohibitively lengthy after the first few intervals, however, while it would be desirable to be able to determine $n(v)$ for remote collision intervals ($u \gg q$, where, as before,

$$u = 2\log(v_0/v) \quad \text{and} \quad q = 2\log[(M+1)/(M-1)])$$

directly. To do this, we may use Mellin transforms with respect to the velocity or, what is the same thing, Laplace transforms with respect to the lethargy. Multiplying (22.11) by $(v/v_0)^{2\eta}$ and integrating over

all v, changing the order of integration and putting

$$\phi(\eta) = \int_0^{v_0} \left(\frac{v}{v_0}\right)^{2\eta} \frac{n(v)}{l(v)}\, dv = \int_0^{\infty} e^{-\eta u} \frac{\bar{n}(u)}{l(u)}\, du, \tag{22.13}$$

we readily obtain

$$\phi(\eta) = \frac{(M+1)^2}{2M(2\eta+1)}\left[1-\left(\frac{M-1}{M+1}\right)^{2\eta+1}\right]\phi(\eta)+\frac{S}{v_0}.$$

Solving the last equation for $\phi(\eta)$ and applying the usual inversion formula, we have

$$\begin{aligned}\frac{n(v)}{l(v)} &= \frac{2}{v}\frac{\bar{n}(u)}{l(u)} \\ &= \frac{2S}{vv_0}\frac{1}{2\pi i}\int_{-i\infty+p}^{i\infty+p} \frac{1}{1-\dfrac{1}{2\eta+1}\dfrac{(M+1)^2}{2M}\left[1-\left(\dfrac{M-1}{M+1}\right)^{2\eta+1}\right]}\left(\frac{v_0}{v}\right)^{2\eta} d\eta,\end{aligned} \tag{22.14}$$

the path of integration being taken to the right of all the singularities of the integrand. These singularities consist only of the poles at the roots of

$$\frac{2M(2\eta+1)}{(M+1)^2} = 1-\left(\frac{M-1}{M+1}\right)^{2\eta+1}. \tag{22.15}$$

The integral in (22.14) is, of course, oscillatory, but this can be dealt with in the same way as in § 5.3. Then, deforming the path of integration and using the method of residues, we easily obtain

$$\frac{n(v)}{l(v)} = \frac{2MS}{v_0^2(M+1)^2}\sum_j \frac{-1}{\left[\dfrac{d}{dy}\left\{\dfrac{1}{y}\left[1-\left(\dfrac{M-1}{M+1}\right)^{y}\right]\right\}\right]_{y=2\eta_j+1}}\left(\frac{v_0}{v}\right)^{2\eta_j+1}, \tag{22.16}$$

where the η_j are the roots of equation (22.15).

One root of (22.15) is $\eta_j = \frac{1}{2}$, and it can easily be shown that all the other roots lie to the left of this. For $v \ll v_0$, the most important contribution to (22.16) therefore comes from $\eta_j = \frac{1}{2}$, and we can rewrite this formula as

$$\frac{n(v)}{l(v)} = \frac{C_M S}{v^2}[1+\text{terms tending to zero as } v/v_0 \to 0]$$

or as

$$n_{as}(v) = C_M S l(v)/v^2, \tag{22.17}$$

where

$$C_M = 2\Big/\left[1-\frac{(M-1)^2}{2M}\log\frac{M+1}{M-1}\right]. \tag{22.18}$$

The formulae (22.17) and (22.18) could have been more simply derived by a physical argument. Applying formula (22.9) for a monochromatic source in a non-capturing medium, we have

$$\chi(v) = \text{constant} = S \quad \text{for all } v < v_0. \tag{22.19}$$

Combining this with (22.7) and taking the case of a single element, we obtain

$$S = v^2 \frac{(M+1)^2}{4M} \int_v^{\frac{M+1}{M-1}v} \frac{n(v')\,dv'}{v'l(v')} - \frac{(M-1)^2}{4M} \int_v^{\frac{M+1}{M-1}v} \frac{v'n(v')\,dv'}{l(v')}. \tag{22.20}$$

This may be regarded as another equation to be satisfied by $n(v)$; $n_{as}(v)$ should clearly satisfy (22.20) and the homogeneous form of (22.11). Since the slowing-down density is constant in the case considered (see (22.19)), and is dimensionally equal to the neutron collision density per unit lethargy interval, it seems likely that the latter quantity $v\tilde{n}(u)/l(u)$ also is constant in the asymptotic region, i.e. $n_{as}(v)$ should be of the form

$$n_{as}(v) = \text{constant} \times l(v)/v^2. \tag{22.21}$$

It can be verified by direct substitution that (22.21) is in fact the solution of the homogeneous form of (22.11). By substituting (22.21) into (22.20) the multiplicative constant can be determined, and we reach (22.17) with C_M given by (22.18), as would be expected.

To determine the point at which $n(v)$ can be replaced by $n_{as}(v)$, the former has been evaluated from one collision interval to the next, starting from $v = v_0$, i.e. following the method described in connexion with (22.12), and the results compared with (22.17). This has been done for a number of values of M in the range $[2, \infty]$, and it was found that, for all M in this range, the difference between $n(v)$ and $n_{as}(v)$ after three collision intervals (i.e. for $v < [(M-1)/(M+1)]^3 v_0$) is negligibly small for most practical purposes (Placzek (41)). Since the determination of $n(v)$ by this method is fairly easy for the first three or four collision intervals, (22.16) need never be used except when it reduces to (22.17).

For hydrogen ($M = 1$) the collision interval is infinite, and the first differential equation derived from (22.11) gives $n(v)/l(v)$ immediately for all v. The solution of this differential equation is of the form (22.17). This implies that, for $M = 1$, the series (22.16) reduces to one term, and this is easily seen to be true, since (22.15) has only one root for $M = 1$

To conclude the discussion of a single element with no capture, let us examine the analytical character of $n(v)$. We can take $M \geqslant 2$, since the exact solution for $M = 1$ is already known. Here it will be more

convenient to revert to the procedure described in connexion with (22.12), i.e. to consider $n(v)$ in each collision interval in turn, starting from v_0. Within each collision interval $n(v)/l(v)$ is analytic, since it is the solution of a differential equation with an analytic free term, and it is necessary to examine only the singularities of $n(v)/l(v)$ at the ends of the collision intervals, i.e. for $v = [(M-1)/(M+1)]^n v_0$. Since $n(v)$ contains a term proportional to $\delta(v-v_0)$, the integrand in (22.11) contains a delta function when v is in the first collision interval. At the end of the first collision interval, this delta function disappears from the integrand, and $n(v)/l(v)$ therefore has a discontinuity for $v = [(M-1)/(M+1)]v_0$. For all smaller v it will be continuous. At the end of the second collision interval, the integrand in the second term of (22.12) is discontinuous, and therefore $n(v)/l(v)$ has a discontinuity in its derivative at this point. For all smaller v, the derivative also is continuous. Similarly, at the end of the third collision interval the second derivative of $n(v)/l(v)$ will be discontinuous, and so on. This implies that the series (22.16) can converge only for $v < [(M-1)/(M+1)]v_0$, its derivative series only for $v < [(M-1)/(M+1)]^2 v_0$, and so on. Thus, although the first term of this series is, as we have seen, a good approximation for all $v < [(M-1)/(M+1)]^3 v_0$, the improvement obtained by taking further terms will seldom be worth the trouble.

22.2.2. *A mixture with no capture*

We now turn from equation (22.11) to the more general (22.10). Since the scattering cross-section is almost constant for many substances, we shall first consider the case where capture is absent ($l_c(v) = \infty$), whilst the scattering mean free paths for all the elements present are either constant or given by the same law of variation, i.e.

$$l_{s,k}(v) \sum_{k'} \frac{1}{l_{s,k'}(v)} = \text{constant}. \tag{22.22}$$

By analogy with the previous case (a single element), we should expect that the deviations of $n(v)$ from $n_{as}(v)$ will disappear after a few collision intervals. Except in hydrogenous media, the collision interval is usually very small compared with the energy range concerned, and so $n_{as}(v)$ is usually the quantity required. The expression for $n_{as}(v)$ can be obtained as in the preceding case, and is

$$n_{as}(v) = 2Sl(v)\left[1 - \sum_k \frac{l(v)}{l_{s\,k}(v)} \frac{(M_k-1)^2}{2M_k} \log \frac{M_k+1}{M_k-1}\right]^{-1} v^{-2}, \tag{22.23}$$

where, as before, S is the source strength and $l(v)$ the total mean free path in the mixture, i.e. $1/l(v) = \sum 1/l_{s,k}(v)$.

As with (22.17), (22.23) can be derived in two ways, either by Mellin transforms or by using the slowing down density, (22.19) and (22.7). The latter derivation is much simpler and clearer, and shows the range of applicability of (22.23). It is seen from this derivation that it does not matter whether the condition (22.22) is satisfied for all velocities between v and v_0; the only relevant conditions are that the medium is non-capturing over this velocity range, and that (22.22) is satisfied at v and over a few collision intervals above it. This, of course, greatly extends the range of applicability of (22.23). It also suggests that (22.23) is a fairly good approximation even when the quantities

$$l(v)/l_{s,k}(v) \tag{22.22'}$$

vary near v, provided that they vary smoothly and change relatively little in one collision interval. Such a conjecture should be qualified, since the condition of smooth variation implies the smallness of the higher derivatives with respect to some variable. It is found that this variable should be the lethargy, if the conjecture is to hold for all $M_k \geqslant 2$. The conjecture can then be verified by direct calculation, substituting (22.23) into the equation to be solved (i.e. the homogeneous form of (22.10) with no capture), and expanding the various $l(v')/l_{s,k}(v')$ in powers of the lethargy difference:

$$l(v')/l_{s,k}(v') = l(v)/l_{s,k}(v) - \log\frac{v}{v'} \cdot \frac{d}{d\log v}[l(v)/l_{s,k}(v)] + \ldots; \tag{22.24}$$

$(d^k/d(\log v)^k)[l(v)/l_{s,k}(v)]$ is regarded as a quantity of the kth order of smallness. On calculating the terms of various orders, it is found that they vanish in the zero and first orders, and thus the extent to which (22.23) fails to satisfy the equation to be solved varies as the square of the relative changes in (22.22') in one collision interval. These changes must therefore be very considerable before (22.23) becomes seriously inaccurate.

22.2.3. *The effects of capture*

If capture is present, the relation (22.19) no longer holds, and the more general relation (22.9) cannot be utilized so easily. It is therefore necessary to use either less accurate approximations or more laborious methods. There are, however, two cases where the presence of capture introduces no new difficulties. One is the case of pure hydrogen, where

(22.10) can be converted immediately into a differential equation valid for all v, which can easily be solved. The other case is where all the cross-sections involved follow the same law of variation, i.e. their ratios are constant. Here the Mellin transform method can again be applied. Putting for brevity $l(v)/l_c(v) = 1-c$, and $l(v)/l_{s,k}(v) = cp_k$, where $l(v)$ is the total mean free path (i.e. $\sum p_k = 1$), we obtain

$$n_{as}(v)/l(v) = Av^{-y}, \tag{22.25}$$

where y is the root of

$$y = c\sum_k p_k \frac{(M_k+1)^2}{2M_k}\left[1-\left(\frac{M_k-1}{M_k+1}\right)^y\right] \tag{22.26}$$

having the greatest real part, and the constant A is determined by evaluating the residue of the Mellin transform at the pole corresponding to this root.

If the ratios of the cross-sections are not constant, approximate methods must be used. Three kinds of situation have received particular attention, namely:

(1) the case where the ratios of the cross-sections vary little in one collision interval;
(2) the case where the cross-sections vary appreciably over only one (or at most a few) collision intervals;
(3) the case of a single element when the ratio of capture to scattering varies as some negative power of the velocity, i.e. $l_c(v)/l_s(v) = (v/w)^k$, say, where $k > 0$.

Let us take the first of these three cases. If the ratios of the cross-sections become constant for $v < v_1$, say, then $n_{as}(v)$ should, by our previous arguments, take the form (22.25) a few collision intervals below v_1. The coefficient A will, of course, be different. Placzek (41) has suggested the following approximate method of determining A. Substituting (22.25) into (22.7), we obtain, since the ratios of the cross-sections are assumed constant in the collision intervals concerned,

$$\chi(v) = Bv^{2-y}, \tag{22.27}$$

where B is another constant whose ratio to A can easily be determined. We have derived (22.27) for v a few collision intervals below v_1. However, there are good reasons for believing that, if capture is fairly small, formula (22.27) is valid much closer to v_1 than (22.25), and that the error in applying (22.27) up to v_1 is much smaller than the corresponding error in (22.25). This conclusion is suggested firstly by the fact that, if

there is no capture for $v < v_1$, (22.27) is rigorously true, as we have seen, for all $v \leqslant v_1$, and secondly by the analogy, pointed out in connexion with (22.8), between the slowing-down density and the current, together with the fact that the assumption of constant ratios of cross-sections is the counterpart of the diffusion approximation. It is known that the relative deviation in the diffusion approximation for the current at the interface between two media is much less than the deviation in the flux; see § 8.4. Thus (22.27) can be applied up to and including v_1 without serious error. We can then imagine that the cross-sections vary stepwise, as in multi-group theory, and apply (22.27) to each group. The values of B and y are different, of course, for each group, those of B being related by the condition that $\chi(v)$ is continuous. Eliminating the B by differentiation, and writing y as $y(v)$ to indicate the group concerned, we have for each group the relation

$$\frac{v}{\chi(v)}\frac{d\chi(v)}{dv} = 2-y(v). \tag{22.28}$$

If (22.28), together with the continuity of $\chi(v)$, gives a good approximation in the case of stepwise variation of the ratios of the cross-sections, it should clearly give at least as good an approximation when they vary continuously. Integrating the differential equation (22.28), we have, since by (22.9) $\chi(v_0) = S$,

$$\chi(v) = S\exp\left[-\int_v^{v_0}[2-y(v')]\frac{dv'}{v'}\right]. \tag{22.29}$$

Having found $\chi(v)$, we can determine $n_{as}(v)$ by assuming that the ratio

$$\frac{v^2 n_{as}(v)}{\chi(v)l(v)}$$

has approximately the same value as it would have if the cross-sections were constant and equal to their actual values at the velocity considered. This assumption introduces a slight error, but the error is not cumulative.

In applying formula (22.29), we should recall that the derivation of (22.28) implied that the change in the cross-sections at each step is small and that these steps do not occur very frequently. This means that in the case of continuous variation we should assume the changes in the cross-sections in one collision interval to be small. The above derivation is valid only if the ratio of capture to scattering is also small.

If these two assumptions hold, and the variation of the cross-sections is sufficiently smooth, there is another way of solving the equations.

This consists in expanding the ratios

$$v'^2 n_{as}(v')/l(v'), \qquad l(v')/l_c(v'), \quad \text{etc.}, \tag{22.30}$$

in powers of the lethargy difference (as in (22.24)), terminating the expansions after a few terms, and substituting into the equations. The integration over v' is carried out, and thus an approximate differential equation for $v^2 n_{as}(v)/l(v)$ is obtained. There are three reasons why the quantities (22.30) are expanded in powers of the lethargy difference and not the velocity difference. Firstly, the collision interval is constant on the lethargy scale; secondly (a closely related reason), the quantities (22.30) have some singularities at $v' = 0$ which may affect the convergence in powers of $v'-v$, whereas the corresponding singularity a $u' = \infty$ cannot have any effect on the convergence; thirdly, the results of (22.23) and (22.24) suggest that simple yet (relatively) accurate results can be obtained if the higher derivatives with respect to lethargy are of higher orders of smallness.

In the present problem, this alternative procedure is not so satisfactory as the one given previously. Unless we work to the first order, we reach a differential equation which cannot be solved analytically, and has to be integrated numerically. If, on the other hand, we work to the first order, i.e. retain only the first derivative of $n_{as}(v)$, the accuracy obtained is considerably inferior to that of (22.29). However, the procedure is of interest because of the ease with which it can be extended to space-dependent slowing-down problems. We shall return to it in Chapter XXIII.

Hitherto we have considered only the case where capture is small and the cross-sections vary relatively little within one collision interval. It has been remarked in § 1.3.2, however, that in practice we often find capture resonances, where the capture cross-section rises to a sharp maximum at some energy. In such a case formula (22.29) is no longer a valid approximation in the resonance region, and an alternative means of calculating $\chi(v)$ in the resonance region is necessary. If the resonance is narrow, lying entirely within a few collision intervals, this can most naturally be done as follows. We calculate $\chi(v)$ from (22.29) up to the point where the resonance becomes appreciable. These $\chi(v)$ over one collision interval above this point are converted into $n_{as}(v)$ (see the discussion following (22.29)). We then return to the original equation (22.10), and solve it from one collision interval to the next as explained in connexion with (22.12), using the values of $n_{as}(v)$ just found. This process is continued until the resonance has been passed and (22.28) can

again be used. Then $\chi(v)$ is determined from (22.9), and thereafter we again use (22.28), i.e. (22.29), but with a modified value of S. A more detailed account of this procedure, together with information about practical devices, is given by Placzek (41).

22.2.4. *The case of* $1/v$ *capture*

Another important case, besides resonance capture, in which the cross-section varies too rapidly for (22.29) to be applicable, is that of $1/v$ capture at low energies (§ 1.3.2) and in time-dependent problems (the fictitious capture of § 3.2), the scattering cross-sections being approximately constant. For simplicity we shall assume that these laws are exactly satisfied, and that only one element is present. Taking the scattering mean free path as the unit of length, and using $n_{as}(v)$ rather than $n(v)$ (which is equivalent to putting $v_0 = \infty$), we obtain from (22.10):

$$\left(1+\frac{w}{v}\right)n_{as}(v) = \frac{(M+1)^2}{2M}\int_v^{\frac{M+1}{M-1}v} n_{as}(v')\frac{dv'}{v'}, \tag{22.31}$$

where w is the velocity for which the capture cross-section is equal to the scattering cross-section. For $v \gg w$ we may regard the capture as a small perturbation, and expand the solution of (22.31) in powers of w. This gives

$$n_{as}(v) = \frac{SC_M}{v^2}\left[1+\sum_{n=1}^{\infty} A_n\left(-\frac{w}{v}\right)^n\right], \tag{22.32}$$

where, as in (22.17), S is the source strength, C_M is given by (22.18), and the A_n can be determined by substituting (22.32) into (22.31) and comparing the coefficients of powers of w. An examination of these coefficients shows that, for M fixed and $n \to \infty$, $A_{n+1}/A_n \to 1$, so that (22.32) is convergent for all $v > w$. However, the relation $A_{n+1} \cong A_n$ holds only for $n \gg M$; for $n < M$, A_n increases rapidly with n, so that (22.32) is of little use for large M. This could have been foreseen on physical grounds, since for large M a neutron can lose very little energy in one collision; many collisions are thus needed for a considerable energy loss, and so capture may be important even if the capture probability per collision is small. The series (22.32), on the other hand, can possess reasonable initial convergence only if the effects of capture are small.

It can easily be shown that, for n fixed and $M \to \infty$, $n!A_n/A_1^n \to 1$. In view of this result and of the conclusions of § 22.2.3, Placzek (41) has suggested that $\log n_{as}(v)$, and not $n_{as}(v)$ itself, should be expanded in

powers of w, putting

$$n_{as}(v) = \frac{SC_M}{v^2}\exp\left[\sum_{n=1}^{\infty} B_n\left(-\frac{w}{v}\right)^n\right], \tag{22.33}$$

where

$$B_n = A_n - \tfrac{1}{2}\sum_{n'} A_{n'}A_{n-n'} + \tfrac{1}{3}\sum_{n'}\sum_{n''} A_{n'}A_{n''}A_{n-n'-n''} - \ldots.$$

On calculating the B_n, we find that every B_n is of the order of M when M is large. The maximum value of (22.33) is thus reached for v of the order of Mw. For v of this order, all the B_n being of the same order, the higher terms of (22.33) are unimportant, whilst beyond the maximum $n_{as}(v)$ decreases very rapidly and is insignificant for v of the order of w. Thus, for most purposes, we can terminate the series in (22.33) at the first term, using simply

$$n_{as}(v) = \frac{SC_M}{v^2}\exp\left[-\frac{M(M+1)}{M-\frac{1}{3}}\frac{w}{v}\right], \tag{22.34}$$

where we have inserted the actual value of B_1.

Although the conclusion that (22.34) is usually adequate has been reached by arguments concerning orders of magnitude, it appears to be valid not only for heavy nuclei, but also for fairly light nuclei. It has been shown by Davison and Mandl (14) that, if $g_2(v)$ is defined by

$$\int_v^\infty [n_{as}(v')]_{exact}\,dv' = g_2(v)\int_v^\infty [n_{as}(v')]_0\,dv', \tag{22.35}$$

where $[n_{as}(v)]_0$ is given by (22.34), then the ratio $g_2(v)$ is a monotonic function of v lying within the limits†

$$1 = g_2(\infty) < g_2(v) < g_2(0)$$
$$= \frac{M(M+1)}{2(M-\frac{1}{3})}\left[1 - \frac{(M-1)^2}{2M}\log\frac{M+1}{M-1}\right] \cong \left(1+\frac{1}{3M}\right)^2. \tag{22.36}$$

The above analysis can be immediately extended to the case where $l_c(v)/l_s(v) = (v/w)^k$, k being any positive number. The case of a mixture, however, has not been investigated in such detail.

† The proof of (22.36) is too long to be given in full, but we shall outline it. We first put $[n_{as}(v)]_{exact} = g_1(v)[n_{as}(v)]_0$, and convert (22.31) into an equation for $g_1(v)$. It can then be shown from this equation that $g_1(v)$ is always greater than a certain weighted mean of $g_1(v')$ over the collision interval above v, i.e. $g_1(v)$ increases 'on the average' as v decreases. Similarly, if we suppose that $g_1(v)$ has a maximum, say at v_1, it can be shown that $g_1([(M+1)/(M-1)]v_1)$ is greater than the weighted mean of $g_1(v')$ over the collision interval $v_1 < v' < [(M+1)/(M-1)]v_1$. There is therefore another maximum for some $v_2 > v_1$, and so on. That is, if $g_1(v)$ has any maxima, either it must have them for arbitrarily large v, which is impossible by (22.32), or there must be a real positive $v = v^*$, say, in the neighbourhood of which $g_1(v)$ has an infinite number of maxima and minima. In this case $g_1(v)$, and therefore $n_{as}(v)$, has an essential singularity at v^*; therefore, by (22.31), $n_{as}(v)$ has essential singularities at $v^*(M+1)/(M-1)$, $v^*(M+1)^2/(M-1)^2$, etc., which is impossible, by (22.32). Thus $g_1(v)$ increases monotonically as v decreases, and

22.3. The spatial moments of the neutron distribution. The slowing-down length

22.3.1. *Definitions*

We here conclude our discussion of the neutron energy spectrum integrated over all space, and we now turn to the determination of the spatial moments of the neutron distribution. For a plane source, these are defined as

$$m_{s,pl}(v) = \int_{-\infty}^{\infty} n(x,v)x^s\,dx, \tag{22.37}$$

where $n(x,v) = N_0(x,v)$ has the same meaning as in (22.2). The extension of the definition to the case of a point source is evident. Since the integral (22.37) is not one-valued if s is not integral, and vanishes by symmetry when s is an odd integer, we consider only even integral values of s. The zero-order moment is, of course, the same as $n(v)$ of the preceding section. The ratio of the sth moment to the zero moment is the expectation of the sth power of the distance from the source to a neutron of velocity v. This can be written

$$\left.\begin{aligned} m_{s,pl}(v)/n(v) &= \langle x^s(v)\rangle_{\mathrm{av}} \\ \text{and}\qquad m_{s,pl}(v)/n(v) &= \langle r^s(v)\rangle_{\mathrm{av}} \end{aligned}\right\}. \tag{22.38}$$

Since $r^2 = x^2+y^2+z^2$, while the values of $\langle x^2(v)\rangle_{\mathrm{av}}$, etc., are obviously

the same must be true of $g_3(v)$. Since (22.34) is exact for $v = \infty$, $g_3(\infty) = 1$. It only remains to evaluate $g_3(0)$, i.e. to find

$$\int_0^{\infty} [n_{as}(v)]_{\mathrm{exact}}\,dv.$$

This can be done by taking the Mellin transform of (22.31) and noting that

$$\lim_{\eta\to 1-}\left[(1-\eta)\int_0^{\infty} v^{\eta} n_{as}(v)\,dv\right] = \lim_{v\to\infty} v^3 n_{as}(v).$$

This gives

$$w\int_0^{\infty} [n_{as}(v)]_{\mathrm{exact}}\,dv = \lim_{\eta\to 1}\left\{\frac{\{(M+1)^3/2M(\eta+1)\}[1-\{(M-1)/(M+1)\}^{\eta+1}]-1}{1-\eta}\right\}\lim_{v\to\infty} v^3 n_{as}(v),$$

while (22.34) gives at once

$$w\int_0^{\infty} [n_{as}(v)]_3\,dv = \frac{M-\frac{1}{3}}{M(M+1)}\lim_{v\to\infty} v^3 n_{as}(v).$$

This completes the proof of (22.36).

independent of whether the source is a point or a plane, we have

$$\left.\begin{aligned}\langle r^2(v)\rangle_{\mathrm{av}} &= 3\langle x^2(v)\rangle_{\mathrm{av}}\\ \langle r^4(v)\rangle_{\mathrm{av}} &= 3\langle x^4(v)\rangle_{\mathrm{av}}+6[\langle x^2(v)\rangle_{\mathrm{av}}]^2, \text{ etc.}\end{aligned}\right\}, \tag{22.39}$$

so that it is sufficient to consider the spatial moments for a plane source.

Some authors define the spatial moments in a slightly different manner, using the slowing-down density instead of the neutron density. We distinguish the moments defined in this manner by asterisks, putting

$$m^*_{s,pl}(v) = \int_{-\infty}^{\infty} \chi(x,v)x^s\,dx, \tag{22.37'}$$

$$m^*_{s,pl}(v)/\chi(v) = \langle x^s(v)\rangle^*_{\mathrm{av}}, \text{ etc.} \tag{22.38'}$$

The quantities (22.38′) are not identical with (22.38), since the latter relate to the distances at which neutrons of velocity v are found, while the former relate to the distances at which neutrons are slowed down from velocities greater than v to those less than v. These latter neutrons, however, may have any velocity lying in the collision interval above v. Instead of being equal to (22.38), the quantities (22.38′) are related to them only by certain inequalities, which for a single element can be expressed as

$$\langle x^s(v)\rangle^*_{\mathrm{av}} \textit{ lies between } \langle x^s(v)\rangle_{\mathrm{av}} \textit{ and } \left\langle x^s\left(\frac{M+1}{M-1}v\right)\right\rangle_{\mathrm{av}}. \tag{22.40}$$

In fact, we shall see shortly that

$$\langle x^s(v)\rangle_{\mathrm{av}} > \langle x^s(v)\rangle^*_{\mathrm{av}} > \left\langle x^s\left(\frac{M+1}{M-1}v\right)\right\rangle_{\mathrm{av}}.$$

The relation (22.40) holds for mixtures also, provided that M is understood as the mass of the lightest nucleus present. Normally, however, the changes in $\langle x^s(v)\rangle_{\mathrm{av}}$ over one collision interval are very small, so that (22.40) is practically an equation. This argument does not hold for hydrogenous media, but it is usually possible to disregard the difference between (22.38) and (22.38′) even in this case.

We have introduced both these sets of moments because (22.38) are more convenient for numerical work, whereas the general physical arguments are clearer when applied to (22.38′).

One further general remark should be made. Let the slowing-down density be normalized to unit source strength, and let the position of the source and the initial neutron energy be explicitly shown. Then, disregarding the anisotropy of scattering in the L system for collisions

which decrease the neutron energy from above v' to below v', but not for any other collisions, we clearly have

$$\chi(x_0 \to x, v_0 \to v) = \int_{-\infty}^{\infty} \chi(x_0 \to x', v_0 \to v')\chi_0(x' \to x, v' \to v)\, dx' \quad \text{for } v_0 > v' > v.$$

Using the definitions (22.37′) and (22.38′), and recalling that all odd-order spatial moments vanish by symmetry, we obtain

$$\left.\begin{aligned} \langle x^2(v_0 \to v)\rangle^*_{\mathrm{av}} &= \langle x^2(v_0 \to v')\rangle^*_{\mathrm{av}} + \langle x^2(v' \to v)\rangle^*_{\mathrm{av}} \\ \langle x^4(v_0 \to v)\rangle^*_{\mathrm{av}} &= \langle x^4(v_0 \to v')\rangle^*_{\mathrm{av}} + 6\langle x^2(v_0 \to v')\rangle^*_{\mathrm{av}} \times \\ &\qquad \times \langle x^2(v' \to v)\rangle^*_{\mathrm{av}} + \langle x^4(v' \to v)\rangle^*_{\mathrm{av}}, \text{ etc.} \end{aligned}\right\} \quad (22.41)$$

To disregard the anisotropy in only one collision for each neutron cannot lead to a great error, and so the formulae (22.41) are nearly exact. However, if they are applied repeatedly, a considerable error may be accumulated. For instance, if N is large and γ is very close to unity, it would not be correct to calculate $\langle x^2(v_0 \to \gamma^N v_0)\rangle^*_{\mathrm{av}}$ as $\sum_{n=0}^{N-1} \langle x^2(\gamma^n v_0 \to \gamma^{n+1} v_0)\rangle^*_{\mathrm{av}}$, though this would give a very good order-of-magnitude estimate.

Two corollaries follow from formula (22.41). The first is that the quantities (22.38′) increase with decreasing v, i.e. the inequalities (22.40) are in the direction stated. The second is that this increase is roughly proportional to the lethargy; thus, from (22.40), it is seen that the difference between the quantities (22.38) and (22.38′) can be disregarded for lethargies large compared with the collision interval, though not for lethargies comparable with the collision interval.

The ratio of the second spatial moment to the zero-order moment is of particular importance. For this reason, we introduce the *slowing-down length* L_s, defined by

$$L_s^2(v) = \tfrac{1}{6}\langle r^2(v)\rangle^*_{\mathrm{av}} = \tfrac{1}{2}\langle x^2(v)\rangle^*_{\mathrm{av}}. \quad (22.42)$$

The first of (22.41) can then be written

$$L_s^2(v_0 \to v) = L_s^2(v_0 \to v') + L_s^2(v' \to v). \quad (22.43)$$

22.3.2. *The equations for the spatial moments*

As we have already remarked, it is sufficient to evaluate the spatial moments for the plane case, and we shall omit the suffix *pl* from now on. The quantities (22.37) are a particular case of the more general quantities

$$m_{s,n}(v) = \int_{-\infty}^{\infty} N_n(x, v) x^s\, dx, \quad (22.44)$$

where the $N_n(x,v)$ are defined as in (22.2), i.e. $m_{s,n}(v)$ is the sth spatial moment of the nth spherical harmonic moment of the neutron distribution (for velocity v) due to a plane source of neutrons of velocity v_0 at $x = 0$. Alternatively, $m_{s,n}(v)$ may be interpreted as the value at $x = 0$ of the nth spherical harmonic moment of the distribution of neutrons with velocity v, due to a distributed source of neutrons of velocity v_0, the strength per unit volume varying as x^s. In the latter case, the spatial distribution of any spherical harmonic moment is given by a polynomial in x of order at most s. The angular distribution at any point is then a polynomial in μ of order at most s, and consequently it cannot involve spherical harmonics of order higher than s. That is,

$$m_{s,n}(v) = 0 \quad \text{for } n > s. \tag{22.45}$$

Moreover, $m_{s,n}(v)$ vanishes, by symmetry, unless n and s are either both even or both odd.

The calculation of the second spatial moment (say), in terms of the quantities (22.44), can be carried out as follows. We multiply the first of (22.2) by x^2, integrate over all space, and eliminate $\partial/\partial x$ by integration by parts. This gives

$$-2m_{1,1}(v)+\left[\frac{1}{l_c(v)}+\sum_k \frac{1}{l_{s,k}(v)}\right]m_{2,0}(v) = \sum_k \frac{(M_k+1)^2}{2M_k}\int_v^{\frac{M_k+1}{M_k-1}v} \frac{m_{2,0}(v')}{l_{s,k}(v')}\frac{dv'}{v'}, \tag{22.46a}$$

where $m_{2,0}(v)$ is what we previously called $m_{2,p_l}(v)$. Similarly, multiplying the second of (22.2) by $\frac{1}{3}x$ and using (22.45), we obtain

$$-\tfrac{1}{3}m_{0,0}(v)+\left[\frac{1}{l_c(v)}+\sum_k \frac{1}{l_{s,k}(v)}\right]m_{1,1}(v) = \sum_k \frac{(M_k+1)^2}{2M_k}\int_v^{\frac{M_k+1}{M_k-1}v} \frac{m_{1,1}(v')}{l_{s,k}(v')}\frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'}\frac{dv'}{v'}, \tag{22.46b}$$

where $m_{0,0}(v)$ is what we previously called $n(v)$. Unless $n(v)$ has already been determined, the equations (22.46) must be supplemented by (22.10).

The equations to determine the higher moments can be constructed similarly. However, it is more convenient to avoid the repetition of identical algebraic operations, as follows. Let $M_n(p,v)$ be the Fourier transform of $N_n(x,v)$, i.e.

$$M_n(p,v) = \int_{-\infty}^{\infty} N_n(x,v)e^{ipx}\,dx;$$

then

$$m_{s,n}(v) = (-i)^s[\partial^s M_n(p,v)/\partial p^s]_{p=0}.$$

Thus, taking the Fourier transform of (22.2), expanding the $M_n(p, v)$ in powers of p, and collecting the terms in the $(n+1)$th equation which contain $p^n, p^{n+2}, \ldots, p^{s-n}$, we obtain the required set of equations to determine $m_{s,2i}(v)$.

22.3.3. *The second spatial moment in a single element without capture*

Since no new difficulties arise for higher moments, it is sufficient to discuss the determination of the second spatial moment, i.e. the solution of (22.46). As with (22.10), we take first the simple case of a single element with no capture. It is seen from the equations (22.46) that they can be solved one by one, whether or not capture is present, for either a single element or a mixture. We first solve (22.10), then (22.46 b), and lastly (22.46 a). For a single element without capture, each of these equations is of the form

$$\int_0^\infty f(v'/v)\psi(v')a(v')\,dv' = b(v),$$

where a, b, and f are known functions and ψ is unknown; these can always be solved by the Mellin transform method (see § 22.2.1). The amount of algebra required can, however, be minimized as follows Since $\delta(v_0-v) = (1/v_0)\delta(v/v_0-1)$, $m_{0,0}(v)$ can be expressed as

$$m_{0,0}(v) = n(v) = \frac{Sl(v)}{v_0^2} h_0\left(\frac{v}{v_0}\right), \tag{22.47}$$

where $h_0(y)$ satisfies

$$h_0(y) = \frac{(M+1)^2}{2M} \int_y^{\frac{M+1}{M-1}y} h_0(y')\frac{dy'}{y'} + \delta(y-1). \tag{22.48 a}$$

The free term of (22.46 b) can be written as

$$-\tfrac{1}{3}\int_0^{v_0} m_{0,0}(v_1)\delta(v/v_1-1)\frac{dv_1}{v_1},$$

and so, if h_1 is the solution of

$$h_1(y) = \frac{(M+1)^2}{2M} \int_y^{\frac{M+1}{M-1}y} h_1(y')\frac{(M+1)y^2-(M-1)y'^2}{2yy'}\frac{dy'}{y'} + \delta(y-1), \tag{22.48 b}$$

the solution of (22.46 b) can be written as

$$m_{1,1}(v) = \frac{l(v)}{3}\int_v^{v_0} m_{0,0}(v_1)h_1\left(\frac{v}{v_1}\right)\frac{dv_1}{v_1}, \tag{22.49 b}$$

the limits of integration being determined by the fact that $m_{0,0}(v_1)$ vanishes for $v_1 > v_0$, while $h_1(v/v_1)$ vanishes for $v_1 < v$. Similarly, we obtain from (22.46 a) and (22.48 a):

$$m_{2,0}(v) = 2l(v) \int_v^{v_0} m_{1,1}(v_1) h_0\left(\frac{v}{v_1}\right) \frac{dv_1}{v_1}. \qquad (22.49\,\text{a})$$

Combining (22.47), (22.49 b), and (22.49 a), we have the final expression for $m_{2,0}(v)$. The slowing-down length is found by substituting these results into (22.38), disregarding the difference between the latter and (22.38′), and using (22.42):

$$L_s^2(v) = \frac{1}{3h_0(v/v_0)} \int_v^{v_0} h_0\left(\frac{v}{v'}\right) l(v') \frac{dv'}{v'} \int_{v'}^{v_0} h_1\left(\frac{v'}{v''}\right) h_0\left(\frac{v''}{v}\right) l(v'') \frac{dv''}{v''}. \qquad (22.50)$$

Here $h_0(y)$ is, of course, the function $n(v)/l(v)$ already studied in § 22.2.1, and is best determined, over the first three collision intervals, by starting at $y = 1$ and solving (22.48 a) in each successive collision interval; for lower energies, the asymptotic expression found by the Mellin transform method can be used. The function $h_1(y)$ satisfies a similar equation and can be similarly determined.

22.3.4. *The case of a mixture and the higher spatial moments*

For a mixture, with or without capture, where the ratios of the cross-sections are constant, the method of the previous section may be applied. Otherwise, the solution is much more difficult. Although the use of the properties of $\chi(v)$ (see § 22.2.2) is some help in determining $n(v)$ for this case, and the same technique can be applied to equation (22.46 a), no counterpart of this method for (22.46 b) is known. Multi-group theory may be used, assuming the ratios of the cross-sections constant in each group. The slowing-down length in each group can then be determined as in § 22.3.3, and the slowing-down length over the entire range by (22.43), for instance. Another method, which can be used in conjunction with this, is to regard the deviations of the cross-section ratios from their mean values as small perturbations. It is beyond the scope of this book to discuss the methods which have been developed to solve (22.46) for certain particular cases; a detailed account is given by Marshak (38, especially pp. 203 ff.), who also discusses many other aspects of slowing-down problems.

We have seen that the equations (22.46) can be solved one by one. This is true for the higher moments also. The equation for $m_{s,n}(v)$ involves no other moments except $m_{s-1,n+1}(v)$ and $m_{s-1,n-1}(v)$.

XXIII

AGE THEORY

23.1. The assumptions underlying age theory

LET us now consider the spatial distribution of neutrons that have been slowed down. As has been pointed out in § 22.1.1, two types of problems are to be distinguished; those concerning neutrons which have travelled very much farther than the average distance from the source in reaching the energy in question, and those concerning typical neutrons. If the nuclei in the medium are sufficiently heavy, there is a comparatively simple and yet fairly accurate approximate method of solution of the latter type of problem, called *age theory*. It was first used by Bethe, Korff, and Placzek (2), who derived it from a simplified physical model. The formulation of age theory as an approximate method of solution of the general transport equations (22.2″), as well as the rigorous investigation of its range of applicability, are due to Marshak (38), whose treatment we shall largely follow.

The method is based on the following assumptions:

(A) The angular distribution of neutrons is nearly isotropic.

(B) The quantity $v^2N(\mathbf{r}, v\Omega)$, for fixed $\mathbf{r}$ and Ω, varies smoothly with lethargy and changes only slightly in one collision interval.

(C) The relevant cross-sections vary smoothly with lethargy and change only slightly in one collision interval.

(D) Capture is small.

(E) The same orders of smallness are referred to in the explicit forms of (A) and (B). Those referred to in the explicit forms of (C) and (D) will, of course, be of at least the same order as those in (A) and (B), if not higher.

The assumption (A) is understood to mean that $|\mathbf{j}(\mathbf{r}, v)|$ is small compared with $n(\mathbf{r}, v)$, while the higher spherical harmonic moments are of higher orders of smallness. The phrase 'varies smoothly with lethargy' in (B) and (C) will be understood to mean that the second derivatives, with respect to the lethargy, of the quantities concerned, multiplied by

the square of the collision interval in lethargy, are of at least the second order of small quantities.†

The assumptions (C) and (D) refer only to the medium concerned. The assumption (B) is a more precise expression of the statement that the problem concerns regions populated by typical neutrons. The assumptions (A) and (E) are intermediate in nature, depending on both the medium and the distances involved.

It will be convenient to determine first the form of the solution of equations (22.2″) under the above assumptions, and then to assess therefrom the range of applicability of these assumptions. However, we shall first make some remarks about the derivation of the conditions (A), (B), and (E), and the sense in which they represent the assumption of typical neutrons.

The velocity dependence in regions populated by typical neutrons should not differ very greatly from that which would have existed if the sources were uniformly distributed in all space. In the latter case, $N(\mathbf{r}, v\Omega)$ becomes equal to $n(v)$, and the analysis of § 22.2 can be used. If v is so close to v_0 (the initial velocity) that $n(v)$ differs appreciably from $n_{as}(v)$, it is hardly possible to speak of 'typical neutrons'. We shall therefore consider velocities for which $n(v)$ is practically the same as $n_{as}(v)$, i.e. those which lie three collision intervals or more below v_0. We know that, if the cross-sections are constant and capture is absent, $n_{as}(v)$ is proportional to $1/v^2$, i.e. $v^2 n_{as}(v)$ is constant. If there is some variation in the cross-sections and some capture, but these lie within the limits imposed by the conditions (C) and (D), then $v^2 n_{as}(v)$ should vary smoothly with v, and change only slightly in one collision interval. Thus assumption (B) may be regarded as a definition, or part-definition, of the regions of the four-dimensional $(\mathbf{r}, v)$ space occupied by the typical neutrons of the system.

The assumptions (B) and (C) have referred to smooth variation with lethargy, rather than with energy or velocity. This is partly because of the analysis of formula (22.23), and partly because of the more general considerations given in connexion with (22.30).

† The *smoothness* of a function $\phi(x)$ in the interval (a, b) may be defined as

$$(b-a)\int_a^b \left(\frac{d\phi}{dx}\right)^2 dx \Big/ \int_a^b |\phi(x)|\, dx \int_a^b \left|\frac{d^2\phi}{dx^2}\right| dx.$$

Since the quantities referred to in (B) and (C) are assumed to change only slightly in one collision interval, our use of the expression 'varies smoothly' is in accordance with this definition.

The anisotropy of the neutron angular distribution in a volume-velocity element $dVdv$ of the region occupied by typical neutrons cannot be much higher than that of neutrons that are scattered into $dVdv$. That is, the anisotropy of $N(\mathbf{r}, v\Omega)$ in this region is mainly governed by the anisotropy of the right side of (22.1) for the given $\mathbf{r}$ and v, and this, if (B) holds, is determined mainly by the anisotropy of

$$\int f(v'\Omega' \to v\Omega)\, dv'. \tag{23.1}$$

If the scattering is isotropic in the C system and the nuclei involved are fairly heavy, the nth spherical harmonic moment of (23.1) is of the order of $1/M^n$ (see § 17.1). These considerations lead to the assumption (A). If $v^2N(\mathbf{r}, v\Omega)$ and its derivatives with respect to lethargy are mutually comparable, the quantities referred to as 'small' in the explicit form of (B) are essentially of the order of the collision interval in lethargy, and this is of the order of $1/M$ for large M. Thus in this case the quantities referred to as small in (B) are of the same order of magnitude as those referred to as small in (A). This explains the introduction of assumption (E).

It is seen from the above considerations that, the heavier the nuclei involved, the better are the assumptions (A) and (E) satisfied. The same is true of (C), for a given rate of variation of the cross-sections. However, we prefer not to introduce directly the condition that the nuclei should be heavy, since this would misleadingly underrate the range of applicability of age theory. In the appropriate region of $(\mathbf{r}, v)$ space, the assumptions (A) to (E) are found to be well satisfied even for M of the order of 7, which can hardly be called heavy.

It would have been possible to avoid the introduction of assumption (D). However, if (D) is violated in some medium over a considerable range of velocities, the neutron population in that medium will be so small that it will be of no interest to obtain a quantitative solution of the slowing-down problem for this medium. If the capture is appreciable only over a narrow resonance, then it is assumption (C), rather than (D), which is violated.

23.2. The basic equation of age theory

23.2.1. *The reduction of the Boltzmann equation*

We shall now apply the above assumptions to simplify the transport equation (22.2″). As has already been remarked, assumption (A) means that, in the notation of (22.2″), $[N_1(\mathbf{r}, v, \mathbf{W})]_{W=1}$ (i.e. $\Omega \cdot \mathbf{j}(\mathbf{r}, v)$) is small compared with $n(\mathbf{r}, v)$, $[N_2(\mathbf{r}, v, \mathbf{W})]_{W=1}$ is of the second order of smallness,

and so on. Thus, if we work to the first order, we shall need only the first two equations (22.2″), i.e. only (22.3), and the first term in the second of (22.3) is omitted also.

The assumptions (B) and (C) imply that, if we expand the quantities $v'^2 n(\mathbf{r}, v')$, $v'^2 \mathbf{j}(\mathbf{r}, v')$, $l_c(v')$ and $l_{s,k}(v')$ in powers of the lethargy difference $u'-u = 2\log(v/v')$ about the point $v' = v$, v and v' lying in the same collision interval ($|u'-u| < \max q_k$), the second term in each series will be small compared with the first, while the third and subsequent terms will be of at least the second order of smallness.

Substituting these series in (22.3) and using assumption (E), we collect the terms of the zero and first orders of smallness. The former cancel by the identity

$$\frac{(M+1)^2}{2M} \int_v^{\frac{M+1}{M-1}v} \frac{dv'}{v'^2} \equiv \frac{1}{v^2}.$$

Putting

$$\left.\begin{aligned} b_k &= v^2 \frac{(M_k+1)^2}{2M_k} \int_v^{\frac{M_k+1}{M_k-1}v} \frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'} \frac{dv'}{v'^2}, \\ \xi_k &= v^2 \frac{(M_k+1)^2}{2M_k} \int_v^{\frac{M_k+1}{M_k-1}v} 2\log\left(\frac{v'}{v}\right) \frac{dv'}{v'^2}, \end{aligned}\right\} \tag{23.2}$$

and also

$$\frac{1}{l_s(v)} = \sum_k \frac{1}{l_{s,k}(v)}, \qquad \frac{b(v)}{l_s(v)} = \sum_k \frac{b_k}{l_{s,k}(v)}, \qquad \frac{\xi(v)}{l_s(v)} = \sum_k \frac{\xi_k}{l_{s,k}(v)}, \tag{23.3}$$

we obtain from the first-order terms

$$\left.\begin{aligned} &\operatorname{div} \mathbf{j}(\mathbf{r}, v) + \frac{n(\mathbf{r}, v)}{l_c(v)} = \frac{1}{2v} \frac{\partial}{\partial v}\left\{\frac{v^2 \xi(v) n(\mathbf{r}, v)}{l_s(v)}\right\} \\ \text{and} \qquad &\mathbf{grad}\, n(\mathbf{r}, v) + 3\frac{1-b(v)}{l_s(v)} \mathbf{j}(\mathbf{r}, v) = 0. \end{aligned}\right\} \tag{23.4}$$

The derivatives with respect to v do not enter into the second of (23.4), since each term in the second of (22.3) is itself of at least the first order of smallness, and thus only the leading terms in the expansions of $v'^2 \mathbf{j}(\mathbf{r}, v')$, etc., appear.

There is a simple physical interpretation of the quantities (23.2) and (23.3). If we substitute in (23.2) $v' = v^2/v''$ and use (2.16), we see that b_k is the probability that a neutron of initial velocity v which collides with a nucleus of mass M_k will have a final velocity between v'' and $v''+dv''$,

multiplied by the cosine of the angle of scattering and integrated over all v''. Thus b_k is the mean cosine of the angle of scattering in collisions with nuclei of mass M_k, i.e. nuclei of the kth kind. Since $l_s(v)/l_{s,k}(v)$ is the probability that a given scattering is at a nucleus of the kth kind, it follows that $b(v)$ is the mean cosine of the scattering angle for neutrons of velocity v in the given medium. Similarly, ξ_k is the mean logarithmic energy loss per collision (i.e. the mean lethargy gain per collision) for collisions with the kth kind of nucleus, whilst $\xi(v)$ is the mean logarithmic energy loss for all collisions of neutrons with velocity v in the given medium.

23.2.2. *The equation for the slowing-down density*

The equations (23.4) should now be put in a more convenient form and related to the conditions at the initial energy. This process is simplified if the unknown function is taken as $v^2\xi(v)n(\mathbf{r},v)/l_s(v)$. In general this quantity does not have a simple physical interpretation, but when the assumptions (B) and (C) are satisfied (i.e. when the equations (23.4) form a valid approximation), it is approximately equal to another quantity which has a simple physical meaning. If we start from the expression (22.7) for the slowing-down density $\chi(\mathbf{r},v)$, expand $v'^2n(\mathbf{r},v')$, $l_c(v')$, and $l_{s,k}(v')$ in powers of $\log(v'/v)$ in accordance with assumptions (B) and (C), and integrate, terminating the expansion at the first term, we obtain

$$\chi(\mathbf{r},v) = \tfrac{1}{2}v^2n(\mathbf{r},v)\sum_k\left[1-\frac{(M_k-1)^2}{2M_k}\log\frac{M_k+1}{M_k-1}\right]\frac{1}{l_{s,k}(v)}+$$
$$+\text{terms which are small when (B) and (C) hold.}$$

Evaluating the second integral in (23.2), comparing with this expression, and using (23.3), we get†

$$\chi(\mathbf{r},v) = v^2\xi(v)n(\mathbf{r},v)/2l_s(v)+\text{terms small when (B) and (C) hold.} \quad (23.5)$$

Combining (23.4) and (23.5) and eliminating $n(\mathbf{r},v)$ and $\mathbf{j}(\mathbf{r},v)$, we obtain the following equation for the slowing-down density:

$$\nabla^2\chi(\mathbf{r},v) = \frac{3[1-b(v)]}{l_c(v)l_s(v)}\chi(\mathbf{r},v)-\frac{3}{2}\frac{v\xi(v)[1-b(v)]}{l_s^2(v)}\frac{\partial\chi}{\partial v}, \quad (23.6)$$

† The result (23.5) could have been foreseen as follows. For the case of no capture, constant cross-section ratios and uniform sources, and for velocities where $n(v)$ is practically the same as $n_{as}(v)$, the number of neutrons slowed down past v per unit time and volume is equal to the neutron collision density per unit lethargy interval multiplied by the mean lethargy gain per collision, i.e.

$$\chi(v) = \xi\, v\tilde{n}_{as}(u)/l_s = \xi\, v^2n_{as}(v)/2l_s.$$

When the assumptions (B), (C), and (D) are satisfied, the relation between $\chi(\mathbf{r},v)$ and $n(\mathbf{r},v)$ should be essentially the same as in the idealized situation just discussed. Hence (23.5) follows.

which is valid in the same region and to at least the same accuracy as (23.4).

We now consider the relation of (23.6) to the conditions at the initial energy. This involves the incorporation of a free term in (23.6), which will be proportional to the three-dimensional delta function $\delta_3(\mathbf{r})$, since we have a point source at $\mathbf{r} = 0$. The coefficient of $\delta_3(\mathbf{r})$ is, of course, the same as for sources uniformly distributed in all space. In the latter case formula (22.9) gives

$$d\chi(v)/dv = -S\delta(v-v_0)+vm(v)/l_e(v),$$

where S is the source strength per unit volume; to incorporate the source term into the corresponding equation for $\chi(v)$, it is sufficient to replace $d\chi(v)/dv$ by $[d\chi(v)/dv+S\delta(v-v_0)]$. The corresponding change in equation (23.6) for the case of a point source is to replace $\partial\chi(\mathbf{r},v)/\partial v$ by

$$[\partial\chi(\mathbf{r},v)/\partial v+S\delta(v-v_0)\delta_3(\mathbf{r})],$$

where S is now the strength of this point source. We shall refer to equation (23.6) thus altered as (23.6′).

It might be inferred by analogy with the results of § 22.2.3 that the equation (23.6′) gives a fairly good approximation for all v up to and including v_0. This, however, is not so. Although the energy dependence of $\iiint \chi(\mathbf{r},v)\,dV$ is much smoother than that of $\iiint n(\mathbf{r},v)\,dV$, there is no reason to suppose that, for a highly localized source, the variation of $\chi(\mathbf{r},v)$ for a given $\mathbf{r}$ is any smoother than that of $n(\mathbf{r},v)$. The equation (23.6′), like (23.4), is therefore a valid approximation only when assumption (B) is satisfied, i.e. only for velocities at least a few collision intervals below v_0. However, for velocities many collision intervals below v_0, equation (23.6′) takes into account, to a very good approximation, the strength and position of the initial source. This can be seen as follows. Under the assumptions (C) and (D), it follows from the discussion of § 22.2.3 that equation (23.6′) leads to a fairly good approximation to the slowing-down density integrated over all space. We need therefore consider only the spatial variation of $\chi(\mathbf{r},v)$. Where assumption (B) is inapplicable, i.e. over the first few collision intervals, the spread of this variation is of a few mean paths only, whether it is determined exactly or by applying (23.6′) up to and including $v = v_0$. For lethargies very large compared with the collision interval, it is thus immaterial whether $\chi(\mathbf{r},v)$ is determined in the first few collision intervals exactly or by applying (23.6′) up to and including $v = v_0$.

23.2.3. *The age of the neutrons*

In order to put the equation (23.6′) in a still more convenient form, we introduce, instead of the velocity, a new independent variable

$$\theta = \theta(v_0 \to v) = \frac{2}{3}\int_v^{v_0} \frac{l_s^2(v')}{\xi(v')[1-b(v')]}\,\frac{dv'}{v'}, \tag{23.7}$$

and represent $\chi(\mathbf{r}, v)$ in the form

$$\chi(\mathbf{r}, v) = p(v)\chi_0(\mathbf{r}, v), \tag{23.8}$$

where $p(v)$ is the solution of

$$\frac{p(v)}{l_c(v)} = \tfrac{1}{2}v\frac{\xi(v)}{l_s(v)}\frac{dp}{dv}, \tag{23.9 a}$$

with the initial condition

$$p(v_0) = 1. \tag{23.9 b}$$

The equation for $\chi_0(\mathbf{r}, v)$ is then identical with (23.6′), except that the term proportional to $1/l_c(v)$ is now absent. If θ is taken as the independent variable instead of v, this equation for $\chi_0(\mathbf{r}, v)$ becomes

$$\partial\chi_0/\partial\theta = \nabla^2\chi_0 + S\delta(\theta)\delta_3(\mathbf{r}). \tag{23.10}$$

This equation is identical in form with the time-dependent equation of heat conduction, with θ playing the part of the time. For this reason θ is called the *Fermi age*, the *symbolic age*, or simply the *age* of the neutrons.

A more direct physical interpretation of the age may be given as follows. In an infinite medium, there cannot be any supply of neutrons from infinity in slowing-down problems, and the boundary condition on the solution of (23.10) is clearly

$$\chi_0(\mathbf{r}, v) = 0 \quad \text{for } |\mathbf{r}| = \infty.$$

It can be verified that the solution of (23.10) under this boundary condition is

$$\chi_0(\mathbf{r}, v) = \frac{S}{8(\pi\theta)^{\frac{3}{2}}}e^{-r^2/4\theta}. \tag{23.11}$$

If $\chi(\mathbf{r}, v)$ is given by (23.8) and (23.11), the value of $\langle r^2(v)\rangle_{\mathrm{av}}^*$ as defined by (22.38′) is simply

$$\iiint r^2 e^{-r^2/4\theta}\,dV \Big/ \iiint e^{-r^2/4\theta}\,dV = 6\theta.$$

In other words, according to the definition (22.42), the age $\theta(v_0 \to v)$ is equal to the square of the slowing-down length as calculated on the assumptions of age theory:

$$\theta(v_0 \to v) = [L_s^2(v_0 \to v)]_{\mathrm{age}}. \tag{23.12}$$

The solution of (23.9a) with the initial condition (23.9b) is clearly

$$p(v) = \exp\left[-2\int_{v}^{v_0} \frac{l_s(v')}{l_c(v')}\frac{dv'}{v'\xi(v')}\right], \tag{23.13}$$

and since, by (23.11), $\iiint \chi_0(\mathbf{r}, v)\, dV = S$, we have also

$$Sp(v) = \left[\iiint \chi(\mathbf{r}, v)\, dV\right]_{\text{age}}. \tag{23.14}$$

The result (23.11) can be expressed by saying that, for a point source in an infinite medium, the spatial distribution of neutrons of any energy appreciably below the initial energy, in the region occupied by typical neutrons, is approximately Gaussian. This could have been foreseen, since this spatial distribution arises by the superposition of a large number of random events, namely individual collisions, and this should be approximately Gaussian by the statistical law of large numbers.

23.2.4. *Modified age theory. The effects of resonance capture*

The results of the preceding section allow an *a posteriori* test of the accuracy of age theory over the region populated by typical neutrons of the system, for a given medium. To make this test, it is sufficient to compare the value of $\iiint \chi(\mathbf{r}, v)\, dV$ as given by (23.13) and (23.14) with the more accurate value obtained by the methods of § 22.2, and to compare the value of the slowing-down length as given by (23.7) and (23.12) with the more accurate value obtained by solving the equations of § 22.3.

This also suggests a modification of age theory which is widely used in practice but has not yet acquired a generally accepted name. We shall call it *modified age theory*. It consists in retaining the form (23.8) for the slowing-down density and the equation (23.10) derived above for $\chi_0(\mathbf{r}, v)$, but using for $p(v)$ not (23.13), but the expression $p(v) = \chi(v)/S$, where $\chi(v)$ is the slowing-down density integrated over all space; this is best determined by the methods of § 22.2 or by experiment. For the age θ we use, instead of (23.7), the square of the slowing-down length; this is best determined by solving the equations of § 22.3 or by experiment.

This modification is of particular interest since it allows age theory to be applied even when the assumption (C) is not well satisfied, provided that we are still concerned with regions populated by typical neutrons. For instance, if the medium has a sharp capture resonance extending over a few collision intervals only, the main effect is to reduce the total number of neutrons present, rather than to alter the spatial distribution, since

formula (23.11) shows that the majority of neutrons are found in the region where the spatial variation of $\chi(\mathbf{r}, v)$ is slight. Thus the main effect of violating assumption (C) is that $S^{-1}\iiint \chi(\mathbf{r}, v)\, dV$ differs from $p(v)$ as given by (23.13). The deviations of $S\chi(\mathbf{r}, v)/\iiint \chi(\mathbf{r}', v)\, dV'$ from (23.11) are less important. If, therefore, we determine $p(v)$ by the methods of § 22.2 instead of using (23.13), the main error due to the violation of assumption (C) will be eliminated without introducing any comparable new error.

The second most important effect of the violation of assumption (C) at a capture resonance is the deviation of L_s^2 from the approximate value given by (23.7) and (23.12). This can be seen as follows. The deviation of the true L_s^2 from the approximate value given by (23.7) and (23.12), arising at a capture resonance, will persist for all lower energies, as follows from (22.43). On the other hand, the remarks at the end of § 23.2.3 imply that any deviation of $\chi(\mathbf{r}, v)$ from the best fitting Gaussian, arising at a resonance, will be progressively reduced at lower energies. This confirms that the second most important effect of the violation of assumption (C) is the deviation of L_s^2 from the value given by (23.7) and (23.12). Thus, disregarding for the moment any other deviation of (23.7) from the best available value of L_s^2, we can say that the use of this value for θ instead of (23.7) eliminates the second most important error due to the violation of assumption (C), without introducing any comparable new error.

The following remark may be made regarding the deviations of (23.7) from the best available value of L_s^2 which are not due to the resonances. The only justification for applying (23.6′) over the first few collision intervals below the initial energy is that this can result in an error of at most $O(l_s^2)$ in θ. Thus, if the best available value of L_s^2 differs from (23.7) by a constant multiple of l_s^2, then the use of this value instead of (23.7) for the age will correct the error in question without introducing any comparable new error.

Finally, it must be borne in mind that we have argued in § 23.1 that it is better to assume $v^2 n(\mathbf{r}, v)$ to vary smoothly with lethargy than with velocity; that is, it is more profitable to expand $v'^2 n(\mathbf{r}, v')$ in powers of $\log(v'/v)$ than in powers of $v'-v$. This simply meant that the expansion in powers of $\log(v'/v)$ ensures that the neglected terms are small. An expansion in powers of $v'-v$ would have led to the same equations (23.8) and (23.10), though with inferior $p(v)$ and $\theta(v_0 \to v)$ (i.e. differing more markedly than do (23.13) and (23.7) from the correct values of $S^{-1}\iiint \chi(\mathbf{r}, v)\, dV$ and of L_s^2). However, the arguments of § 23.1 do not

preclude the possibility of the existence of a function $s(v)$ such that expanding $v'^2 n(\mathbf{r}, v')$ in powers of $s(v')-s(v)$ would be still more profitable than expanding it in powers of $\log(v'/v)$. In particular, it should be possible to find $s(v)$ such that the age obtained is identical (apart from an added constant of the order of l_s^2) with the best available value of L_s^2.

These considerations suggest that the modified age theory should always be an improvement, if only a slight one, on the original age theory of § 23.2.3.

23.3. The range of applicability of age theory

We shall now estimate the range of applicability of age theory. To do so, we start with the solution given by age theory and find the region in which it satisfies the assumptions underlying the theory. For the sake of simplicity, we take first a non-capturing medium, where $p(v) = 1$ for all v, so that $\chi(\mathbf{r}, v)$ is the same as $\chi_0(\mathbf{r}, v)$. Combining (23.11) and (23.5), we have

$$n(r, v) = \frac{S l_s(v)}{4\pi^{\frac{3}{2}} v^2 \xi(v) L_s^3(v)} e^{-r^2/4L_s^2(v)}, \tag{23.15}$$

where, in accordance with (23.12), we have written $\theta(v)$ as $L_s^2(v)$. Combining this with the second of (23.4) we obtain

$$j(r, v) = \frac{r l_s(v)}{6[1-b(v)] L_s^2(v)} n(r, v). \tag{23.16}$$

According to assumption (A), $j(r, v)$ should be small compared with $n(r, v)$. The equation (23.16) then shows that assumption (A) can be satisfied only if

$$r \ll 6[1-b(v)] L_s^2(v)/l_s(v). \tag{23.17}$$

To assess the range of applicability of assumption (B), we should compare the value of $\frac{1}{2} q_M d[v^2 n(r, v)]/d(\log v)$ (where q_M is the collision interval in lethargy) with that of $v^2 n(r, v)$ itself, and the value of

$$\tfrac{1}{2} q_M d[v^2 j(r, v)]/d(\log v)$$

with that of $v^2 j(r, v)$. Formula (23.15) gives

$$\tfrac{1}{2} q_M \frac{d[v^2 n(r, v)]}{d \log v} = v^2 n(r, v) \left[\frac{q_M r^2}{8 L_s^4(v)} \frac{d L_s^2(v)}{d \log v} + \frac{q_M}{2} \frac{d \log[l_s(v)/\xi(v) L_s^3(v)]}{d \log v} \right], \tag{23.18}$$

and a similar relation is obtained from (23.16). Using ordinary age theory, i.e. defining L_s^2 by (23.7) and (23.12), the condition that the first term in the brackets in (23.18) is small is clearly

$$r^2 \ll 12\{\xi(v)/q_M\}[1-b(v)][L_s^2(v)/l_s(v)]^2. \tag{23.19}$$

Using modified age theory, i.e. the best available value of L_s^2 instead of (23.7) and (23.12), the same inequality is obtained, since the order of magnitude of $dL_s^2(v)/d(\log v)$ cannot be affected, even though the actual value is somewhat different from that found by ordinary age theory.

The second term in the brackets in (23.18) and in the corresponding expression for $\frac{1}{2}q_M\, d[v^2 j(r,v)]/d(\log v)$ will be small, provided that assumption (C) holds and that $\frac{1}{2}q_M\, d(\log L_s^2)/d(\log v)$ is small. By (23.7) and (23.12), this latter condition is essentially equivalent to

$$q_M l_s^2(v) \ll \int_v^{v_0} l_s^2(v')\frac{dv'}{v'}. \tag{23.20}$$

This condition means simply that age theory cannot be applicable within the first few collision intervals below the initial energy, a fact which has already been pointed out in the derivation.

If hydrogen is present in appreciable amounts, the collision interval q_M in lethargy is infinite, and (23.19) and (23.20) are certainly violated. This means that age theory is at best a poor approximation for hydrogenous media.

For non-hydrogenous media, $\xi(v)$ and q_M are each of the order of $1/M$, i.e. they are mutually comparable, and the condition (23.19) is essentially identical with (23.17). All these considerations can, of course, be immediately extended to the case where capture is present.

We have considered above only the validity of the approximations made for a particular v. However, if the solution of (23.6′) (i.e., effectively, (23.11)) is to give the correct answer, the approximations made must have been valid during almost the whole slowing down of the neutron to velocity v. This will, in general, reduce somewhat the range of applicability of age theory. If the effect in question is taken into account, the inequality (23.17) must be replaced by

$$r \ll 6L_s^2(v)/\max_{v<v'<v_0}\{l_s(v')/[1-b(v')]\}, \tag{23.17′}$$

with a similar alteration of (23.20):

$$q_M \max_{v<v'<v_0} l_s^2(v') \ll \int_v^{v_0} l_s^2(v')\frac{dv'}{v'}. \tag{23.20′}$$

The detailed proof of (23.17′) is rather lengthy, but the necessity of replacing $l_s(v)$ by $\max l_s(v')$ $(v < v' < v_0)$ can be seen as follows. Let us consider neutrons which have been slowed down from v_0 to v' near the source, travelled to r without collision, and been slowed down from v' to

v near $\mathbf{r}$. The contribution of these neutrons to $n(r, v)$, apart from a numerical factor, is given by

$$\frac{S}{L_s^2(v_0 \to v') L_s^2(v' \to v)} \frac{1}{r^2} e^{-r/l_s(v')}. \tag{23.21}$$

For r sufficiently large, (23.11) decreases faster than (23.21) as r increases, and will eventually be smaller than (23.21). At this point, age theory is certainly inapplicable. The two expressions are readily seen to be comparable for r of the order of $L_s^2(v)/l_s(v')$. Since v' can have any value in (v, v_0), this means that age theory will certainly break down for r of the order of

$$L_s^2(v) \Big/ \max_{v<v'<v_0} l_s(v'),$$

which is essentially the same as (23.17′).

The above discussion shows also that, beyond the range of applicability of age theory, i.e. when (23.17′) is no longer satisfied, the decrease of $n(r, v)$ for fixed v and increasing r is retarded, and the Gaussian shape of the curve gradually becomes nearly exponential. We shall give in Chapters XXIV to XXVI a more detailed discussion of the behaviour of $n(r, v)$ in this region.

23.4. The boundary conditions and methods of solution of the age equations

23.4.1. *The boundary conditions*

We shall leave until § 23.6 the discussion of possible improvements in age theory, and examine first the applications of this theory in its present form.

Formula (23.11) for the case of an infinite medium could have been derived directly, without using (23.10). The main advantage of being able to use (23.10) is that it allows solutions to be obtained for finite media and for systems containing more than one medium. The boundary conditions to be used in the solution of (23.10) in such cases are found as follows. Since the higher spherical harmonic moments have been neglected in deriving (23.4), age theory may be regarded as a P_1 approximation, and the boundary conditions for that approximation may be taken over. That is, at an interface we stipulate that $n(\mathbf{r}, v)$ and the normal component of $\mathbf{j}(\mathbf{r}, v)$ are continuous functions of $\mathbf{r}$ for all v. In terms of $\chi_0(\mathbf{r}, v)$, using (23.5), (23.8), and the second of (23.4), these conditions become

$$\left.\begin{aligned} &p(v) l_s(v) \chi_0(\mathbf{r}, v)/\xi(v) \quad \textit{is continuous} \\ \text{and} \quad &\frac{p(v)}{\xi(v)[1-b(v)]} l_s^2(v) \frac{\partial \chi_0(\mathbf{r}, v)}{\partial n} \quad \textit{is continuous} \end{aligned}\right\}, \tag{23.22}$$

where $\partial\chi_0/\partial n$ is the normal component of $\mathbf{grad}\,\chi_0$, and $p(v)$ is, of course, calculated separately for each medium.

For the free surface, we proceed as follows. In the lower-order spherical harmonics approximations, and in particular the P_1 approximation, it is known that Marshak's boundary conditions are always preferable to Mark's boundary conditions (see § 10.3.6). In the P_1 approximation, Marshak's boundary conditions are

$$j_n(\mathbf{r}, v) = \tfrac{1}{2}n(\mathbf{r}, v),$$

where $j_n(\mathbf{r}, v)$ is the component of $\mathbf{j}$ along the outward normal to the surface. Combining this with the second of (23.4) and expressing the result in terms of $\chi_0(\mathbf{r}, v)$, we obtain

$$\chi_0(\mathbf{r}, v)+\frac{2}{3}\frac{l_s(v)}{1-\bar{b}(v)}\frac{\partial\chi_0(\mathbf{r}, v)}{\partial n} = 0 \quad \text{at the free surface,} \quad (23.23)$$

where $\partial\chi_0/\partial n$ is the component of $\mathbf{grad}\,\chi_0$ along the outward normal.

23.4.2. *Analytical methods of solution of the age equations*

If the mean free paths concerned are independent of the velocity, the boundary conditions (23.22) and (23.23) are of the form usually found in problems of heat conduction, and the methods developed in the latter subject can be used to solve (23.10); the most general method is as follows. If all the mean free paths are constant, the age θ as given by (23.7) is proportional in each medium to the lethargy u, while $p(v)$ in each medium is of the form $e^{-\gamma u}$, where γ is constant in each medium. Taking the Laplace transform of (23.10) and of the boundary conditions (23.22) and (23.23) with respect to the lethargy, we obtain a differential equation in the spatial coordinates only. On solving this equation and taking the inverse Laplace transform, we find $\chi_0(\mathbf{r}, v)$.

This method can be immediately extended to the case where the mean free paths follow the same law of variation in each medium, while either the free surfaces are absent or the relevant geometrical dimensions are so large compared with $l_s(v)$ that (23.23) can be approximated by

$$\chi_0(\mathbf{r}, v) = 0 \quad \text{at the free surface,} \quad (23.24)$$

and all the media are non-capturing. In this case, although the ages are no longer proportional to the lethargy, the ages in any two media are proportional. The age in some chosen medium, which we call the *reference medium*, can then be taken as the independent variable throughout the system, and the Laplace transform of equations (23.10) can be taken in every medium with respect to the age in the reference medium. Since all

the nuclei are assumed to be non-capturing, $p(v)$ will not appear, and since $l_s(v)/\xi(v)$ on one side of the interface is proportional to its value on the other side, the first condition (23.22) is equivalent to the condition that $\chi_0(\mathbf{r}, v)$ on one side of the interface is proportional to its value on the other side. Hence the Laplace transform of $\chi_0(\mathbf{r}, v)$ with respect to the age in the reference medium will satisfy the same condition. The second condition (23.22) will be transformed similarly. Thus the solution can be obtained by the same means as in the case of constant mean free paths.

Further simplifications often arise in the above two cases which allow the solution to be obtained without using the method of Laplace transforms. For instance, if we have a bare slab with a point source of fast neutrons inside it, and the distance from this point to either face of the slab is so large compared with $l_s(v)$ that (23.23) may be approximated by (23.24), the solution of (23.10) can easily be obtained by the well-known *method of images*. Similar simplifications occur in several other cases.

23.4.3. *Numerical methods; the direct approach*

If the mean free paths in the different media do not follow the same law of variation, or if (23.23) cannot be safely approximated by (23.24), and the velocity dependence of $l_s(v)$ in (23.23) has to be taken into account, the solution of (23.10) is much more difficult. An analytical solution is hardly ever available, and numerical methods of solution must be employed. The general principles of the numerical methods of solving the heat conduction equation are well known; we introduce a network of points in the $(\mathbf{r}, v)$ space, approximate the derivatives by finite differences, and solve the resulting system of algebraic equations. If v_i is the ith velocity value in the network, $\mathbf{r}_\mathbf{m}$ is the $\mathbf{m}$th point in the spatial network, and $\chi_{i,\mathbf{m}} = \chi_0(\mathbf{r}_\mathbf{m}, v_i)$, the best way of converting (23.10) into a finite difference equation is to represent it as

$$\chi_{i+1,\mathbf{m}} - \chi_{i,\mathbf{m}} = \tfrac{1}{2}\Delta\theta[(\nabla^2\chi)_{i+1,\mathbf{m}} + (\nabla^2\chi)_{i,\mathbf{m}}], \tag{23.25}$$

where

$$(\nabla^2\chi)_{i',\mathbf{m}} = \sum_{\mathbf{m}'} a_{\mathbf{m},\mathbf{m}'}\chi_{i',\mathbf{m}'},$$

the summation being extended over $\mathbf{m}' = \mathbf{m}$ and its nearest neighbours. The step from i to $i+1$ may be made either by solving the simultaneous equations for the $\chi_{i+1,\,\mathbf{m}}$ (16), or by iteration. The equation (23.25) is recommended because of its *stability*; that is, if (23.25)

is used, the accumulated effect of the rounding-off errors can never become appreciable. The equation

$$\chi_{t+1,m} = \chi_{t,m} + \Delta\theta(\nabla^2\chi)_{t,m}$$

is stable only if $\Delta\theta$ is sufficiently small; the stability criterion cannot be given explicitly except in the simplest cases.

If only a single medium is concerned, and the necessity of numerical methods arises only because of the boundary condition (23.23), we should choose the v_t so as to obtain equal intervals of age for this medium. If the system involves several media, and the ages in them, as functions of velocity, are not simply related, the v_t should represent equal intervals of lethargy.

In adopting the numerical procedure described, we no longer take full advantage of the fact that the integrals in (22.3) are expressed in terms of $\chi(\mathbf{r}, v)$ and $\partial\chi/\partial\theta$. Thus, if numerical procedures are to be used, we can equally well start with equations (22.3), neglecting $[N_2(\mathbf{r}, v, \mathrm{W})]_{W=1}$, eliminate $\mathbf{j}(\mathbf{r}, v)$, and express all the integrals involved in terms of the $n(\mathbf{r}, v)$ values at the network points. This more direct procedure has the advantage that the collision interval need not be small, and so hydrogenous media can be dealt with. Inelastic scattering also can be taken into account without additional difficulty, whereas it is very awkward to incorporate it into ordinary age theory. The direct approach was first suggested by Ehrlich and Hurwitz (16), and was applied to the case of inelastic scattering by Mandl (31).

23.5. Application of age theory to spectrum-regeneration problems

23.5.1. *Modified two-group theory: general considerations*

After developing age theory, we return in the present section to spectrum-regeneration problems. In dealing with these it is sometimes necessary to solve an auxiliary slowing-down problem, and age theory is often used for this purpose. For instance, let a body consist of a mixture of fissile material and a *moderator* of fairly light nuclei, the latter having a very small capture probability per collision, with the additional conditions that the thermal fission cross-section is (as is usual) much greater than the fast fission† cross-section, while the mean free path for thermal fission is at most comparable with that for thermal capture in the light nuclei. This system is, of course, a homogeneous thermal reactor. In such a medium, a fast neutron will rapidly lose energy in

† i.e. fission by non-thermal neutrons.

collisions with the moderator, and the probability that it will cause fast fission is negligibly small. The system is, however, spectrum-regenerating, since a thermal neutron has a finite probability of causing fission. It may be advantageous to consider the fast and thermal neutrons separately in such a system. If the spatial distribution of fissions is given, the determination of the fast neutron distribution for any particular energy may be regarded as an auxiliary slowing-down problem. Its solution gives, *inter alia*, the distribution of neutrons which are just becoming thermal. Using these as the thermal neutron sources, we can use the methods of Part II to find the complete distribution of thermal neutrons, and thus of fissions.

This treatment is a form of two-group theory, but whereas the thermal neutrons are treated in the manner of Chapter XIX, a more accurate method is used for the fast group, namely the solution of the auxiliary slowing-down problem. We have seen that the essential limitations on the age-theory treatment of slowing-down problems are given by the inequalities (23.17′) and (23.20′). In the present application, we are concerned only with fast neutrons which are just becoming thermal, and the inequality (23.20′) is certainly satisfied for these, at least for non-hydrogenous media. The inequality (23.17′) has been derived for the case of a point source, for which the neutron density (from a unit source) at the distances where (23.17′) no longer holds is

$$O(e^{-L_1^2/\tau_1})$$

(see (23.11)), and this is usually extremely small. Consequently, if instead of a point source we have fast neutron sources distributed in the medium (as in the case considered), and the source density does not vary rapidly with position, the great majority of fast neutrons of a given velocity v at a point come from sources so near that point that (23.17′) is satisfied. The contribution of the sources which are so remote from the point that (23.17′) does not hold can be neglected. This inequality therefore does not affect the present application. Another restriction on the applicability of age theory is that the cross-sections do not vary rapidly. However, the error incurred by violating this condition can be greatly reduced by using modified age theory (§ 23.2.4) instead of ordinary age theory. Thus age theory should be fully adequate for solving the auxiliary slowing-down problem in non-hydrogenous media.

23.5.2. *Modified two-group theory: the form of the equations*

We shall now write down the equations of modified two-group theory as formulated in the preceding section, and for simplicity we take first

the case of an infinite medium, i.e. where the solution of (23.10) is given by (23.11). Since age theory is essentially equivalent to a solution of equations (22.2′) in the P_1 approximation, it would be somewhat inconsistent to use an approximation higher than P_1 in dealing with the diffusion of thermal neutrons. Since the anisotropy of the fast neutron distribution is small, and so is the anisotropy of scattering, the anisotropy in the angular distribution of neutrons after a collision is of the second order of small quantities, and can be neglected in the present approximation. This applies, in particular, to neutrons which have just become thermal, so that the thermal neutron sources may be regarded as isotropic.† The equations of one-group theory in the P_1 approximation, with anisotropic scattering and isotropic sources, are

$$\operatorname{div}\mathbf{j}(\mathbf{r})+\frac{1-c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}\rho(\mathbf{r})=s_0(\mathbf{r}) \tag{23.26a}$$

and

$$\operatorname{\mathbf{grad}}\rho(\mathbf{r})+3\frac{1-b_{\mathrm{th}}c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}\mathbf{j}(\mathbf{r})=0, \tag{23.26b}$$

from (17.35′), since $\Psi'_0=\rho$, etc.; $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are the flux and current of thermal neutrons, $s_0(\mathbf{r})$ is the thermal source density, that is, the number of neutrons that become just thermal at $\mathbf{r}$ per unit time and volume, and $l_{\mathrm{tot,th}}$, b_{th}, and $c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}$ are respectively the total mean free path, mean cosine of the scattering angle, and mean number of thermal secondaries per collision, for thermal neutrons. The last three quantities should, of course, be interpreted as the averages over the thermal spectrum of the corresponding velocity-dependent quantities $l_{\mathrm{tot}}(v)$, $b(v)$, etc., the averaging being performed as shown in connexion with (4.13). The mean number of secondaries is expressed as the product of two factors for convenience in passing from the auxiliary one-group problem to the actual problem. From the definitions (1.5) and (1.7) we have

$$\frac{1-c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}=\frac{1}{l_{c,\mathrm{th}}}+\frac{1}{l_{f,\mathrm{th}}}. \tag{23.27}$$

We shall now consider $s_0(\mathbf{r})$ more closely. The quantity $\rho(\mathbf{r})/l_{f,\mathrm{th}}$ represents the number of neutrons which cause fissions at $\mathbf{r}$ per unit time and volume, and is therefore the number of fissions, so that the number of neutrons produced in fission at $\mathbf{r}$ per unit time and volume is $\nu\rho(\mathbf{r})/l_{f,\mathrm{th}}$. Since the medium extends to infinity, the slowing-down

† It may seem inconsistent, in view of this, to have retained $b(v)$ in the second of (23.4), but there we were dealing with a large number of collisions and wished to preclude the possibility of an accumulated error; here only a single collision in a neutron's history is concerned. The situation is similar to that encountered in the derivation of (22.41).

density from a point source is given by (23.8) and (23.11), while the slowing-down density from the actual source distribution is

$$\frac{\nu}{8\pi^{\frac{3}{2}}}\frac{p(v)}{L_s^3(v)}\frac{1}{l_{f,\mathrm{th}}}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2(v)}\,dV', \tag{23.28}$$

where, as in (23.15), we have expressed the age in terms of the slowing-down length. On the other hand, $s_0(\mathbf{r})$ in (23.26 a) is the number of neutrons that become thermal at $\mathbf{r}$ per unit time and volume, and it is therefore the slowing-down density for the velocity v^* which is chosen as the boundary between fast and thermal neutrons. Thus $s_0(\mathbf{r})$ in (23.26 a) is the value of (23.28) for $v = v^*$. The best choice of v^* is, of course, rather uncertain, but $p(v^*)$ and $L_s(v^*)$ are comparatively insensitive to the actual choice made, and we can therefore speak simply of the capture escape probability and slowing-down length from fission to thermal energy. We call these p and L_s respectively. Then (23.26 a) becomes

$$\operatorname{div}\mathbf{j}(\mathbf{r})+\left(\frac{1}{l_{c,\mathrm{th}}}+\frac{1}{l_{f,\mathrm{th}}}\right)\rho(\mathbf{r}) = \frac{\nu p}{8\pi^{\frac{3}{2}}L_s^3 l_{f,\mathrm{th}}}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2}\,dV', \tag{23.29 a}$$

where we have also used equation (23.27).

Equations (23.29 a) and (23.26 b) are the basic equations of the modified two-group theory. However, it is often preferred to introduce certain changes in (23.26 b). Firstly, since b_{th} is usually fairly small, while $c_{\mathrm{th}} f_{\mathrm{th}\to\mathrm{th}}$ is close to unity, $1-b_{\mathrm{th}}c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}$ can usually be replaced by $1-b_{\mathrm{th}}$. Secondly, a comparison with (5.15) suggests that, if we replace $l_{\mathrm{tot,th}}$ in (23.26 b) by $l_{s,\mathrm{th}}$ (the thermal scattering mean free path), this will to some extent compensate the error involved in the P_1 approximation. With these modifications, equation (23.26 b) becomes

$$\mathbf{grad}\,\rho(\mathbf{r})+3\frac{1-b_{\mathrm{th}}}{l_{s,\mathrm{th}}}\mathbf{j}(\mathbf{r}) = 0. \tag{23.29 b}$$

Eliminating $\mathbf{j}(\mathbf{r})$ between the equations (23.29) and introducing the abbreviations

$$\left.\begin{aligned} L &= \left[\frac{l_{s,\mathrm{th}}\,l_{c,\mathrm{th}}\,l_{f,\mathrm{th}}}{3(1-b_{\mathrm{th}})(l_{c,\mathrm{th}}+l_{f,\mathrm{th}})}\right]^{\frac{1}{2}} \\ \text{and}\qquad f\eta &= \frac{\nu l_{c,\mathrm{th}}}{l_{c,\mathrm{th}}+l_{f,\mathrm{th}}} \end{aligned}\right\}, \tag{23.30}$$

we obtain

$$\left(\nabla^2-\frac{1}{L^2}\right)\rho(\mathbf{r}) = -\frac{f\eta p}{8\pi^{\frac{3}{2}}L_s^3 L^2}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2}\,dV'. \tag{23.31}$$

On comparing the first of (23.30) with (23.27) and (5.16 b), we can see that L represents an approximation to the diffusion length for thermal

density from a point source is given by (23.8) and (23.11), while the slowing-down density from the actual source distribution is

$$\frac{\nu}{8\pi^{\frac{3}{2}}}\frac{p(v)}{L_s^3(v)}\frac{1}{l_{f,\mathrm{th}}}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2(v)}\,dV', \tag{23.28}$$

where, as in (23.15), we have expressed the age in terms of the slowing-down length. On the other hand, $s_0(\mathbf{r})$ in (23.26 a) is the number of neutrons that become thermal at $\mathbf{r}$ per unit time and volume, and it is therefore the slowing-down density for the velocity v^* which is chosen as the boundary between fast and thermal neutrons. Thus $s_0(\mathbf{r})$ in (23.26 a) is the value of (23.28) for $v = v^*$. The best choice of v^* is, of course, rather uncertain, but $p(v^*)$ and $L_s(v^*)$ are comparatively insensitive to the actual choice made, and we can therefore speak simply of the capture escape probability and slowing-down length from fission to thermal energy. We call these p and L_s respectively. Then (23.26 a) becomes

$$\operatorname{div}\mathbf{j}(\mathbf{r})+\left(\frac{1}{l_{c,\mathrm{th}}}+\frac{1}{l_{f,\mathrm{th}}}\right)\rho(\mathbf{r}) = \frac{\nu p}{8\pi^{\frac{3}{2}}L_s^3 l_{f,\mathrm{th}}}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2}\,dV', \tag{23.29 a}$$

where we have also used equation (23.27).

Equations (23.29 a) and (23.26 b) are the basic equations of the modified two-group theory. However, it is often preferred to introduce certain changes in (23.26 b). Firstly, since b_{th} is usually fairly small, while $c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}$ is close to unity, $1-b_{\mathrm{th}}c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}$ can usually be replaced by $1-b_{\mathrm{th}}$. Secondly, a comparison with (5.15) suggests that, if we replace $l_{\mathrm{tot,th}}$ in (23.26 b) by $l_{s,\mathrm{th}}$ (the thermal scattering mean free path), this will to some extent compensate the error involved in the P_1 approximation. With these modifications, equation (23.26 b) becomes

$$\mathbf{grad}\,\rho(\mathbf{r})+3\frac{1-b_{\mathrm{th}}}{l_{s,\mathrm{th}}}\mathbf{j}(\mathbf{r}) = 0. \tag{23.29 b}$$

Eliminating $\mathbf{j}(\mathbf{r})$ between the equations (23.29) and introducing the abbreviations

$$\left.\begin{aligned} L &= \left[\frac{l_{s,\mathrm{th}}l_{c,\mathrm{th}}l_{f,\mathrm{th}}}{3(1-b_{\mathrm{th}})(l_{c,\mathrm{th}}+l_{f,\mathrm{th}})}\right]^{\frac{1}{2}} \\ \text{and}\qquad f\eta &= \frac{\nu l_{c,\mathrm{th}}}{l_{c,\mathrm{th}}+l_{f,\mathrm{th}}} \end{aligned}\right\}, \tag{23.30}$$

we obtain

$$\left(\nabla^2-\frac{1}{L^2}\right)\rho(\mathbf{r}) = -\frac{f\eta p}{8\pi^{\frac{3}{2}}L_s^3L^2}\iiint \rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|^2/4L_s^2}\,dV'. \tag{23.31}$$

On comparing the first of (23.30) with (23.27) and (5.16 b), we can see that L represents an approximation to the diffusion length for thermal

the case of an infinite medium, i.e. where the solution of (23.10) is given by (23.11). Since age theory is essentially equivalent to a solution of equations (22.2′) in the P_1 approximation, it would be somewhat inconsistent to use an approximation higher than P_1 in dealing with the diffusion of thermal neutrons. Since the anisotropy of the fast neutron distribution is small, and so is the anisotropy of scattering, the anisotropy in the angular distribution of neutrons after a collision is of the second order of small quantities, and can be neglected in the present approximation. This applies, in particular, to neutrons which have just become thermal, so that the thermal neutron sources may be regarded as isotropic.† The equations of one-group theory in the P_1 approximation, with anisotropic scattering and isotropic sources, are

$$\operatorname{div}\mathbf{j}(\mathbf{r})+\frac{1-c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}\rho(\mathbf{r})=s_0(\mathbf{r}) \tag{23.26a}$$

and

$$\operatorname{\mathbf{grad}}\rho(\mathbf{r})+3\frac{1-b_{\mathrm{th}}c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}\mathbf{j}(\mathbf{r})=0, \tag{23.26b}$$

from (17.35′), since $\Psi'_0=\rho$, etc.; $\rho(\mathbf{r})$ and $\mathbf{j}(\mathbf{r})$ are the flux and current of thermal neutrons, $s_0(\mathbf{r})$ is the thermal source density, that is, the number of neutrons that become just thermal at $\mathbf{r}$ per unit time and volume, and $l_{\mathrm{tot,th}}$, b_{th}, and $c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}$ are respectively the total mean free path, mean cosine of the scattering angle, and mean number of thermal secondaries per collision, for thermal neutrons. The last three quantities should, of course, be interpreted as the averages over the thermal spectrum of the corresponding velocity-dependent quantities $l_{\mathrm{tot}}(v)$, $b(v)$, etc., the averaging being performed as shown in connexion with (4.13). The mean number of secondaries is expressed as the product of two factors for convenience in passing from the auxiliary one-group problem to the actual problem. From the definitions (1.5) and (1.7) we have

$$\frac{1-c_{\mathrm{th}}f_{\mathrm{th}\to\mathrm{th}}}{l_{\mathrm{tot,th}}}=\frac{1}{l_{c,\mathrm{th}}}+\frac{1}{l_{f,\mathrm{th}}}. \tag{23.27}$$

We shall now consider $s_0(\mathbf{r})$ more closely. The quantity $\rho(\mathbf{r})/l_{f,\mathrm{th}}$ represents the number of neutrons which cause fissions at $\mathbf{r}$ per unit time and volume, and is therefore the number of fissions, so that the number of neutrons produced in fission at $\mathbf{r}$ per unit time and volume is $\nu\rho(\mathbf{r})/l_{f,\mathrm{th}}$. Since the medium extends to infinity, the slowing-down

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where the suffixes th and fa denote thermal and fast, and the suffix is omitted from ρ_{th} and $\mathbf{j}_{th}$. We have assumed, for the same reasons as before, that the anisotropy in the distribution of neutrons passing from one group to the other can be neglected.

Changing the third and fourth of (23.39) in the same way as (23.26 b), eliminating $\mathbf{j}(\mathbf{r})$ and $\mathbf{j}_{fa}(\mathbf{r})$, and using

$$c_{th}f_{th\to fa}/l_{tot,th} = \nu/l_{f,th}, \qquad f_{fa\to fa}+f_{fa\to th} = 1,$$

and the abbreviations (23.30) together with

$$L_{fa} = \left[\frac{l_{s,fa}\,l_{tot,fa}}{3(1-b_{fa})(1-c_{fa}f_{fa\to fa})}\right]^{\frac{1}{2}} \tag{23.40}$$

and

$$\phi(\mathbf{r}) = \frac{(1-c_{fa}f_{fa\to fa})l_{c,th}\,l_{f,th}}{l_{tot,fa}(l_{c,th}+l_{f,th})}\rho_{fa}(\mathbf{r}),$$

we obtain

$$\left.\begin{aligned}\left(\nabla^2-\frac{1}{L^2}\right)\rho(\mathbf{r}) &= \frac{c_{fa}-c_{fa}f_{fa\to fa}}{1-c_{fa}f_{fa\to fa}}\frac{\phi(\mathbf{r})}{L^2}\\ \text{and}\qquad \left(\nabla^2-\frac{1}{L_{fa}^2}\right)\phi(\mathbf{r}) &= -\frac{f\eta}{L_{fa}^2}\rho(\mathbf{r})\end{aligned}\right\}. \tag{23.41}$$

Let us compare (23.41) with (23.31). Since the latter as it stands is valid only for an infinite medium, we shall make the comparison for this case. To facilitate comparison, we put (23.41) into a form as similar as possible to (23.31), by converting it into an integro-differential equation; we do this by solving the second of (23.41) for $\phi(\mathbf{r})$ in terms of $\rho(\mathbf{r})$, with the appropriate boundary conditions, and then substituting into the first of (23.41). The boundary condition in question for an infinite medium is that, if $\rho(\mathbf{r})$ is bounded at infinity, so is $\phi(\mathbf{r})$. Eliminating $\phi(\mathbf{r})$ in this way, we get

$$\left(\nabla^2-\frac{1}{L^2}\right)\rho(\mathbf{r}) = -\frac{c_{fa}-c_{fa}f_{fa\to fa}}{1-c_{fa}f_{fa\to fa}}f\eta\frac{1}{4\pi L^2L_{fa}^2}\iiint\rho(\mathbf{r}')e^{-|\mathbf{r}-\mathbf{r}'|/L_{fa}}\frac{dV'}{|\mathbf{r}-\mathbf{r}'|}. \tag{23.42}$$

Since the present treatment is much less accurate than that given in § 23.5.2, (23.31) may be regarded as exact. It is then seen that to combine all the fast neutrons into a single group is equivalent to approximating the Gaussian kernel by an exponential kernel. If such an approximation is a reasonable one, we should require, first of all, that the integrals of the two kernels over all space are equal, and this gives

$$\frac{c_{fa}-c_{fa}f_{fa\to fa}}{1-c_{fa}f_{fa\to fa}} = p, \tag{23.43}$$

where p is the capture escape probability. This condition could have been foreseen from the definitions of c_{fa} and $f_{fa\to fa}$. Secondly, both (23.31) and (23.42) should lead to the same solution for an infinite medium. For $f\eta p$ sufficiently near unity, this means that $L_{fa} = L_s$ or, by (23.40),

$$L_s^2 = \frac{l_{s,fa}\, l_{tot,fa}}{3(1-b_{fa})(1-c_{fa}f_{fa\to fa})}. \tag{23.44}$$

If the system concerned has geometrical dimensions large compared with L_s, the solution in it does not differ appreciably from that in an infinite medium. Thus, if (23.43) and (23.44) are satisfied, (23.42) should be a reasonable approximation to (23.31), even though the actual shape of the kernel is rather different.

The conditions (23.43) and (23.44) are easily found to have the following simple meaning. The constants used for the fast group, i.e. c_{fa}, $f_{fa\to fa}$, etc., should be such as to give the correct values of the zero and second spatial moments of the neutrons passing from the fast to the thermal group, for a fast point source in an infinite medium. At the beginning of Chapter XIX we said that these constants might be regarded as given experimentally. In actual fact, at least for the kind of medium considered in the present section, we should simply use (23.43) and (23.44) together with the known values of p and L_s^2.

If, instead of treating all the fast neutrons as a single group, we subdivide them into groups covering about equal intervals of lethargy, and then proceed as in the derivation of (23.42), i.e. start with the P_1 approximation and eliminate all the currents and fast fluxes, obtaining an integro-differential equation of the form

$$\left(\nabla^2 - \frac{1}{L^2}\right)\rho(\mathbf{r}) = \iiint K(|\mathbf{r}-\mathbf{r}'|)\frac{\rho(\mathbf{r}')}{L^2}\,dV' \tag{23.45}$$

for the thermal flux, the kernel $K(|\mathbf{r}-\mathbf{r}'|)$ of this equation will rapidly tend to that of (23.31) as the number of groups increases. This can be seen from the general statistical argument leading to the Gaussian kernel, and has also been confirmed by direct calculation. Conversely, if we start with the method of § 23.5.2 but, instead of attempting to solve (23.10) analytically, replace it by a difference-differential equation (difference in v and differential in space), we shall clearly return to ordinary multi-group theory.

This may be regarded as an alternative derivation of multi-group

theory, although it is restricted to large systems and the P_1 approximation. However, the derivation given in §§ 19.2.2 and 19.2.3 also supposed that large systems were concerned. In the alternative derivation, no appeal is made to the constancy of the cross-sections in each group. This constancy is largely irrelevant to the comparison between (23.45) and (23.31). Thus, except possibly for small systems or approximations higher than P_1, the constancy of the cross-sections in each group does not affect the accuracy attainable by multi-group theory so much as does the number and spacing of the groups in lethargy.

23.6. Improvements in age theory. Other methods for moderate distances

We now return to slowing-down problems and examine methods other than age theory for determining the slowing-down density at moderate distances from the source. There are two kinds of situation here, namely those where the assumptions underlying age theory are valid, but greater accuracy is desired, and those where age theory breaks down altogether. In the former case the procedure is evident, at least in principle. We start with the same equations (22.2″), and with the same estimate of the orders of magnitude involved, but retain terms up to the second or third order of smallness, instead of only the first. The resulting equations can easily be written down, but their analytic solution is hardly ever available, so that numerical methods must be used for their solution. Since the terms taken into account in ordinary age theory are the leading terms (provided that the assumptions underlying that theory remain valid), whereas the terms introduced in the higher approximations represent a small correction, the perturbation method also should prove fruitful. However, not much detailed work has yet been done in this direction. If numerical methods are adopted, it may be advantageous to improve the direct approach of § 23.4.3 rather than age theory itself. Just as the direct approach there outlined represents the P_1 approximation, so we can easily develop a direct approach using the P_3 or higher approximations. Again, not much detailed work has been done here. This is partly because of the laborious calculations involved in such improvements on age theory, and partly because the accuracy of the result given by age theory in the regions where it is applicable is adequate for most practical purposes.

A more important question is that of determining the solution for cases where age theory breaks down. As we have seen from the derivation of age theory, there are four such situations: (1) the distances at

which (23.17′) is violated; (2) the velocities at which (23.20′) is violated; (3) hydrogenous media; (4) too rapid variation of the cross-sections.

The first of these will be discussed in Chapters XXIV to XXVI. The most important of the other three is the case of hydrogenous media. A semi-empirical method of the kind mentioned in § 19.7 is the most successful in finding the distribution of neutrons just becoming thermal, but it does not give the spectrum for intermediate energies. For the latter purpose, the direct approach of § 23.4.3 seems to be promising, although it is still largely untried, having been suggested comparatively recently. Another possibility is the Monte Carlo method. The polynomial approximations method of § 20.4.3 has proved successful in determining the spectrum of neutrons reflected from a hydrogenous medium, though it probably cannot give high accuracy in the interior of a hydrogenous medium.

For hydrogenous media, unlike many non-hydrogenous ones, $L_s(v)$ is not much greater than $\max l_s(v')$ $(v < v' < v_0)$, and thus (23.17′) will be violated even at moderate distances from the source. We shall therefore discuss hydrogenous media again in Chapter XXV.

The case (4), i.e. where the cross-sections vary rapidly, can generally be dealt with by modified age theory (see § 23.2.4). If higher accuracy is needed, the perturbation method can be applied to the P_1 approximation form of equations (22.2″), i.e. (22.3).

The case (2), i.e. where the velocities are such that (23.20′) is violated, has not yet received much attention as regards non-hydrogenous media. This is primarily due to the lack of interest in this situation. Monochromatic sources are seldom found, and one is seldom interested in the slowing-down density within a few collision intervals of the source energy.

XXIV

SLOWED-DOWN NEUTRONS AT LARGE DISTANCES FROM THE SOURCE. THE CASE OF CONSTANT CROSS-SECTIONS

24.1. A qualitative discussion

We here conclude our discussion of the distribution of slowed-down neutrons at moderate distances from the source, and turn to the third and last type of slowing-down problem, that of determining the distribution at large distances from the source. By 'large' distances we mean those at which the condition (23.17) does not hold, and consequently age theory is no longer applicable. This condition was derived from the assumption that the neutron distribution is nearly isotropic for each $\mathbf{r}$ and v. For the distances which we now consider, this assumption is no longer valid; on the contrary, we should expect $N(\mathbf{r}, v\boldsymbol{\Omega})$ to be strongly anisotropic for a given $\mathbf{r}$ and v. The physical reason for this is as follows. The regions concerned are occupied by neutrons which have penetrated exceptionally far from the source in losing a given amount of energy. It follows from (2.16) that, the smaller the scattering angle, the less the energy loss in the collision. Thus a neutron is most likely to travel far without losing much energy if it is scattered through only a small angle in each collision; further, a neutron for which the small deflexions at each collision neutralize one another will travel farther from the source than one for which these small deflexions reinforce one another. Thus, for any given velocity, the neutrons moving directly away from the source predominate at sufficiently large distances, and this effect is the more pronounced the greater the distance from the source. The effect is most distinctly seen when the mean free paths for the various processes are independent of the energy, and as the distance from the source tends to infinity, we should expect the angular dependence for each v to tend to a delta function. If, however, the mean free path varies with the energy, another effect will come into play. If the mean free path has its largest value in the velocity interval (v, v_0) at v_1, say, then a neutron is most likely to travel far from the source in reaching velocity v if it is slowed down from v_0 to $v' \cong v_1$ near the source, travels with velocity v' to the neighbourhood of the point $\mathbf{r}$ concerned, and is then slowed down from v' to v near $\mathbf{r}$. This effect tends to diminish somewhat the predominance of the direc-

tions away from the source, as compared with the case of constant mean free path.

The anisotropy of the angular distribution and its dependence on the variation of the cross-sections with energy have no direct effect on the use of the Monte Carlo method; the main difficulty in this type of problem is that the number of neutrons found at large distances in a given velocity interval is very small in comparison with the initial output of the source. It is therefore imperative to use the most refined methods of sampling described in § 16.9. Otherwise, there is no appreciable difference from the application of Monte Carlo in other types of problem. Analytical methods, however, are noticeably complicated by the strong anisotropy of the angular distribution and its sensitivity to the law of variation of the cross-sections, and only a few comparatively simple cases have so far been investigated. In particular, only the case of an infinite homogeneous medium has so far been considered. The only types of energy dependence of the mean free path that have been discussed are those where the mean free path either is constant† or decreases monotonically with decreasing energy, and can be approximated by a simple analytical function of a certain type (see (25.1) and (25.3) below).

Thus the Monte Carlo method seems more effective than the analytical methods in determining the distribution of slowed-down neutrons at large distances from the source. However, since the Monte Carlo method has already been described in Chapter XVI, we shall now discuss only the analytical methods. In the present chapter we shall consider the case of constant cross-sections, and in the next chapter that of variable cross-sections.

Only the quantity $n(\mathbf{r}, v)$, rather than the angular distribution $N(\mathbf{r}, v\boldsymbol{\Omega})$, is usually of interest. Since for an infinite homogeneous medium $n(\mathbf{r}, v)$ for any source distribution can be expressed in terms of that for a plane source (see the footnote to (5.44)), we shall discuss only the case of a plane source.

The following order of presentation will be convenient. First, disregarding the above qualitative considerations, we shall construct a formal solution of the equations (22.2) for the case of a plane source and

† It might be supposed that the case of constant mean free path has been adequately examined, for the case of an infinite homogeneous medium, in Chapters V and XVII. However, we were there concerned only with the neutron flux, and no attention was paid to the variation of the energy spectrum with position, which is now our main concern. In Chapter V, moreover, the anisotropy of scattering in the L system was neglected altogether, whilst in Chapter XVII only a few leading terms were taken into account; here we have to take the anisotropy into account as accurately as possible.

constant cross-sections. This solution will not be immediately suitable for numerical work, but it will include most of the techniques used in the more efficient methods. Next, using the qualitative considerations given above, we shall examine the possible modifications in the formal solution which would improve its ability to deal with the main difficulty in these problems, i.e. the strong anisotropy of the neutron distribution at large distances from the source. Several alternatives corresponding to different situations will be noticed, and we shall then describe these in greater detail.

24.2. The formal solution

24.2.1. *The method of Fourier–Mellin transforms*

To obtain a formal solution of the equations (22.2) for the case of a plane source in an infinite homogeneous medium with constant cross-sections, we can proceed as follows. We take the Fourier transform of these equations with respect to the spatial coordinate x (measured perpendicular to the source plane) and their Mellin transform with respect to the velocity, putting

$$\phi_n(p,\eta) = \frac{1}{v_0^\eta}\int_0^\infty v^{\eta-1}\,dv \int_{-\infty}^{\infty} N_n(x,v)e^{ipx}\,dx, \tag{24.1}$$

where p and η are the Fourier and Mellin transform variables respectively, and obtain

$$ip[(n+1)\phi_{n+1}(p,\eta)+n\phi_{n-1}(p,\eta)]+\{(2n+1)/l\}[1-g_n(\eta)]\phi_n(p,\eta) = (S/v_0^2)\,\delta_{n0}, \tag{24.2}$$

where we have put for brevity

$$g_n(\eta) = \sum_k \frac{(M_k+1)^2}{2M_k}\frac{l}{l_{s\,k}}\int_{\frac{M_k-1}{M_k+1}}^{1} y^{\eta-1}P_n\left[\frac{(M_k+1)y^2-(M_k-1)}{2y}\right]dy, \tag{24.3}$$

and l is the total mean free path $\left(l^{-1} = l_c^{-1}+\sum_k l_{s,k}^{-1}\right)$. The way in which p and η appear in (24.2) could have been foreseen from what was said in §§ 4.3 and 5.3; they occur as parameters, i.e. (24.2) can be solved for each pair of values of p and η independently, unlike the situation we shall find in Chapter XXV.

The equations (24.2) are very similar to those found in Chapter X, and by proceeding as in the derivation of (10.22) we can show that the

solution of (24.2) for $\phi_0(p, \eta)$, for example, can be expressed in the form (given by Waller (51)):

$$\phi_0(p,\eta) = \frac{Sl/v_0^2}{1-g_0(\eta)+}\,\frac{p^2l^2}{3[1-g_1(\eta)]+}\,\frac{4p^2l^2}{5[1-g_2(\eta)]+}\,\frac{9p^2l^2}{7[1-g_3(\eta)]+}\cdots. \tag{24.4}$$

It is seen from (24.3) that, for any fixed η, $g_n(\eta)$ tends to zero as n tends to infinity. The region of convergence of the continued fraction forming the denominator of (24.4) is therefore the same as that of (10.22), except that ν is replaced by ipl. That is, (24.4) converges in the complex p-plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l$ and from $p = i/l$ to $p = i\infty$. The quantity $\phi_0(p, \eta)$ is thus determined in the cut p-plane, and the other $\phi_n(p, \eta)$ can then be readily determined by using the equations (24.2) one by one. Having determined the $\phi_n(p, \eta)$, we can find the corresponding $N_n(x, v)$ by inverting the double transform (24.1).

24.2.2. *The inversion of the Fourier transform*

A double numerical integration would be required in order to obtain $N_n(x, v)$ exactly. However, this can be avoided if only the leading term in $N_0(x, v)$ for large x is of interest. By taking the Mellin transform of the equations (22.2) with respect to the velocity, we reduce the problem to a one-velocity problem (see § 4.3), for which the equation (18.3) always has a solution.† In other words, for every η there is at least one value of p, which we denote by $i/L(\eta)$, such that the homogeneous system of equations corresponding to (24.2) has a solution, whilst the inhomogeneous system (24.2) itself has not. Thus, as p tends to $i/L(\eta)$, $\phi_0(p, \eta)$ should tend to infinity, and therefore has a pole at $p = i/L(\eta)$. Since $\phi_0(p, \eta)$ is an even function of p (see (24.4)), $p = -i/L(\eta)$ is another pole, so that the poles appear in pairs, as in Chapter XVII. To distinguish the poles forming a pair, we shall take $\mathrm{im}[i/L(\eta)] \geqslant 0$.

If η is real and positive, and the integral (24.1) converges, $|\phi_0(p, \eta)|$ will obviously be greater for p purely imaginary than for any complex p with the same imaginary part and non-zero real part. If η is real and positive, the poles nearest the real axis therefore lie on the imaginary p-axis, i.e. $L(\eta)$ is real and, by the above convention, positive. Since the poles concerned must lie in the cut plane, $L(\eta) > l$. If η is complex but its imaginary part is small, re η being positive, the position of the poles should be near their position for real positive η, i.e. the pair of poles

† The fact that (18.3) for the original problem has no solution simply means that the eigenvalue of (18.3) for the transformed problem is a function of η.

nearest the real p-axis will be the one tending to the pair described above for $\operatorname{im}\eta = 0$. It can also be shown that these two poles are necessarily simple poles.

The conclusions stated imply that $\phi_0(p, \eta)$ can be expressed as

$$\phi_0(p,\eta) = \frac{iY(\eta)}{p+i/L(\eta)} + \\ + \text{terms regular in the strip } 0 \geqslant \operatorname{im} p \geqslant -[\operatorname{re}\{1/L(\eta)\}+\epsilon'], \tag{24.5}$$

where ϵ' is any small positive quantity. Here the values of $L(\eta)$ and of the residue $iY(\eta)$ of $\phi_0(p, \eta)$ at the pole concerned will be determined from (24.4).

We now take the case $x > 0$, say, substitute (24.5) into the Fourier transform inversion formula and deform the path of integration as in § 5.3, obtaining

$$\frac{1}{v^{\eta}}\int_0^{\infty} v^{\eta-1}N_0(x,v)\,dv = e^{-x/L(\eta)}Y(\eta)+T(x,\eta), \tag{24.6}$$

where $T(x, \eta)$ represents the sum of terms decreasing faster than $e^{-x/L(\eta)}$ as $x \to \infty$, and is consequently negligible for large x. If x is not necessarily positive, then it is replaced by $|x|$ on the right side of (24.6).

24.2.3. *The inversion of the Mellin transform. The method of steepest descent*

Let us now consider the inversion of the Mellin transform. Retaining only the leading term for large x in (24.6), and substituting it into the Mellin transform inversion formula, we obtain

$$N_0(x,v) \cong \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \exp[\eta\log(v_0/v) - |x|/L(\eta)]Y(\eta)\,d\eta. \tag{24.7}$$

If $Y(\eta)$ and $L(\eta)/L_0$, where L_0 is some fixed length, vary with η at a comparable rate, and $|x| \gg L_0$, then the first factor in the integrand of (24.7) varies much more rapidly with η than the second factor. The first factor has a saddle-point at the root $\eta = \eta_0$ of the equation

$$\log(v_0/v) = |x|[d\{1/L(\eta)\}/d\eta]_{\eta=\eta_0}. \tag{24.8}$$

A more detailed examination of the continued fraction (24.4) shows that

$$d^2\{1/L(\eta)\}/d\eta^2 < 0 \quad \text{for } \eta \text{ real and positive,} \tag{24.9}$$

and the first factor in the of (24.7) therefore integrand behaves as follows

for η near η_0. For real η, the factor has a minimum for $\eta = \eta_0$, whilst if $\eta = \eta_0 + i\zeta$ where ζ is real, the factor has a maximum for $\zeta = 0$. Thus, if the path of integration in (24.7) is chosen to pass through $\eta = \eta_0$, i.e. we make the substitution $\eta = \eta_0 + i\zeta$, where ζ is real, the first factor in the integrand can be approximated for small ζ by a Gaussian, and the slowly varying second factor by its value at $\eta = \eta_0$, whilst, when ζ is not small, the first factor in the integrand becomes so small that the contribution of this region to the value of (24.7) can be neglected. Thus (24.7) can be approximated by

$$N_0(x,v) \cong Y(\eta_0)\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\tfrac{1}{2}\zeta^2|x|\left[\frac{d^2}{d\eta^2}\frac{1}{L(\eta)}\right]_{\eta=\eta_0}\right\}d\zeta\exp[\eta_0\log(v_0/v)-|x|/L(\eta_0)]$$

$$\cong \frac{Y(\eta_0)}{\sqrt{[-2\pi|x|d^2\{1/L(\eta)\}/d\eta^2]_{\eta=\eta_0}}}\exp[\eta_0\log(v_0/v)-|x|/L(\eta_0)], \qquad (24.10)$$

the quantity under the radical sign being positive, by (24.9). The method just described for reducing (24.7) to (24.10) is called the *saddle-point method* or the *method of steepest descent.*† The error involved in applying this method can be estimated as follows. After expanding $[\eta\log(v_0/v)-|x|/L(\eta)]$ in powers of $\eta-\eta_0$, we do not terminate the expansion at the quadratic term, but include some further terms; we then expand

$$\exp\left[-|x|\frac{(\eta-\eta_0)^3}{3!}\left\{\frac{d^3}{d\eta^3}\frac{1}{L(\eta)}\right\}_{\eta=\eta_0}+\ldots\right]$$

in powers of $\eta-\eta_0$, substitute in (24.7) and integrate term by term. The relative correction to (24.10) is thus found to be of the order

$$O\left(\frac{l}{|x|}\frac{\{d^3[l/L(\eta)]/d\eta^3\}^2_{\eta=\eta_0}}{\{d^2[l/L(\eta)]/d\eta^2\}^3_{\eta=\eta_0}}\right)+O\left(\frac{l}{|x|}\frac{\{d^4[l/L(\eta)]/d\eta^4\}_{\eta=\eta_0}}{\{d^2[l/L(\eta)]/d\eta^2\}^2_{\eta=\eta_0}}\right). \qquad (24.10')$$

The coefficients of $l/|x|$ in these expressions depend on η_0, which depends on x. However, using a result obtained in § 24.5 below, it can be shown that, as $|x|$ tends to infinity, these coefficients tend to finite limits, and then (24.10′) provides a rigorous justification of the method of steepest descent.

† This latter term is due to the fact that, with the contour chosen, the first factor in the integrand decreases more rapidly as we move away from the saddle-point than it would along any other path through the saddle-point. If $d^2[l/L(\eta)]/d\eta^2$ were complex for $\eta = \eta_0$, the path of steepest descent would no longer be parallel to the imaginary axis.

24.2.4. *Discussion. An alternative form of the results*

Formula (24.8) allows us to establish a qualitative relation between η_0 and x. For $L(\eta)$ is clearly the diffusion length in the transformed problem. If η is taken large, this means that, in arriving at the transformed problem, a high weight is given to neutrons with nearly the initial velocity. In the limit as $\eta \to \infty$, only the neutrons of the initial velocity will give any contribution. In that case, however, every scattered neutron will be treated as though it were captured, and the diffusion length therefore coincides with the mean free path (cf. (5.5) with $c = 0$). Thus

$$\lim_{\eta \to \infty} L(\eta) = l, \tag{24.11}$$

and it can easily be seen that the approach to this limit should be monotonic. Since, as $\eta \to \infty$, $l/L(\eta)$ tends monotonically to a finite limit, its derivative should tend to zero, and then the formula (24.8) implies that, as $|x|$ tends to infinity, so does η_0. A more detailed analysis on these lines shows that η_0 increases monotonically with $|x|$.

The fact that η_0 tends monotonically to infinity with $|x|$ could have been foreseen as follows. As we move away from the source, the neutron energy spectrum becomes more and more degraded. Consequently, if $|x|$ increases while η is fixed, the relative contribution of a given velocity interval dv decreases to zero as $|x| \to \infty$. For a given velocity interval it is therefore necessary to increase η as $|x|$ increases.

Since $d[l/L(\eta)]/d\eta$ is a monotonic function of η, (24.10) can be rewritten in a slightly more convenient form. Solving (24.8) for η_0 and substituting in (24.10), we can express the result in the form

$$N_0(x, v) = \frac{1}{\sqrt{[2\pi \log(v_0/v)]}} F\left(\frac{|x|}{l \log(v_0/v)}\right) \exp\left[-H\left(\frac{|x|}{l \log(v_0/v)}\right) \log \frac{v_0}{v}\right]. \tag{24.12}$$

The functions F and H in (24.12) are defined by

$$\left.\begin{aligned} \frac{l}{L(\eta)} \frac{d\eta}{d[l/L(\eta)]} - \eta &= H\left(\frac{d\eta}{d[l/L(\eta)]}\right) \\ \text{and} \qquad Y(\eta) \Big/ \sqrt{\left\{\frac{d}{d\eta} \log \frac{d\eta}{d[l/L(\eta)]}\right\}} &= F\left(\frac{d\eta}{d[l/L(\eta)]}\right) \end{aligned}\right\}, \tag{24.13}$$

as may easily be verified. To evaluate H and F, of course, we first evaluate $d\eta/d[l/L(\eta)]$ and the left sides of (24.13) as functions of η, and then eliminate η.

The solution derived above is rigorous, but we have referred to it as a formal solution, since the procedure given above to determine $L(\eta)$ is

rather unsuitable for numerical work. Since we are interested in large values of $|x|$, η_0 will be large also, by the conclusions derived from (24.11). For large η and n fairly small, $g_n(\eta)$ will be comparable with unity, as may be seen from (24.3). Thus, if (24.4) is to be used to determine $L(\eta)$, a large number of $g_n(\eta)$ will have to be evaluated, which is very troublesome except in the case of pure hydrogen.

24.3. Preliminary survey of the more effective analytical methods available

We now turn to the more effective analytical methods, i.e. those which aim at removing the difficulty just mentioned, and at putting the calculation in a more manageable form. The basic features, such as the use of Fourier and Mellin transforms, the method of residues and the saddle-point method, will, of course, be retained. We shall seek more effective methods of determining the various $\phi_n(p, \eta)$. In doing so, we shall bear in mind that it is the anisotropy which brings about the difficulty mentioned above.

It has been remarked in § 24.1 that, for a constant mean free path, the angular distribution of $N(x, v, \mu)$ should tend to $\delta(\mu-1)$ as x tends to infinity. Since $\delta(\mu-1) = \frac{1}{2}\sum(2n+1)P_n(\mu)$, this means that, as $x \to \infty$, the various coefficients $N_n(x, v)$ in the expansion

$$N(x, v, \mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) N_n(x, v) P_n(\mu) \tag{24.14}$$

tend to become equal, and the main contribution to (24.14) will come from very large values of n. The series (24.14) can therefore be approximated, for sufficiently large x, by the integral

$$N(x, v, \mu) = \frac{1}{2\pi} \int_0^{\infty} \nu N_\nu(x, v) P_\nu(\mu)\, d\nu, \tag{24.15}$$

where ν is a continuous variable which replaces the subscript n in (24.14). For such an angular distribution, moreover, the main contribution to $n(x, v)$ comes from the directions for which $\mu \cong 1$, i.e. from small values of the angle $\theta = \cos^{-1}\mu$ between the direction Ω of the neutron and the x-axis. For large ν and small θ, $P_\nu(\cos\theta)$ is approximately equal to $J_0(\nu\theta)$, where J_0 is the Bessel function (see (22), p. 99), so that (24.15) can be approximated by

$$N(x, v, \cos\theta) \cong \frac{1}{2\pi} \int_0^{\infty} N_\nu(x, v) J_0(\nu\theta) \nu\, d\nu. \tag{24.16}$$

Similarly, the quantity

$$\frac{n+1}{2n+1}N_{n+1}(x,\nu)+\frac{n}{2n+1}N_{n-1}(x,\nu) \tag{24.17}$$

can be approximated by

$$\left[1+\frac{1}{2}\left(\frac{d^2}{d\nu^2}+\frac{1}{\nu}\frac{d}{d\nu}\right)\right]N_\nu(x,\nu), \tag{24.18}$$

so that (22.2) and (24.2) become differential equations in ν. This is the basis of Wick's asymptotic solution, which will be described in § 24.5.

Apart from the above limiting case discussed by Wick, we shall have to deal with the situation where, although a large number of terms in the series (24.14) have to be taken into account, it cannot yet be approximated by an integral, and neither can (24.17) be approximated by (24.18). Several methods of dealing with this situation have been suggested. The simplest of these, for the case of constant cross-sections, is due to Bethe, who begins by observing that, as we have remarked in § 23.5.2, the angular distribution of neutrons emerging from a collision is rather less anisotropic than that of neutrons undergoing a collision. Proceeding as in § 2.5 or § 17.2, we can express the angular distribution of neutrons at any point in terms of the angular distribution of neutrons emerging from collisions throughout space. This transformation, however, amounts to passing from the Boltzmann integro-differential equation to the integral equation. If, therefore, we start from the integral equation instead of the corresponding Boltzmann equation, we have instead of (24.14) another series which converges rather better. The details of this procedure are given in § 24.4 below.

The ranges of values of $|x|$ considered by Bethe's and Wick's methods almost, but not quite, meet. The gap can be bridged by interpolation in suitable variables. A more systematic method of doing so has been developed by Holte; his ideas, however, are most naturally applied to problems where the cross-sections are variable. We shall therefore leave the discussion of Holte's method until Chapter XXVI.

24.4. Bethe's method

24.4.1. *General outline of the method*

In this section we shall discuss the method suggested by Bethe (1). As already mentioned, the essential feature of this method which distinguishes it from that given in § 24.2 is that the Boltzmann equation is first converted into an integral equation. To do so, it is convenient to

write the Boltzmann equation in the form

$$\mu\frac{\partial}{\partial x}N(x,v,\mu)+\frac{1}{l}N(x,v,\mu)=\sum_{n=0}^{\infty}(2n+1)P_n(\mu)\sum_k\frac{(M_k+1)^2}{8\pi M_k l_{s,k}}\int_v^{\frac{M_k+1}{M_k-1}v}\frac{dv'}{v'}\times$$

$$\times N_n(x,v')P_n\left[\frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'}\right]+\frac{S}{4\pi v_0}\delta(v-v_0)\,\delta(x). \quad (24.19)$$

This is, of course, an intermediate stage in the transformation of (22.1) into (22.2), for the case of a plane source and constant cross-sections. Taking the Fourier and Mellin transforms of (24.19) as in the derivation of (24.2), using the abbreviations (24.3) and multiplying by l, we obtain

$$(1-ipl\mu)\phi(p,\eta,\mu)=\frac{1}{4\pi}\sum_n(2n+1)g_n(\eta)\phi_n(p,\eta)P_n(\mu)+\frac{Sl}{4\pi v_0^2}, \quad (24.20)$$

where $\phi(p,\eta,\mu)$ is defined, similarly to (24.1), by

$$\phi(p,\eta,\mu)=\frac{1}{v_0^\eta}\int_0^\infty v^{\eta-1}\,dv\int_{-\infty}^{\infty}N(x,v,\mu)e^{ipx}\,dx, \quad (24.21)$$

so that

$$\phi_n(p,\eta)=\int\phi(p,\eta,\mu)P_n(\mu)\,d\mu. \quad (24.22)$$

Converting (24.19) into an integral equation and then taking the Fourier and Mellin transforms is clearly equivalent to starting with (24.20) and dividing by $1-ipl\mu$. On doing this, expanding the resulting equation in spherical harmonics, writing

$$A_{nn'}(ipl)=\frac{1}{4\pi}\int\int\frac{P_n(\mu)P_{n'}(\mu)}{1-ipl\mu}\,d\Omega=\frac{1}{2}\int_{-1}^{1}\frac{P_n(\mu)P_{n'}(\mu)}{1-ipl\mu}\,d\mu, \quad (24.23)$$

and using (24.22), we obtain

$$\phi_n(p,\eta)=\frac{Sl}{v_0^2}A_{n0}(ipl)+\sum_{n'=0}^{\infty}(2n'+1)g_{n'}(\eta)\phi_{n'}(p,\eta)A_{nn'}(ipl), \quad (24.24)$$

which replace the equations (24.2) of § 24.2.

We have seen that, to invert the double transform (24.1), we need to know the poles of $\phi_0(p,\eta)$, while the position of these poles is given by the values of p for which the homogeneous forms of (24.2), and therefore the homogeneous forms of (24.24), have a solution. Since the determinant

of the coefficients of the ϕ_n should then vanish, we obtain, denoting this value of p by $-i/L(\eta)$ as before, the following determinantal equation for $L(\eta)$:

$$\begin{vmatrix} g_0(\eta)A_{00}[l/L(\eta)]-1 & 3g_1(\eta)A_{01}[l/L(\eta)] & 5g_2(\eta)A_{02}[l/L(\eta)] \ldots \\ g_0(\eta)A_{01}[l/L(\eta)] & 3g_1(\eta)A_{11}[l/L(\eta)]-1 & 5g_2(\eta)A_{12}[l/L(\eta)] \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0. \quad (24.25)$$

In practice, of course, this infinite-order determinant will be replaced by the finite-order determinant formed by the first few rows and columns of (24.25). The convergence of these determinants for any given η, however, is much better, as we should expect, than that of the continued fraction (24.4).

24.4.2. *The evaluation of the residue*

In order to apply the formula (24.12), the residue $iY(\eta)$ of $\phi_0(p, \eta)$ at the pole determined above must also be evaluated. This is most simply done by eliminating all the $\phi_n(p, \eta)$ except $\phi_0(p, \eta)$ from the equations (24.24), thus obtaining an equation for $\phi_0(p, \eta)$, and determining the required residue from the latter equation. To eliminate $\phi_n(p, \eta)$ for $n \geqslant 1$, we notice that the free terms in (24.24) are proportional to the coefficients of $\phi_0(p, \eta)$ on the right sides of those equations. Thus, introducing new unknown functions

$$\omega_n(p, \eta) = \frac{\phi_n(p, \eta)}{g_0(\eta)\phi_0(p, \eta)+S/v_0^2}, \quad (24.26)$$

and dividing the equations (24.24) for $n \geqslant 1$ by $[g_0(\eta)\phi_0(p, \eta)+S/v_0^2]$, we obtain

$$\omega_n(p, \eta) = A_{n0}(ipl)+\sum_{n'=1}^{\infty}(2n'+1)g_{n'}(\eta)A_{nn'}(ipl)\omega_{n'}(p, \eta) \quad (n \geqslant 1), \quad (24.27)$$

which is a set of inhomogeneous equations to determine $\omega_n(p, \eta)$. Expressing $\phi_n(p, \eta)$ for $n \geqslant 1$ in terms of $\omega_n(p, \eta)$ in the first equation (24.24), we find

$$\phi_0(p, \eta) = [g_0(\eta)\phi_0(p, \eta)+Sl/v_0^2]\Big[A_{00}(ipl)+\sum_{n=1}^{\infty}(2n+1)g_n(\eta)A_{n0}(ipl)\omega_n(p, \eta)\Big],$$

and solving this for $\phi_0(p, \eta)$ we have

$$\phi_0(p,\eta) = -\frac{Sl}{v_0^2 g_0(\eta)} - \frac{Sl/v_0^2 g_0^2(\eta)}{A_{00}(ipl) - [1/g_0(\eta)] + \sum_{n=1}^{\infty}(2n+1) g_n(\eta) A_n(ipl)\omega_n(p,\eta)}. \quad (24.28)$$

On the other hand, $Y(\eta)$, which we require, was defined by

$$iY(\eta) = \lim_{p \to -i/L(\eta)} [\{p + i/L(\eta)\}\phi_0(p,\eta)]$$

(see (24.5)). Substituting from (24.28) and using de l'Hôpital's rule to evaluate the indefinite expressions of the form 0/0, we obtain

$$Y(\eta) = iSl \Big/ \left\{ v_0^2 g_0^2(\eta) \frac{d}{dp}\left[A_{00}(ipl) + \sum_{n=1}^{\infty}(2n+1) g_n(\eta) A_{0n}(ipl)\omega_n(p,\eta) \right] \right\}_{p=-i/L(\eta)}. \quad (24.29)$$

This formula might at first sight seem to necessitate the differentiation of quantities obtained numerically. This, however, is easily avoided. If we expand the $\omega_n(p, \eta)$ in powers of $[ipl - l/L(\eta)]$, putting

$$\omega_n(p,\eta) = \omega_{n0}(\eta) + [ipl - l/L(\eta)]\omega_{n1}(\eta) + O\{[ipl - l/L(\eta)]^2\}, \quad (24.30)$$

the equations (24.27) give at once

$$\omega_{n0}(\eta) = A_{n0}\{l/L(\eta)\} + \sum_{n'=1}^{\infty}(2n'+1) g_{n'}(\eta) A_{nn'}\{l/L(\eta)\}\omega_{n'0}(\eta) \quad (24.31)$$

and

$$\omega_{n1}(\eta) = \left[\frac{d}{d\nu}\left\{ A_{n0}(\nu) + \sum_{n'=1}^{\infty}(2n'+1) g_{n'}(\eta)\omega_{n'0}(\eta) A_{nn'}(\nu) \right\} \right]_{\nu = l/L(\eta)} + \\ + \sum_{n'=1}^{\infty}(2n'+1) g_{n'}(\eta) A_{nn'}\{l/L(\eta)\}\omega_{n'1}(\eta), \quad (24.32)$$

and substituting (24.30) in (24.29), we have

$$Y(\eta) = \frac{S}{v_0^2 g_0^2(\eta)} \left[\sum_{n=1}^{\infty}(2n+1) g_n(\eta) A_{0n}\{l/L(\eta)\}\omega_{n1}(\eta) + \right. \\ \left. + \frac{d}{d\nu}\left\{ A_{00}(\nu) + \sum_{n=1}^{\infty}(2n+1) g_n(\eta) A_{0n}(\nu)\omega_{n0}(\eta) \right\} \right]_{\nu = l/L(\eta)}^{-1}. \quad (24.33)$$

The actual calculation is then as follows. For each η, $l/L(\eta)$ is calculated from (24.25); then $\omega_{n0}(\eta)$ is found by solving (24.31), and the free term in (24.32) is determined. Then (24.32) can be solved, and $Y(\eta)$ determined

from (24.33). The functions H and F can then be evaluated from (24.13), and (24.12) can be applied.

24.4.3. *A single fairly heavy element*

The actual form of the functions H and F defined in (24.13) depends on the masses of the nuclei present, and on the relative probabilities of collision with each kind of nucleus. If only a single fairly heavy element is present, however, the evaluation of the functions (24.13) for each mass number separately can be replaced by the evaluation of certain functions independent of M. This can be done as follows. We first notice that η appears in $L(\eta)$ and $Y(\eta)$ only through the solutions of (24.25), (24.31), and (24.32), i.e. only as $g_n(\eta)$. Thus, if we put $\eta = K\bar{\eta}$, $L(\eta) = \bar{L}(\bar{\eta})$, and $Y(\eta) = \bar{Y}(\bar{\eta})$, and introduce new functions $\bar{H}$ and $\bar{F}$ defined by relations analogous to (24.13), the new functions will be related to the old ones by

$$K\bar{H}(w) = H(Kw) \quad \text{and} \quad \sqrt{K}\,\bar{F}(w) = F(Kw). \tag{24.34}$$

It can easily be verified that, for $M \gg 1$ and $\eta \ll M^2$, $g_n(\eta)$ depends, to a first approximation, only on η/M. For, transforming (24.3) in the same manner as in the derivation of (17.5), we obtain for the case of a single element

$$g_n(\eta) = \tfrac{1}{2}c \int_{-1}^{1} P_n(s) \frac{(M+1)^2}{M(M^2-1+s^2)^{\frac{1}{2}}} \left[\frac{(M^2-1+s^2)^{\frac{1}{2}}+s}{M+1}\right]^{\eta} ds, \tag{24.35}$$

where $c = l/l_s$ is the probability that a neutron is scattered and not captured. Neglecting quantities of the orders of η/M^2, $1/M^2$, and $(1-c)/M$, we can approximate the above expression by

$$g_n(\eta) \cong \left[1+\frac{2}{M}-(1-c)\right] \cdot \frac{1}{2} \int_{-1}^{1} P_n(s) \left[\frac{1+s/M}{1+1/M}\right]^{\eta} ds, \tag{24.36}$$

or by

$$g_n(\eta) \cong \left[1+\frac{2}{M}-(1-c)\right] \cdot \frac{1}{2} \int_{-1}^{1} P_n(s) e^{-(1-s)\eta/M}\, ds. \tag{24.37}$$

Also, since M appears only in $g_n(\eta)$, it follows from (24.37) that, if we take $K = 1/M$ in (24.34), $\bar{H}(w)$ and $\bar{F}(w)$ should be of the form

$$\left.\begin{aligned} \bar{H}(w) &= \bar{H}_0(w) - \left(\frac{2}{M}-1+c\right)\bar{H}_1(w) + \dots \\ \bar{F}(w) &= \bar{F}_0(w) - \left(\frac{2}{M}-1+c\right)\bar{F}_1(w) + \dots \end{aligned}\right\}, \tag{24.38}$$

where $\bar{H}_0$, $\bar{H}_1$, $\bar{F}_0$, and $\bar{F}_1$ are independent of M and c, and $\bar{H}_1$ is positive. In terms of these functions, the formula (24.12) can be rewritten

$$N_0(x,v) = \frac{1}{(2\pi\tilde{u})^{\frac{1}{2}}} \bar{F}_0\left(\frac{|x|}{l\tilde{u}}\right) \exp\left[-\tilde{u}\bar{H}_0\left(\frac{|x|}{l\tilde{u}}\right) + \left(\frac{2}{M} - 1 + c\right)\tilde{u}\bar{H}_1\left(\frac{|x|}{l\tilde{u}}\right)\right], \quad (24.39)$$

where

$$\tilde{u} = M \log(v_0/v), \quad (24.40)$$

and terms of the order of $[(2/M)-1+c]$ are neglected, except in the exponent. The functions $\bar{H}_0$, $\bar{H}_1$, and $\bar{F}_0$ have been tabulated by Bethe, Tonks, and Hurwitz (3) and by Price (42), whose values are given in Table IV.

TABLE IV

Values of the functions used in Bethe's method

w	$H_0(w)$	$H_1(w)$	$F_0(w)$
0	0	0	1·225
0·06748	0·003394	0·00126	1·210
0·14005	0·014346	0·00899	1·165
0·22415	0·035501	0·02177	1·095
0·32968	0·072671	0·04171	1·002
0·47460	0·13833	0·07217	0·892
0·69736	0·26176	0·12041	0·766
1·0995	0·52545	0·20294	0·625
2·0518	1·2478	0·3724	0·469
6·315	4·937	0·926	0·291

The successive entries correspond to equal intervals of $l/L(\eta)$.

24.4.4. *The connexion with age theory*

As was pointed out in connexion with (24.11), the less the distance from the source, the smaller is η_0. On the other hand (24.37) shows, on account of the orthogonality of the Legendre polynomials, that $g_n(\eta) = O[(\eta/M)^n]$ for small η/M. Thus, the smaller η_0/M, the sooner we can terminate the infinite determinant (24.25) and the infinite sums in (24.31)–(24.33). In particular, for sufficiently small x and large M, we can approximate (24.25) by

$$\left.\begin{aligned} g_0(\eta)A_{00}[l/L(\eta)] - 1 &\cong 0, \\ \text{while (24.33) can be approximated by} \quad Y(\eta) &\cong \frac{S}{v_0^2 g_0^2(\eta)}\left[\frac{d\nu}{dA_{00}(\nu)}\right]_{\nu=l/L(\eta)}. \end{aligned}\right\} \quad (24.41)$$

Using the explicit expressions for A_{00} and $g_0(\eta)$ from (24.23) and (24.37), performing the integration and expanding the resulting expressions in

powers of $l/L(\eta)$ and η/M (which are small in the present case), and retaining only the first few terms, we have

$$\left.\begin{aligned} A_{00}[l/L(\eta)] &= 1+\tfrac{1}{3}l^2/L^2(\eta) \\ g_0(\eta) &= \sum_k \frac{l}{l_{s,k}}\left(1+\frac{2-\eta}{M_k}\right) \end{aligned}\right\}, \tag{24.42}$$

where we now again consider a mixture of elements. The coefficient of η in the second of (24.42) is closely related to $\xi(v)$, the mean logarithmic energy loss per collision, defined by (23.3). If all the M_k are large, and the cross-sections are independent of the energy, the expression for $\xi(v)$ in the first approximation is

$$\xi = 2l_s \sum_k \frac{1}{M_k l_{s,k}}, \tag{24.43}$$

so that the second of (24.42) can be rewritten

$$g_0(\eta) = c[1+\tfrac{1}{2}\xi(2-\eta)].$$

Combining this with the first of (24.42) and the equations (24.41), and evaluating the corresponding expressions for H and F from the definitions (24.13), retaining only the first term, we obtain

$$\begin{aligned} H(w) &= \tfrac{3}{8}\xi w^2+2(1-c)/\xi-2, \\ F(w) &= (S/lv_0^2)(3/\xi)^{\frac{1}{2}}, \end{aligned}$$

where c is replaced by unity except in $(1-c)$.

With these expressions for H and F, the formula (24.12) becomes

$$N_0(x,v) = \frac{S}{\xi v^2}\frac{l}{L_s\sqrt{\pi}}e^{-(x^2/4L_s^2)-\{2(1-c)/\xi\}\log(v_0/v)}, \tag{24.44}$$

where we have used the fact that in the present case formulae (23.7) and (23.12) give

$$L_s^2 = \frac{2l^2}{3\xi}\log\frac{v_0}{v}. \tag{24.45}$$

From (23.8), (23.11), (23.13), and (23.5), we see that (24.44) coincides with the expression given by age theory for this case. That is, for small distances Bethe's theory reduces to age theory, as it should.

24.5. Wick's asymptotic solution for constant cross-sections

24.5.1. *The approximate form of the equation*

Bethe's method, described above, is fairly successful in dealing with the case of constant cross-sections, provided that $|x|/l\tilde{u}$, i.e. $|x|l/3L_s^2$, is small (see (24.45), together with (24.40) and (24.43)). As this quantity increases, however, the values of η/M_k increase also (see the discussion

following (24.11)), and more terms have to be retained in the infinite determinant (24.25). For $|x|l/3L_s^2 = 6{\cdot}315$ (the last value given in Table IV), the calculation is quite laborious, and for somewhat higher values of $|x|$ the method is quite impracticable. We shall therefore now consider a different method, designed to give the asymptotic solution for $|x| \to \infty$. This method is due to Wick (**54**); a preliminary account was given by Marshak (38, p. 231). For the sake of simplicity, we shall describe it for the case of a single element with no capture. As already mentioned in § 24.3, the basic idea of Wick's method consists in approximating the sum (24.14) by the integral (24.15), or rather by (24.16), while (24.17) is replaced by (24.18). This latter procedure means that (24.2) are approximated by differential equations in the parameter ν introduced in (24.15). To justify these approximations as rigorously as possible we proceed as follows.

Let us take the Fourier–Mellin transform of equation (22.1) instead of (22.2). Putting

$$\phi(p, \eta, \cos\theta) = \bar{\phi}(p, \eta, \theta), \tag{24.46}$$

where $\theta = \cos^{-1}\mu$ and ϕ is defined by (24.21), we then have

$$(1 - ipl\cos\theta)\bar{\phi}(p, \eta, \theta) = \int\int \bar{\phi}(p, \eta, \theta')g(\Theta, \eta)\, d\Omega' + Sl/4\pi v_0^2, \tag{24.47}$$

where $\Theta = \cos^{-1}(\Omega \cdot \Omega')$, and

$$g(\Theta, \eta) = \frac{(M+1)^2}{4\pi M} \int_{\frac{M-1}{M+1}}^{1} \delta\left[\cos\Theta - \frac{(M+1)y^2-(M-1)}{2y}\right] y^{\eta-1}\, dy$$

$$= \frac{(M+1)^2}{4\pi M(M^2-\sin^2\Theta)^{\frac{1}{2}}} \left[\frac{(M^2-\sin^2\Theta)^{\frac{1}{2}}+\cos\Theta}{M+1}\right]^{\eta}, \tag{24.48}$$

as can easily be seen by using the substitution

$$[(M^2-1+t^2)^{\frac{1}{2}}+t]/(M+1) = y.$$

In order to put the integral equation (24.47) in a more manageable form, we first examine more closely (24.48). We already know, from the discussion following (24.11), that, as $|x|$ tends to infinity, η_0 (i.e. the value of η which interests us) also tends to infinity. The quantity in (24.48) which is to be raised to the power η is unity for $\Theta = 0$, less than unity for $\Theta \neq 0$, and regular near $\Theta = 0$. It is known that, under such conditions, the function can always be approximated by a Gaussian. If we put

$$g(\Theta, \eta) = Ae^{-B\Theta^2}[1 + C_1\Theta^4 + C_2\Theta^6 + \ldots], \tag{24.49}$$

therefore, the quantities $C_1\Theta^4$, $C_2\Theta^6$, etc., can be regarded as perturbations of increasing orders of smallness.

On evaluating the coefficients in (24.49), we obtain

$$\left.\begin{aligned} A &= \frac{1}{4\pi}\left(1+\frac{1}{M}\right)^2, \\ B &= \frac{1}{2M}\left(\eta-\frac{1}{M}\right), \\ C_1 &= \frac{1}{24M}\left[\left(1-\frac{3}{M^2}\right)\eta-\left(2-\frac{3}{M^2}\right)\frac{2}{M}\right], \text{ etc.} \end{aligned}\right\} \tag{24.50}$$

We now look for further possible simplifications in equation (24.47). It has already been mentioned that, as $x \to \infty$, the angular dependence of $N(x, \nu, \mu)$ should tend to a delta function. For $\eta \to \infty$, the angular dependence of $\bar{\phi}(p, \eta, \theta)$ should therefore also tend to a delta function, a conclusion which could have been more directly reached from the arguments leading to (24.11). Thus the quantities appearing in (24.47) need be accurately treated only near $\theta = 0$. We can therefore put

$$\left.\begin{aligned} \cos\theta &\cong 1-\tfrac{1}{2}\theta^2, \\ \Theta^2 &\cong \theta^2+\theta'^2-2\theta\theta'\cos(\phi-\phi'), \\ d\Omega' &\cong \theta'\,d\theta'\,d\phi' \end{aligned}\right\} \tag{24.51}$$

(where ϕ and ϕ' are the azimuthal angles of Ω and Ω'), treating the further terms not given on the right side as small perturbations.

We have seen in § 24.2 that the value of p is needed only in the vicinity of $-i/L(\eta)$, and, according to (24.11), $L(\eta)$ tends to l as η tends to infinity. Thus ipl will be very close to unity, and can be replaced by unity except in the difference $1-ipl$. Combining this with the first of (24.51), we have

$$1-ipl\cos\theta \cong (1-ipl)+\tfrac{1}{2}\theta^2. \tag{24.52}$$

Since the values of $1-ipl$ concerned are themselves small, the second term in the expansion of $\cos\theta$ has to be taken into account, as well as the term equal to unity.

Substituting (24.52), the last two of (24.51), and the leading term of (24.49) into (24.47), we obtain, as a first approximation to the latter, the equation

$$\begin{aligned}(1-ipl)\bar{\phi}^0(p,\eta,\theta) = &-\tfrac{1}{2}\theta^2\bar{\phi}^0(p,\eta,\theta)+Sl/4\pi\nu_0^2+ \\ &+\frac{1}{4\pi}\left(1+\frac{1}{M}\right)^2\int\int e^{-(1/2M)(\eta-1/M)(\theta^2+\theta'^2-2\theta\theta'\cos(\phi-\phi'))}\bar{\phi}^0(p,\eta,\theta')\theta'\,d\theta'\,d\phi',\end{aligned} \tag{24.53}$$

where $\bar{\phi}^0(p, \eta, \theta)$ is the first approximation to $\bar{\phi}(p, \eta, \theta)$, the 0 being placed as a superscript to avoid any possible confusion with the spherical

harmonic moments; the difference $[\bar{\phi}(p,\eta,\theta)-\bar{\phi}^0(p,\eta,\theta)]$ can be obtained by means of the perturbation method. The integration over θ' in (24.53) can be extended to infinity.

24.5.2. *The application of Bessel transforms*

The approximation of $d\Omega'$ by $\theta' d\theta' d\phi'$ and the extension of the integration over θ' to infinity is equivalent to replacing the sphere $|\Omega| = 1$, i.e. the locus of the end-point of the unit vector Ω, by a plane; here θ plays the part of the radial coordinate. Thus $\bar{y} = \theta\cos\phi$ and $\bar{z} = \theta\sin\phi$ may be regarded as the Cartesian coordinates in this plane. Taking the two-dimensional Fourier transform with respect to $\bar{y}$ and $\bar{z}$, we can easily convert (24.53) into a differential equation. Let ν_y and ν_z be the corresponding transform parameters, and $\nu^2 = \nu_y^2+\nu_z^2$; since $\bar{\phi}^0(p,\eta,\theta)$ depends only on the radial coordinate and not on $\bar{y}$ and $\bar{z}$ separately, its transform depends only on ν and not on ν_y and ν_z separately. Thus we can put

$$\iint \bar{\phi}^0(p,\eta,\sqrt{(\bar{y}^2+\bar{z}^2)})e^{i(\bar{y}\nu_y+\bar{z}\nu_z)}\,d\bar{y}\,d\bar{z} = \Phi^0(p,\eta,\nu), \qquad (24.54)$$

while the application of the two-dimensional Fourier transform to (24.53) then gives

$$(1-ipl)\Phi^0(p,\eta,\nu) = \left[\frac{1}{2}\left\{\frac{d^2}{d\nu^2}+\frac{1}{\nu}\frac{d}{d\nu}\right\}+\frac{(M+1)^2}{2(M\eta-1)}e^{-M^2\nu^2/2(M\eta-1)}\right]\Phi^0(p,\eta,\nu)+$$
$$+Sl\delta(\nu)/2\nu\nu_0^2, \qquad (24.55)$$

where we have used the relations

$$\int_{-\infty}^{\infty} e^{i\bar{y}\nu_y}\,dy = 2\pi\delta(\nu_y) \quad \text{and} \quad \delta(\nu_y)\delta(\nu_z) = \delta(\nu)/2\pi\nu,$$

and the fact that $(\partial^2/\partial\nu_y^2)+(\partial^2/\partial\nu_z^2)$, operating on a function of ν only, is equal to $(d^2/d\nu^2)+(1/\nu)(d/d\nu)$.

Using the integral representation of the Bessel functions, we can easily see that the definition (24.54) is the same as

$$\Phi^0(p,\eta,\nu) = 2\pi\int_0^{\infty}\bar{\phi}^0(p,\eta,\theta)J_0(\nu\theta)\theta\,d\theta. \qquad (24.56)$$

The application of the inversion theorem to (24.56) gives

$$\bar{\phi}^0(p,\eta,\theta) = \frac{1}{2\pi}\int_0^{\infty}\Phi^0(p,\eta,\nu)J_0(\nu\theta)\nu\,d\nu,$$

which is, of course, equivalent to (24.16). Similarly, the appearance of the operator $\frac{1}{2}(d^2/d\nu^2)+(1/2\nu)(d/d\nu)$ was anticipated in (24.18).

It is usually preferable to change the scale of ν by putting

$$s = M\nu[2/(M\eta-1)]^{\frac{1}{2}} = \nu/\nu_0. \tag{24.57}$$

The equation (24.55) then becomes

$$\frac{(M\eta-1)(1-ipl)}{M^2}\Phi^0(p,\eta,\nu_0 s)$$

$$= \left\{\frac{d^2}{ds^2}+\frac{1}{s}\frac{d}{ds}+\frac{1}{2}\left(1+\frac{1}{M}\right)^2 e^{-\frac{1}{2}s^2}\right\}\Phi^0(p,\eta,\nu_0 s)+Sl\delta(s)/s\nu_0^2. \tag{24.58}$$

Since $\bar{\phi}^0(p,\eta,\theta)$ remains finite for all θ, its transform $\Phi^0(p,\eta,\nu_0 s)$ should tend to zero as s tends to infinity. This provides the boundary condition on the solution of (24.58).

24.5.3. *The solution of the transformed equation: the homogeneous case*

We now consider the solution of equation (24.58). We have already seen in connexion with (24.7) that this solution is not needed for all values of the parameter p, but only near the value of p where $\Phi^0(p,\eta,\nu_0 s)$ has a pole. That is, only the position of the pole and the corresponding residue need be determined. We shall first consider the determination of the position of the pole. This is given by the value of p, say $-i/L^0(\eta)$, for which the corresponding homogeneous equation has a solution (see the arguments of § 24.2.2). We use $L^0(\eta)$ instead of $L(\eta)$ in order to distinguish the eigenvalue of the approximate equation (24.53) from that of the exact equation (24.47). By the solution of the homogeneous equation we mean, of course, the solution regular at the origin, since a solution which is singular there does not satisfy the homogeneous equation at $s = 0$. Since the solution of (24.58) must vanish at infinity, so must that of the corresponding homogeneous equation. This gives two boundary conditions which define the eigenvalue problem.

The homogeneous form of (24.58),

$$\frac{(M\eta-1)(1-ipl)}{M^2}\Phi^0(p,\eta,\nu_0 s)$$

$$= \left\{\frac{d^2}{ds^2}+\frac{1}{s}\frac{d}{ds}+\frac{1}{2}\left(1+\frac{1}{M}\right)^2 e^{-\frac{1}{2}s^2}\right\}\Phi^0(p,\eta,\nu_0 s), \tag{24.58'}$$

is of the same form as Schrödinger's equation for a particle in a two-dimensional Gaussian potential field, and the boundary conditions are the same. No simple analytical solution of this equation is available, but

the required eigenvalue can be obtained by means of the variational method. It can be verified in the usual manner (§ 15.1) that

$$\frac{(M\eta-1)\{1-l/L^0(\eta)\}}{M^2} = \max_U \frac{\int_0^\infty \{\tfrac{1}{2}(1+1/M)^2 e^{-\frac{1}{2}s^2} U^2(s) - (dU/ds)^2\} s\, ds}{\int_0^\infty U^2(s) s\, ds}, \tag{24.59}$$

where $U(s)$ is the trial function. Wick suggests the use of the trial function

$$U(s) = e^{-\frac{1}{2}as^2}, \tag{24.60}$$

where a is the variable parameter. The integrals in (24.59) are then elementary, owing to the appearance of the factor s in their integrands. The same will be true if the more general trial function

$$U(s) = \sum_j A_j e^{-\frac{1}{2}a_j s^2}$$

is used. Once the eigenvalue of (24.58′) is known, we can obtain the corresponding eigenfunction by numerical integration of that equation, and the difference between $L(\eta)$ and $L^0(\eta)$ can then be determined to various orders of accuracy by means of ordinary perturbation methods, as follows. By retaining more terms in the expansions (24.49), (24.51), and (24.52), and transforming them as before, we obtain the equation (24.58′) with a correction term, and there will also be a correction to the eigenvalue. We represent the solution of (24.58′) with the correction term as $\Phi^0+\Phi^1+\ldots$, where Φ^0 is the function already known. This gives an inhomogeneous equation for Φ^1, which has a solution if its free term is orthogonal to Φ^0. The first correction to the eigenvalue is thus determined, and so on. In this manner we find

$$\frac{l}{L(\eta)} = 1 - \frac{M^2}{M\eta-1} k_1 - \frac{M^4}{(M\eta-1)^2} k_2 - \ldots, \tag{24.61}$$

where $k_1(M)$ is the maximum of the functional (24.59), while $k_2(M)$, $k_3(M)$, etc., are determined by the perturbation method.

The series (24.61) is actually needed only to determine the functions H and F defined in (24.13), and for that purpose it is more convenient to express η in terms of $l/L(\eta)$. Solving (24.61) for η we have

$$\eta = \frac{Mk_1}{1-l/L(\eta)} + \eta^* + [1-l/L(\eta)]\eta^{**} + \ldots, \tag{24.61′}$$

where η^* can readily be expressed in terms of M, k_1, and k_2, η^{**} in terms of M, k_1, k_2, and k_3, etc. In practice, η^*, η^{**}, etc., would be worked out

directly, using an obvious modification of the perturbation method described above. As x tends to infinity, i.e. for the region with which this method is intended to deal, the terms involving η^{**}, η^{***}, etc., are negligibly small. These terms are therefore usually disregarded, and only the first-order perturbation is considered.

24.5.4. *The evaluation of the residue. Final results*

In order to apply formula (24.12), we must also determine the residue of $\bar{\phi}(p, \eta, \theta)$ at its pole $p = -i/L(\eta)$. Since this residue gives rise only to a slowly varying factor in $N(x, v, \cos\theta)$, it is sufficient to determine it in the zero-order approximation, i.e. to determine the residue of $\bar{\phi}^0(p, \eta, \theta)$. Moreover, only the value of $N_0(x, v)$ is usually of interest, and not the complete angular distribution $N(x, v, \cos\theta)$; consequently, we need only the residue of $\iint \bar{\phi}^0(p, \eta, \theta)\, d\Omega$, and not that of $\bar{\phi}^0(p, \eta, \theta)$ for all θ. In dealing with integrals over Ω, we can and should approximate $d\Omega$ by $\theta\, d\theta d\phi$, so that we have by (24.56)

$$\iint \bar{\phi}^0(p, \eta, \theta)\, d\Omega = \Phi^0(p, \eta, 0). \tag{24.62}$$

Thus only the residue of the latter quantity is needed. However, it is more convenient to evaluate the residue of $\Phi^0(p, \eta, \nu_0 s)$ for general s. If we call this residue $iY(\eta, s)$ in conformity with the notation of (24.5), it is

$$iY(\eta, s) = \lim_{p \to -i/L^0(\eta)} \{[p + i/L^0(\eta)]\Phi^0(p, \eta, \nu_0 s)\}. \tag{24.63}$$

The quantity in the braces in (24.63) is the solution of the equation obtained from (24.58) by multiplying the free term by $\{p + i/L^0(\eta)\}$. Thus $iY(\eta, s)$ is the solution of (24.58′), i.e. of the homogeneous form of (24.58). The solution of the latter, however, is determined to within an arbitrary factor, and one solution of (24.58′) is already known from the calculations of § 24.5.3; we call it $U(s)$, the same notation as for the trial function in (24.59), since the ideal trial function is simply the solution of the equation in question. Thus we have

$$iY(\eta, s) = CU(s), \tag{24.64}$$

where C is a normalization factor, and we have only to determine this factor. To do so, we proceed as follows. We multiply (24.58) by $sU(s)$, subtract (24.58′) multiplied by $s\Phi^0$, and integrate. The quantity

$$\int \left\{\Phi^0 \cdot \left[\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right] U(s) - U(s)\left[\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right]\Phi^0\right\} s\, ds$$

vanishes by Gauss's theorem, since Φ^0 and $U(s)$ vanish at infinity, whilst

the singularity of Φ^0 at $s = 0$ is taken into account by the free term $Sl\delta(s)/s\,v_0^2$ in (24.58). There remains

$$-\frac{il(M\eta-1)}{M^2}\int[p+i/L^0(\eta)]\Phi^0(p,\eta,\nu_0 s)U(s)s\,ds = \frac{Sl}{v_0^2}\int\frac{\delta(s)}{s}U(s)s\,ds.$$

Taking the limit as $p \to -i/L^0(\eta)$, and using (24.63) and (24.64), we find the equation for C, while, substituting this in (24.64) and putting $s = 0$, we have

$$Y(\eta,0) = SMG/v_0^2\,\eta, \tag{24.65}$$

where

$$G = U^2(0)\Big/\int_0^\infty U^2(s)s\,ds \tag{24.66}$$

and $\eta - 1/M$ is replaced by η; this is permissible to the same accuracy as replacing the residue of $\bar{\phi}$ by that of $\bar{\phi}^0$.

Substituting (24.61′) and (24.65) in the definitions (24.13), we obtain

$$\left.\begin{aligned} H(w) &= w - 2(Mk_1w)^{\frac{1}{2}} - \eta^* + O(1/\sqrt{w}) \\ \text{and}\quad F(w) &= \frac{SMG}{v_0^2(4Mk_1w)^{\frac{1}{2}}}[1+O(1/\sqrt{w})] \end{aligned}\right\}, \tag{24.67}$$

and hence (24.12) gives†

$$N_0(x,v) = \frac{SMG}{2\sqrt{\pi}\,v_0^2}\left[\frac{l}{Mk_1|x|\log(v_0/v)}\right]^{\frac{1}{4}}\exp\left[-\frac{|x|}{l}+2\left\{\frac{Mk_1|x|}{l}\log\frac{v_0}{v}\right\}^{\frac{1}{2}}+\eta^*\log\frac{v_0}{v}\right]. \tag{24.68}$$

On examining the order of magnitude of the terms neglected, we see that formula (24.68) as it stands is valid for

$$|x| \gg l[M\log(v_0/v)]^3. \tag{24.69}$$

If the second perturbation were taken into account in (24.61′), and the first perturbation in the evaluation of the residue, a formula valid for $|x| \gg l[M\log(v_0/v)]^2$ would be obtained; taking account of the next order would give a formula valid for $x \gg l[M\log(v_0/v)]^{\frac{1}{2}}$, and so on. However, these higher perturbations are extremely laborious to calculate, and we shall give in Chapter XXVI a better method of bridging the gap between the range of applicability of (24.68) and that of Bethe's method.

In the case of hydrogen ($M = 1$), however, as Wick points out, the second perturbation term is quite small, and in this case formula (24.68) is applicable over a wider range of values of x than that given by (24.69).

† There are certain misprints in this formula as quoted by Marshak (38, p. 234).

XXV

SLOWED-DOWN NEUTRONS AT LARGE DISTANCES FROM THE SOURCE. THE CASE OF VARIABLE CROSS-SECTIONS

25.1. Preliminary survey

We shall now consider the extension of the analysis of the preceding chapter to the case where the cross-sections depend on the velocity. In this case, taking the Mellin transform of equations (22.2) with respect to the velocity does not reduce the number of independent variables, since the Mellin transform variable η now appears as an independent variable, and not as a parameter. In general, the transformed equations will be integral equations in η. However, if the velocity dependence of the cross-sections concerned is given by

$$\frac{1}{l_c(v)} = \frac{1}{l(v_0)} \sum_j A_{c,j} \left(\frac{v}{v_0}\right)^{\alpha_j}, \qquad \frac{1}{l_{s,k}(v)} = \frac{1}{l(v_0)} \sum_j A_{s,k,j} \left(\frac{v}{v_0}\right)^{\alpha_j}, \quad (25.1)$$

where α_j, $A_{c,j}$, and $A_{s,k,j}$ are some constants, $l(v_0)$ is the total mean free path at the initial velocity, and the summation over j involves only a finite number of terms, the integral equation in η reduces to a difference equation. As in the derivation of (20.10), we can verify that, with $l_c(v)$ and $l_{s,k}(v)$ given by (25.1), the equations (24.2) are replaced by

$$ip[(n+1)\phi_{n+1}(p,\eta)+n\phi_{n-1}(p,\eta)]+\frac{2n+1}{l(v_0)}\sum_j A_{c,j}\phi_n(p,\eta+\alpha_j)+$$
$$+\frac{2n+1}{l(v_0)}\sum_k\sum_j A_{s,k,j}[1-g_{n,M_k}(\eta)]\phi_n(p,\eta+\alpha_j)=\frac{S}{v_0^2}\delta_{n0}, \quad (25.2)$$

where the definition of $g_{n,M_k}(\eta)$ is evident from (24.3) (since (24.3) can be rewritten as $g_n(\eta) = \sum_k g_{n,M_k}(\eta) l/l_{s,k}$), while the $\phi_n(p,\eta)$ are defined by the same formula (24.1) as before. If the summations over j and k involve only a small number of terms, there is some hope of success in attempting to solve (25.2).

Alternatively, if we have a single non-capturing element, and the variation of the mean free path with energy is given by

$$l(v) = l(v_0)\sum_j B_j(v/v_0)^{\beta_j}, \quad (25.3)$$

where β_j and B_j are some constants, the Mellin transform equations can

again be put into the form of difference equations; the $\phi_n(p, \eta)$ are now replaced by

$$\pi_n(p, \eta) = \frac{1}{v_0^{\eta}} \int_0^{v_0} v^{\eta-1}\, dv \int_{-\infty}^{\infty} \frac{N_n(x, v)}{l(v)} e^{ipx}\, dx, \tag{25.4}$$

and the Mellin transform equations are

$$ipl(v_0) \sum_j B_j[(n+1)\pi_{n+1}(p, \eta+\beta_j)+n\pi_{n-1}(p, \eta+\beta_j)]+ \\ +(2n+1)[1-g_n(\eta)]\pi_n(p, \eta) = S\delta_{n0}/v_0^2. \tag{25.5}$$

The equations (25.5), with $g_n(\eta)$ defined by (24.3), will of course apply to mixtures, with or without capture, provided that all the cross-sections follow the same law. The majority of investigations have therefore assumed that the cross-sections are given either by (25.1) or by (25.3).

Another important limitation which has always been imposed hitherto is that the total mean free path should decrease monotonically with the velocity, and that the value of its derivative at the initial velocity (i.e. $dl(v)/dv$ for $v = v_0$) should not be zero. When these conditions are satisfied, the form of the solution at large distances will differ markedly from that in the case of constant cross-sections. This may be seen as follows. As was said in § 24.1, the best chance for a neutron to travel a very large distance from the source is for it to travel the greater part of the distance with a velocity where the total mean free path is near its maximum value. In the present case, this means that the neutron would travel the greater part of the distance with its initial velocity. In the constant cross-section case there was no such restriction, and therefore the chance of reaching very large distances for a given v and $l(v_0)$ was much greater, and more so as the distances involved increased. In other words, in the present case, $N_0(x, v)$ decreases much faster with increasing distance than it did in the case of constant cross-sections. The number of non-scattered neutrons decreases faster still, so that the ratio

$$N_0(x, v)/N_0(x, v_0) \tag{25.6}$$

actually increases with distance, but very much more slowly than in the case of constant cross-sections.

For any given v_0 and $l(v_0)$, the greater $dl(v)/dv$ for $v = v_0$, the narrower is the range of velocities within which the neutrons have a fair chance of reaching very large distances, and therefore the fewer neutrons penetrate to such distances, and the slower the increase of (25.6). On the other hand, since a neutron has to travel most of the distance with its initial velocity, the variation of $l(v)$ for $v < v_0$ ought to have little effect on the

rate of variation of (25.6) with x for a given v. The same argument shows that the rate of variation of (25.6) ought to be independent of v for large x. At very large distances, therefore, $N_0(x, v)$ should be of the form

$$N_0(x, v) \cong n_0(x) f(v), \tag{25.7}$$

where, for a single non-capturing element, the form of $n_0(x)$ depends only on v_0, $l(v_0)$, $dl(v)/dv$ for $v = v_0$, and M. If we introduce the dimensionless quantity

$$\gamma = \frac{v_0}{l(v_0)} \left[\frac{dl(v)}{dv}\right]_{v=v_0}, \tag{25.8}$$

the form of $n_0(x)$, as a function of $x/l(v_0)$, can depend only on γ and M. Of course, if there is capture or more than one element is present, the form of $n_0(x)$ depends on the ratios $l(v_0)/l_c(v_0)$ and $l(v_0)/l_{s,k}(v_0)$, and on the parameters

$$\gamma_c = \frac{v_0}{l(v_0)} \left[\frac{dl_c(v)}{dv}\right]_{v=v_0} \quad \text{and} \quad \gamma_{s,k} = \frac{v_0}{l(v_0)} \left[\frac{dl_{s,k}(v)}{dv}\right]_{v=v_0}$$

instead of γ only.

Some analytical consequences of the approximate relation (25.7) will be discussed in § 25.2. The actual determination of $n_0(x)$ will be outlined in § 25.4, although we shall take only the case of a single non-capturing element.

The behaviour of $f(v)$, the second factor in (25.7), near $v = v_0$ can be assessed at the same time as $n_0(x)$ is determined. For lower values of v, the form of $f(v)$ depends on the actual variation of $l_c(v)$ and of $l_{s,k}(v)$. The determination of $f(v)$ in the latter region has not yet been much investigated. Although the approximation (25.7) becomes better as x increases, it becomes worse as v decreases. This follows since the smaller v the sooner a neutron can leave its initial energy and yet have a fair chance of reaching x with the velocity v. The arguments leading to (25.7) are therefore invalid over a greater fraction of the path, and for fairly small v (25.7) breaks down altogether. We shall discuss the latter region in § 25.3 and Chapter XXVI, while in § 25.5 we shall consider some consequences of (25.7) for hydrogenous media.

25.2. Singularities of the Fourier–Mellin transform of the density when the cross-section varies monotonically

We have said above that, if the total mean free path decreases monotonically with energy and $\gamma \neq 0$, $N_0(x, v)$ can be approximated in the form (25.7) for fairly large distances and not very small velocities. Then, if $\mathrm{re}\,\eta$ is fairly large ($> \sigma_\eta$, say), so that the high-velocity contribution

predominates, and if $\operatorname{im} p$ is fairly near to $\pm 1/l(v_0)$ ($\sigma_p < |\operatorname{im} p| \leqslant 1/l(v_0)$, say), so that the large-distance contribution predominates, the Fourier–Mellin transform $\phi_0(p, \eta)$ of $N_0(x, v)$ should be approximately given by

$$\phi_0(p, \eta) \cong \phi_0(p) F(\eta) \quad [\operatorname{re} \eta > \sigma_\eta, \quad \sigma_p < |\operatorname{im} p| \leqslant 1/l(v_0)], \quad (25.9)$$

where $\phi_0(p)$ is the Fourier transform of $n_0(x)$, and $F(\eta)$ is the Mellin transform of $f(v)$. The formula (25.9) implies that, for $\operatorname{re} \eta > \sigma_\eta$, the position of the singularities of $\phi_0(p, \eta)$ in the two strips $\sigma_p < |\operatorname{im} p| \leqslant 1/l(v_0)$ of the complex p-plane should be independent of η. Since σ_p is defined in such a way that† $\phi_0(p, \eta)$ is regular in the strip $-\sigma_p \leqslant \operatorname{im} p \leqslant \sigma_p$, (25.9) therefore implies that, for $\operatorname{re} \eta > \sigma_\eta$, the position of the singularities of $\phi_0(p, \eta)$ in the strip

$$-1/l(v_0) \leqslant \operatorname{im} p \leqslant 1/l(v_0) \quad (25.10)$$

is independent of η.

As η tends to infinity, $\phi_0(p, \eta)$ clearly tends to the Fourier transform $M_0(p, v_0)$ (say) of $N_0(x, v_0)$, i.e. of the density of neutrons which have not undergone any collisions. The function $M_0(p, v_0)$ is regular in the complex p-plane cut along the imaginary axis from $-i\infty$ to $-i/l(v_0)$ and from $i/l(v_0)$ to $i\infty$, and in particular it is regular in the strip (25.10). Thus, for $\operatorname{re} \eta > \sigma_\eta$, $\phi_0(p, \eta)$ should be regular in that strip, and the same is true of all other $\phi_n(p, \eta)$ and of all $\pi_n(p, \eta)$.

This conclusion could have been reached by purely mathematical means, without using the physical considerations leading to (25.7). For the sake of simplicity, we shall give the alternative proof only for a single example, namely a single element with no capture, assuming that (25.3) holds and that the summation over j in (25.3) contains only two terms: $j = 0$, $\beta_0 = 0$ and $j = 1, \beta_1 = \beta > 0$. In this case we have to deal with $\pi_n(p, \eta)$, and the equations (25.5) become

$$ipl(v_0)B_0[(n+1)\pi_{n+1}(p, \eta) + n\pi_{n-1}(p, \eta)] + (2n+1)[1 - g_n(\eta)]\pi_n(p, \eta)$$
$$= S\delta_{n0}/v_0^2 - ipl(v_0)B_1[(n+1)\pi_{n+1}(p, \eta+\beta) + n\pi_{n-1}(p, \eta+\beta)], \quad (25.11)$$

with $B_0 + B_1 = 1$, $B_0 \geqslant 0$ (since $l(v)$ is non-negative) and $B_1 > 0$ (from the assumption that $dl(v)/dv > 0$). The above conditions imply that

$$0 \leqslant B_0 < 1. \quad (25.12)$$

The equations (25.11) may be regarded as a system of equations to determine the $\pi_n(p, \eta)$ when $\pi_n(p, \eta+\beta)$ is known. They then represent

† If $-\sigma_p \leqslant \operatorname{im} p \leqslant \sigma_p$, the contribution from large distances is not dominant, while that from smaller distances cannot cause a divergence of the integral (24.1) defining $\phi_0(p, \eta)$, if $\operatorname{re} \eta > \sigma_\eta$.

a difference equation in only one variable, namely n. It is well known that such equations can always be solved, provided that the corresponding homogeneous system

$$ipl(v_0)B_0[(n+1)\bar{\pi}_{n+1}(p,\eta)+n\bar{\pi}_{n-1}(p,\eta)]+(2n+1)[1-g_n(\eta)]\bar{\pi}_n(p,\eta)=0 \tag{25.13}$$

has no solution. Thus, for a given η, the singularities of $\pi_n(p,\eta)$ as a function of p can arise only in one of two ways: either p is an eigenvalue of (25.13), or $\pi_n(p,\eta+\beta)$ has a singularity for the given p. If the first of these holds, since the equations (25.13) are identical with the homogeneous form of (24.2) in a medium where the total mean free path is $B_0 l(v_0)$, the eigenvalues tend to $\pm i/B_0 l(v_0)$ as $\mathrm{re}\,\eta \to \infty$. By (25.12), these lie some distance outside the strip (25.10), and therefore, for $\mathrm{re}\,\eta >$ some σ_η, (25.13) has no eigenvalues in the strip (25.10). Therefore, if $\pi_n(p,\eta)$, for some η such that $\mathrm{re}\,\eta > \sigma_\eta$, has a singularity at some point $p = p_0$, say, in the strip (25.10), then $\pi_n(p,\eta+\beta)$, $\pi_n(p,\eta+2\beta)$, etc., would have singularities there, and this is impossible in view of the behaviour of $\pi_n(p,\eta)$ as $\mathrm{re}\,\eta \to \infty$.

The extension of this proof to more complex situations is evident.

If B_0 in (25.13) is zero, p does not appear in this equation. Instead of the eigenvalues p for a given η, we then have the eigenvalues η, which are independent of p. In this case, a refinement of the above arguments seems to show that, for η sufficiently large ($\mathrm{re}\,\eta > \sigma_\eta$), the $\pi_n(p,\eta)$ are regular functions of p, not only in the strip (25.10), but throughout the complex plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l(v_0)$ and from $p = i/l(v_0)$ to $p = i\infty$.

25.3. The method of Verde and Wick

25.3.1. *The singularities*

The above conclusion has been used by Verde and Wick (50) in the procedure they suggested for determining $N_0(x,v)$ for small v when

$$\lim_{v\to 0} l(v) = 0. \tag{25.14}$$

It will be assumed, of course, that either (25.1) holds or all the cross-sections follow the same law and (25.3) holds. In the latter case (25.14) implies that all the β_j are positive. If the various cross-sections follow different laws, so that (25.1) is used, we shall assume that the ratio of the capture and scattering cross-sections remains finite. If (25.1) is used, (25.14) implies that at least one α_j is negative, while the assumption that the capture-scattering ratio remains finite means that, if α_0 is the

largest (in modulus) negative α_j, $\sum_k A_{s,k,0} \neq 0$. The procedure is very similar, whether we assume (25.1) and start from (25.2), or assume (25.3) and start from (25.5). For definiteness, we shall take the latter case, and suppose $l(v)$ proportional to v, so that the sum in (25.3) reduces to a single term with $\beta = 1$. The equations (25.5) then become

$$(2n+1)[1-g_n(\eta)]\pi_n(p,\eta) = \frac{S}{v_0^2}\delta_{n0} - ipl(v_0)[(n+1)\pi_{n+1}(p,\eta+1)+n\pi_{n-1}(p,\eta+1)], \quad (25.15)$$

and the corresponding equations (25.13) are simply

$$(2n+1)[1-g_n(\eta)]\bar{\pi}_n(p,\eta) = 0. \quad (25.16)$$

Let us examine the solution of (25.15) for a particular p in the cut plane. According to the conclusions of § 25.2, especially the remark at the end of that section, for $\mathrm{re}\,\eta >$ some σ_η, $\pi_n(p,\eta)$ should be regular. As $\mathrm{re}\,\eta$ decreases to values $\leqslant \sigma_\eta$, the first singularity encountered clearly occurs when η is an eigenvalue of (25.13), i.e. (25.16) in our case; see § 25.2. The eigenvalues of (25.16), however, are given simply by the roots of one of the equations

$$1-g_n(\eta) = 0. \quad (25.17)$$

As $\mathrm{re}\,\eta$ decreases further, other singularities may arise in two ways: either from further roots of (25.17) or if $\pi_{n\pm1}(p,\eta+1)$ is itself singular. All these singularities are poles, though they need not be simple ones; multiple poles may arise either from the multiple roots, if any, of (25.17), or if (25.17) has a root at $\eta = \eta'$, say, whilst $\pi_{n\pm1}(p,\eta)$ has a pole at $\eta = \eta'+1$.

The position of these poles is independent of p. This is not only a feature of the particular example we have chosen, but is found whenever (25.14) holds. For, if all the β_j are positive, the equation (25.13) always has the form (25.16), whilst if we start from (25.2), at least one of the α_j is negative, and the homogeneous equation corresponding to (25.13) is then†

$$\frac{2n+1}{l(v_0)}[A_{s,0}+\sum_k A_{s,k,0}\{1-g_{n,M_k}(\eta)\}]\bar{\phi}_n(p,\eta+\alpha_0) = 0,$$

where α_0 is the largest (in modulus) negative α_j, and these equations again do not involve p.

The following remark may be made concerning the distribution of these poles of the $\pi_n(p,\eta)$. Each $g_n(\eta)$ is a rational algebraic function of η (see

† The unknown functions in this equation are denoted by $\bar{\phi}_n$, and not by $\bar{\pi}_n$, since, if we assume (25.1) instead of (25.3), we must use (24.1) instead of (25.4).

(24.3)), so that each equation (25.17) can have only a finite number of roots. None of the roots of (25.17) for any n lies to the right of $\mathrm{re}\,\eta = \sigma_\eta$ (this can be proved directly by using (24.3) and evaluating σ_η explicitly). If, for some n, (25.17) has a root $\eta = \eta'$, say, the resulting poles of $\pi_{n'}(p, \eta)$ lie at $\eta = \eta' - |n-n'|, \eta' - |n-n'| - 2$, etc. For any particular n and σ', say, the number of poles of $\pi_n(p, \eta)$ to the right of $\mathrm{re}\,\eta = \sigma_\eta - \sigma'$ is therefore finite.

If $\eta_{n,s}$ is the sth pole of $\pi_n(p, \eta)$ in order of decreasing $\mathrm{re}\,\eta$, i.e. $\mathrm{re}\,\eta_{n,s} \leqslant \mathrm{re}\,\eta_{n,s-1}$, and $m_{n,s}$ is the multiplicity of this pole, then an expansion of $\pi_n(p, \eta)$ in partial fractions gives

$$\pi_n(p, \eta) = \sum_s \sum_{q=1}^{m_{n,s}} \frac{\omega_{n,s,q}(p)}{(\eta - \eta_{n,s})^q} + U_n(p, \eta), \tag{25.18}$$

provided that the sum over s converges, where $\omega_{n,s,q}$ are some functions of p alone, regular in the cut p-plane, and the $U_n(p, \eta)$ are regular for all η and any p in the cut plane. It is difficult to obtain any information regarding the functions $U_n(p, \eta)$ in (25.18), but it has been conjectured that they are zero. The solution of (25.15) should be unique, according to the general properties of slowing-down problems (see Chapter XVIII). Consequently, the conjecture is proved if a solution satisfying all the conditions on the $\pi_n(p, \eta)$ can be obtained with the $U_n(p, \eta)$ omitted. Instead of (25.18), we shall therefore assume henceforth simply

$$\pi_n(p, \eta) = \sum_s \sum_{q=1}^{m_{n,s}} \frac{\omega_{n,s,q}(p)}{(\eta - \eta_{n,s})^q}. \tag{25.18'}$$

25.3.2. *The evaluation of the coefficients*

We thus have to determine only the functions $\omega_{n,s,q}(p)$. Applying the inverse Mellin transformation to (25.18'), we have

$$\frac{1}{l(v)} \int_{-\infty}^{\infty} N_n(x, v) e^{ipx}\, dx = \sum_s \left(\frac{v}{v_0}\right)^{-\eta_{n,s}} \sum_{q=1}^{m_{n,s}} \frac{\log^{q-1}(v_0/v)}{(q-1)!}\, \omega_{n,s,q}(p), \tag{25.19}$$

and this shows that, for any n and sufficiently small v, only the first few terms in (25.19) will give a non-negligible contribution. To determine $N_0(x, v)$, for example, for small v, we need only know the $\omega_{0,s,q}(p)$ for the first few values of s.

The equations to determine† $\omega_{n,s}(p)$, assuming $\eta_{n,s}$ to be a simple pole, are obtained by substituting (25.18') into (25.15) and taking the limit

† The third subscript is omitted when $m_{n,s} = 1$, i.e. when the pole is simple.

as $\eta \to \eta_{n,s}$. This gives

$$-(2n+1)[dg_n(\eta)/d\eta]_{\eta=\eta_{n,s}}\omega_{n,s}(p) = S\delta_{0n}/v_0^2 - \\ -ipl(v_0)\Big\{(n+1)\sum_{s'}^{m_{n+1,s'}}\sum_{q=1}\omega_{n+1,s',q}(p)+n\sum_{s'}^{m_{n-1,s'}}\sum_{q=1}\omega_{n-1,s',q}(p)\Big\}. \quad (25.20)$$

The modifications necessary in (25.20) when $\eta_{n,s}$ is a multiple pole are evident.

It seems hopeless to attempt to solve the system (25.20) directly for arbitrary p. However, the coefficients of $\omega_{n+1,s',q}(p)$ and $\omega_{n-1,s',q}(p)$ on the right side of (25.20) are proportional to p. Thus, for small p, the equations (25.20) can be solved by expanding in powers of p. Since the $\omega_{n,s,q}(p)$ are regular in the cut p-plane, this series will converge for $|p| < 1/l(v_0)$. This is not yet sufficient for our purposes, but a slight modification of the procedure gives an expansion valid throughout the cut plane. Thus, let $\epsilon(p)$ be the function which maps the region of regularity of $\omega_{n,s,q}(p)$ conformally on the circle $|\epsilon| < 1$, such that $\epsilon(0) = 0$; then, expanding the above series for $\omega_{n,s,q}(p)$ in powers of ϵ, we obtain a series which converges for all ϵ in this circle, and putting $\epsilon = \epsilon(p)$ we obtain an expression for $\omega_{n,s,q}(p)$ valid throughout the region where these functions are regular. This region being as stated, the function $\epsilon(p)$ is

$$\left.\begin{aligned} \epsilon(p) &= \frac{pl(v_0)}{[\{pl(v_0)\}^2+1]^{\frac{1}{2}}+1}, \\ \text{and thus} \qquad pl(v_0) &= 2\epsilon/(1-\epsilon^2). \end{aligned}\right\} \quad (25.21)$$

In practice, of course, we should not expand $\omega_{n,s,q}(p)$ first in powers of p and then in those of ϵ, but substitute $pl(v_0)$ from the second of (25.21) in (25.20), expand $\omega_{n,s,q}(p)$ directly in powers of ϵ, and then substitute for ϵ from the first of (25.21). In evaluating the coefficients in these expansions, it is often convenient to redefine the functions $\omega_{n,s,q}(p)$ slightly by including in them certain frequently occurring numerical factors. The details of such calculations may be found in Verde and Wick's paper (**50**).

25.3.3. *The inversion of the Fourier transforms*

When $\omega_{n,s,q}(p)$ has been determined, we have still to invert the Fourier transform, i.e. to evaluate the integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ipx}\omega_{n,s,q}(p)\,dp, \quad (25.22)$$

which gives the coefficients of

$$\frac{1}{(q-1)!}\left(\frac{v}{v_0}\right)^{-\eta_{n,s}}\log^{q-1}\left(\frac{v_0}{v}\right)$$

in the final expression for $N_n(x, v)$. Neither the method of residues nor that of steepest descent is applicable here, and some numerical procedure is necessary. If $|x|$ is not very large, numerical integration along the real p-axis is practicable. If $|x|$ is very large, the path of integration has to be deformed, and in so doing we have to compromise between two requirements. In order to avoid subtracting nearly equal quantities, the path should closely follow the two sides of the cut, as in the evaluation of $\rho_{tx}(r)$ in (5.23) (Fig. 3); but, in order to minimize the error due to terminating the expansion of $\omega_{n,s,q}(p)$ after a finite number of terms, the path should lie as far as possible in the region of small $|\epsilon|$. If $\Xi(|\epsilon|)$ is an upper bound of the modulus of the remainder term in the expansion of $\omega_{n,s,q}(p)$ in powers of ϵ, the ideal path would be that which minimizes the integral

$$\int_{\substack{|p|=\infty,\,\mathrm{re}\,p>0\\ |p|=\infty,\,\mathrm{re}\,p<0}} e^{-x\,\mathrm{im}\,p}\,\Xi\left(\left|\frac{pl(v_0)}{[\{pl(v_0)\}^2+1]^{\frac{1}{2}}+1}\right|\right)|dp|.$$

Here we can estimate $\Xi(|\epsilon|)$ by means of the coefficients that have been evaluated, noticing that $\omega_{n,s,q}(p)$ should not have singularities of very high order on the cut, and therefore should not increase rapidly.

Verde and Wick themselves adopted a different method of evaluating the integral (25.22) for large $|x|$. In the particular example they considered, it was observed that, if the expansion is

$$\omega_{0,0}(p) = \sum_j w_j \epsilon^j,$$

all the w_j that had been evaluated numerically could be represented in the form

$$w_j = w_{j0} + w_{j1}, \tag{25.23}$$

where the w_{j0} were the corresponding coefficients in the expansion of a comparatively simple algebraic function, namely

$$C(1+\epsilon^2)^g,$$

for a certain C and g, while the ratio $|w_{j1}/w_{j0}|$ decreased rapidly and was soon very small compared with unity. Verde and Wick conjectured that the relation (25.23), with the above w_{j0} and w_{j1}, holds for all j, and therefore put

$$\omega_{00}(p) = \omega_{00,0}(p) + \omega_{00,1}(p).$$

It can then be easily shown that, as $|x| \to \infty$, the contribution of $\omega_{00,1}(p)$ to (25.22) becomes negligibly small in comparison with that of $\omega_{00,0}(p)$,

and the latter can be found analytically. The asymptotic behaviour of the latter as $|x| \to \infty$ is given by

$$C'[|x|/l(v_0)]^{g'} e^{-|x|/l(v_0)}, \tag{25.24}$$

which is of essentially the same form as the expression for $n_0(x)$ in (25.7) (see (25.44) below). However, Verde and Wick do not state whether the value of g' in (25.24) is the same as that of $-1-g_0/\gamma$ in (25.44).

It is to be hoped that a representation of the type (25.23) for the coefficients in the expansion of $\omega_{n,s,q}$ in powers of ϵ will exist in all applications of Verde and Wick's method. It is not known, however, whether this is actually so. If not, one can always use deformation of the contour and numerical integration, as described earlier in this section.

We have seen that all the equations (25.15) can be satisfied on the assumption of (25.18′). If (25.18′) converges, it therefore gives the required solution, and this proves the conjecture that all the $U_n(p, \eta)$ in (25.18) vanish. However, it has not yet been possible to prove the convergence of (25.18′).

The method of Verde and Wick is limited, as we have seen, to the case where $l(0) = 0$. If the mean free path tends to a non-zero limit as v tends to zero, another closely related method, due to Holte, is available. Before discussing this, however, we should investigate the behaviour of the solution for v fairly close to v_0, and in particular determine $n_0(x)$ in (25.7).

25.4. Wick's asymptotic solution

25.4.1. *The reduction of the equations*

The determination of $n_0(x)$ in (25.7), which is more troublesome, has also been carried out by Wick (54). As with the derivation of (25.7), the following analysis applies only to the case where the cross-sections decrease monotonically with increasing velocity. For simplicity we take the case of a single non-capturing element. The procedure is very similar to that of § 24.5. Taking the Fourier–Mellin transform of (22.1) and putting, similarly to (24.21) and (24.46),

$$\bar{\pi}(p, \eta, \theta) = \frac{1}{v_0^{\eta}} \int_0^{v_0} v^{\eta-1}\, dv \int_{-\infty}^{\infty} \frac{N(x, v, \cos\theta)}{l(v)} e^{ipx}\, dx, \tag{25.25}$$

we obtain

$$\bar{\pi}(p, \eta, \theta) - ip\cos\theta\, \bar{\phi}(p, \eta, \theta) = \iint g(\Theta, \eta)\bar{\pi}(p, \eta, \theta')\, d\Omega' + S/4\pi v_0^3, \tag{25.26}$$

where $\bar{\phi}(p, \eta, \theta)$, Θ, and $g(\Theta, \eta)$ have the same significance as in § 24.5.

The next step is to find the relation between $\bar{\pi}$ and $\bar{\phi}$. In deriving (25.7) we have remarked that $n_0(x)$ depends only on $l(v_0)$ and $[v/l(v)][dl(v)/dv]$ for $v = v_0$, and not on the actual form of $l(v)$ for $v < v_0$. We can therefore expand $l(v)$ in powers of $v_0 - v$, or in powers of the lethargy, the latter being more convenient, and terminate the expansion at the second term. In other words, with the notation of (25.8), we may put

$$l(v) = l(v_0)[1+\gamma \log(v/v_0)]. \tag{25.27}$$

With this $l(v)$, a comparison of (25.25) with (24.21) and (24.46) gives at once

$$\bar{\phi}(p, \eta, \theta) = l(v_0)\left(1+\gamma \frac{\partial}{\partial \eta}\right)\bar{\pi}(p, \eta, \theta),$$

and equation (25.26) then becomes

$$[1 - ipl(v_0)\cos\theta - ipl(v_0)\gamma \cos\theta\, \partial/\partial\eta]\bar{\pi}(p, \eta, \theta)$$
$$= \iint g(\Theta, \eta)\bar{\pi}(p, \eta, \theta')\, d\Omega' + S/4\pi v_0^2. \tag{25.28}$$

The only important difference between (24.47) and (25.28) is the appearance in the latter of the extra term

$$-ipl(v_0)\gamma \cos\theta\, \partial\bar{\pi}/\partial\eta, \tag{25.29}$$

but this changes the analytical nature of the solution.

Let us first consider what are the ranges of p and η for which the solution of (25.28) is relevant. For large $|x|$, as is easily shown, the main contribution to $N(x, v, \cos\theta)$, as given by the inverse Fourier transform of (25.25), comes from values of p in the vicinity of $\pm i/l(v_0)$, i.e. the situation is much the same as in the case of constant cross-sections.† For η, however, the position is rather different. The formula (25.9) implies that the range of values of η which, for a given v, give the main contribution to the inverse of the Mellin transform in (25.25) is independent of x. Thus, instead of determining the asymptotic behaviour of the solution of (25.28) as $\eta \to \infty$, as in the constant cross-section case, we have to find this solution for some fixed η. However, if we wish to determine only $n_0(x)$ in (25.7), it does not matter what value of η is chosen, and it is convenient to take the case of large η. In fact, this is necessary because of the approximation (25.27). Once η has been assumed large

† If x is large and positive, for instance, we should use, in evaluating the inverse transform of (25.25), a contour in the p-plane along which im p is negative and as large as possible. Since $\bar{\pi}$ is regular in the strip (25.10) (see § 25.2), we therefore take a contour with im $p \cong -1/l(v_0)$. If (25.25) converges, its values for p purely imaginary should be much greater than those for complex p with the same imaginary part and a non-negligible real part; the difference is particularly marked if $|x|$ is large. This confirms that the main contribution to $N(x, v, \cos\theta)$ for large $|x|$ is from the vicinity of $p = -i/l(v_0)$ if x is positive, or of $p = i/l(v_0)$ if x is negative.

and $p \cong -i/l(v_0)$, we can introduce the same approximations in (25.28) as in (24.47). With the same accuracy, and noticing that (25.29) is the only term in (25.28) which involves $\gamma\partial/\partial\eta$, we see that in this term also we can approximate $\cos\theta$ by unity. This gives, similarly to (24.53),

$$[1-ipl(v_0)]\bar{\pi}^0(p,\eta,\theta) = -(\tfrac{1}{2}\theta^2-\gamma\partial/\partial\eta)\bar{\pi}^0(p,\eta,\theta)+$$
$$+\frac{1}{4\pi}\left(1+\frac{1}{M}\right)^2\int\int e^{-\frac{1}{2}i(\eta/M)-(1/M^2)][\theta^2+\theta'^2-2\theta\theta'\cos(\phi-\phi')]}\bar{\pi}^0(p,\eta,\theta')\theta'd\theta'd\phi'+$$
$$+S/4\pi v_0^2, \quad (25.30)$$

where $\bar{\pi}^0(p,\eta,\theta)$ is the first approximation to $\bar{\pi}(p,\eta,\theta)$. Taking the Bessel transform of (25.30), we obtain, similarly to (24.55),

$$[1-ipl(v_0)]\Pi^0(p,\eta,\nu)$$
$$=\left\{\frac{1}{2}\left(\frac{\partial^2}{\partial\nu^2}+\frac{1}{\nu}\frac{\partial}{\partial\nu}\right)+\gamma\frac{\partial}{\partial\eta}+\frac{(M+1)^2}{2(M\eta-1)}e^{-M^2\nu^2/2(M\eta-1)}\right\}\Pi^0(p,\eta,\nu)+\frac{S\delta(\nu)}{2v_0^2\nu}, \quad (25.31)$$

where the definition of $\Pi^0(p,\eta,\nu)$ is evident by analogy with (24.54).

25.4.2. *An eigenvalue problem*

By means of the substitution (24.57), it was possible to put the equation (24.55) into a more convenient form. The same change of variables in (25.31) renders the homogeneous form of the latter equation separable. We denote the function $\Pi^0(p,\eta,\nu)$ as a function of p, η, and s by a new symbol $V(p,\eta,s)$, so that

$$\Pi^0(p,\eta,\nu) = V[p,\eta,M\nu\{2/(M\eta-1)\}^{\frac{1}{2}}]. \quad (25.32)$$

Then

$$\frac{\partial}{\partial\eta}\Pi^0(p,\eta,\nu) = \left\{\frac{\partial}{\partial\eta}-\frac{Ms}{2(M\eta-1)}\frac{\partial}{\partial s}\right\}V(p,\eta,s) \quad \text{for } s = M\nu[2/(M\eta-1)]^{\frac{1}{2}},$$

and the equation (25.31) becomes

$$\frac{(M\eta-1)}{M^2}\left\{[1-ipl(v_0)]-\gamma\frac{\partial}{\partial\eta}\right\}V(p,\eta,s)$$
$$=\left\{\frac{\partial^2}{\partial s^2}+\frac{1}{s}\frac{\partial}{\partial s}-\frac{\gamma}{2M}s\frac{\partial}{\partial s}+\frac{1}{2}\left(1+\frac{1}{M}\right)^2e^{-\frac{1}{2}s^2}\right\}V(p,\eta,s)+\frac{S\delta(s)}{v_0^2 s}. \quad (25.33)$$

The solution of the corresponding homogeneous equation is given by superposition of expressions of the form $\bar{a}(\eta,g)w(s,g)$, where $\bar{a}$ and w are the solutions of the equations

$$(\mathrm{K}+g)\bar{a}(\eta,g) = 0 \quad (25.34\,\text{a})$$

and

$$(\Lambda+g)w(s,g) = 0, \quad (25.34\,\text{b})$$

K and Λ being the operators on $V(p, \eta, s)$ on the left and right sides respectively of (25.33), and g being an adjustable parameter.

It can readily be seen, as in the case of equation (24.58′), that the required solutions of (25.34 b) vanish for $s = \infty$ and are regular for $s = 0$, so that (25.34 b) leads to an eigenvalue problem. This immediately suggests the solution of (25.33) by an expansion in terms of the eigenfunctions of (25.34 b). Before doing this, however, we shall examine equation (25.34 b) more closely. To bring it into a more familiar form, we put

$$w(s, g) = e^{\gamma s^2/8M} \tilde{w}(s, g). \tag{25.35}$$

The equation for $\tilde{w}(s, g)$ is then

$$\left\{\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} + \frac{1}{2}\left(1 + \frac{1}{M}\right)^2 e^{-\frac{1}{4}s^2} - \frac{\gamma^2 s^2}{16M^2}\right\}\tilde{w}(s, g) = -\left(g + \frac{\gamma}{2M}\right)\tilde{w}(s, g), \tag{25.36}$$

and this may be regarded as the Schrödinger equation for a particle in a two-dimensional potential field which, at large distances, tends to that for a harmonic oscillator. This equation has the following properties:

(1) The eigenvalues g form a discrete set. This can be proved in the same way as the theorem that the eigenvalues for a harmonic oscillator form a discrete set.

(2) The eigenfunctions of (25.36) corresponding to different eigenvalues are orthogonal with weight factor s, as may easily be verified. The corresponding eigenfunctions of (25.34 b), if suitably normalized, therefore satisfy the orthogonality condition

$$\int_0^\infty w(s, g_m) w(s, g_{m'}) e^{-\gamma s^2/4M}\, s\, ds = \delta_{mm'}, \tag{25.37}$$

where g_m is the mth eigenvalue of (25.36).

(3) There is a good deal of evidence that the eigenfunctions of any equation of Schrödinger type form a complete set. This may therefore be assumed regarding (25.36), and hence also for (25.34 b).

We shall see shortly that, for our present purpose, only the lowest eigenvalue is of interest. This can be determined by means of the variational method, as in § 24.5.3.

25.4.3. *Final results*

From the above remarks concerning equations (25.34 b) and (25.36), we can put

$$V(p, \eta, s) = \sum_{m'} w(s, g_{m'}) a(p, \eta, g_{m'}). \tag{25.38}$$

Substituting this into (25.33), multiplying by

$$w(s, g_m)e^{-\gamma s^2/4M}\, s,$$

integrating over all s, and using (25.37), we obtain

$$\left\{\frac{1}{M^2}(M\eta-1)\left[1-ipl(v_0)-\gamma\frac{\partial}{\partial\eta}\right]+g_m\right\}a(p,\eta,g_m)=\frac{S}{v_0^2}\tilde{w}(0,g_m). \tag{25.39}$$

From its definition, $\Pi^0(p,\eta,\nu)$, should remain finite as $\eta\to+\infty$ for any given $p\neq -i/l(v_0)$ and any given $\nu\neq 0$. This implies that the required solution of (25.39) should also remain finite as $\eta\to+\infty$. This solution is therefore

$$a\left(p,\eta+\frac{1}{M},g_m\right)=\frac{SM}{v_0^2\gamma}\tilde{w}(0,g_m)\int_\eta^\infty\frac{d\eta'}{\eta'}\left(\frac{\eta}{\eta'}\right)^{g_m/\gamma}\exp\left[\frac{1-ipl(v_0)}{\gamma}(\eta-\eta')\right]. \tag{25.40}$$

As has been said previously (see the discussion preceding formula (25.30)), in order to determine $n_0(x)$ for large x we have to examine the asymptotic behaviour of the solution of the transformed equation as $p\to -i/l(v_0)$, whilst η must be kept fixed, though it may be taken as large as we please. Let us consider the asymptotic behaviour of (25.40) as $p\to -i/l(v_0)$, bearing in mind that γ is assumed positive (see (25.8)). Then, if g_m is positive, the integral in (25.40) converges for all $\operatorname{im} p\geqslant -1/l(v_0)$, and as $p\to -i/l(v_0)$ it reduces to

$$\int_\eta^\infty\frac{d\eta'}{\eta'}\left(\frac{\eta}{\eta'}\right)^{g_m/\gamma}. \tag{25.41}$$

If, however, $g_m\leqslant 0$, (25.41) diverges and (25.40) tends to infinity as $p\to -i/l(v_0)$. To estimate its asymptotic behaviour in this case, we put

$$z=[1-ipl(v_0)]\eta/\gamma,\qquad z'=[1-ipl(v_0)]\eta'/\gamma,$$

and notice that, if p tends to this limit while η is fixed, z tends to zero. If g_m is negative, the integral in (25.40) becomes

$$z^{g_m/\gamma}e^z\int_z^\infty e^{-z'}z'^{-1-(g_m/\gamma)}\,dz'\cong z^{g_m/\gamma}\int_0^\infty e^{-z'}z'^{-1-(g_m/\gamma)}\,dz'=z^{g_m/\gamma}\Gamma(-g_m/\gamma), \tag{25.42}$$

i.e.

$$[1-ipl(v_0)]^{g_m/\gamma}.(\eta/\gamma)^{g_m/\gamma}\Gamma(-g_m/\gamma). \tag{25.42'}$$

The modification needed if g_m is zero is evident.

Substituting (25.42′) in (25.40) and combining it with (25.38) and (25.32), we see that for $g_m < 0$ the corresponding term in $\Pi^0(p, \eta, \nu)$ behaves as

$$[1 - ipl(v_0)]^{g_m/\gamma} \times \text{a function of } \eta \text{ and } \nu \text{ only}$$

when $p \to i/l(v_0)$, so that g_m makes a more important contribution, the larger its modulus ($g_m < 0$). A closer examination of the equation (25.36) which defines the eigenvalues g_m shows that the lowest eigenvalue, which we call g_0, is always negative. The leading term in $\Pi^0(p, \eta, \nu)$ thus behaves, as p tends to the above limit, like

$$[1 - ipl(v_0)]^{g_0/\gamma} \times \text{a function of } \eta \text{ and } \nu \text{ only.} \tag{25.43}$$

Inverting all the transforms involved, bearing in mind that the difference between $\bar{\pi}(p, \eta, \theta)$ and $\bar{\pi}^0(p, \eta, \theta)$ is negligible for large η, and noting that the inverse Fourier transform of the factor given explicitly in (25.43) is proportional to $x^{-(g_0/\gamma)-1} e^{-x/l(v_0)}$, we obtain

$$N(x, v, \cos\theta) = \left(\frac{|x|}{l(v_0)}\right)^{-1-(g_0/\gamma)} e^{-|x|/l(v_0)} f(v, \theta) + \text{smaller terms,} \tag{25.44}$$

where g_0/γ is negative, as has been said before. This fact could have been foreseen from our discussion of (25.6), since $N_0(x, v_0)$ is proportional to $E_1[|x|/l(v_0)]$ (see the derivation of equation (6.1)), while the latter function behaves as

$$\{l(v_0)/|x|\} e^{-|x|/l(v_0)}$$

for $|x| \gg l(v_0)$.

The formula (25.44) is, of course, of exactly the same form as (25.7), and thus gives the expression for $n_0(x)$. An examination of the derivation of (25.44) gives a clearer idea of the range of validity of (25.7). In particular, the approximation introduced in the derivation of (25.42) shows that, if v is very close to v_0, (25.44) becomes valid only at very large distances. Values of v which are close to v_0 are, however, seldom of interest in practice.

The behaviour of the other factor in (25.44), i.e. $f(v, \theta)$, as $v \to v_0$ can easily be expressed in terms of the eigenfunction $\tilde{w}(s, g_0)$ of (25.36) which belongs to the eigenvalue g_0. To determine it for smaller v, however, more terms should be taken in the expansion (25.27) and the perturbation method applied. This has not yet been done. If v differs appreciably from v_0, and in particular for the lower end of the spectrum, the present method is no longer suitable, and another must be sought. One such has already been outlined in § 25.3. Further alternatives will be discussed in Chapter XXVI.

25.5. Hydrogenous media: the method of Spencer and Fano

25.5.1. *Introduction*

The existence of the approximate relation (25.7), which was conjectured in § 25.1 and confirmed by the analysis of § 25.4, is of interest in another respect also. We have seen in § 23.6 that age theory, besides breaking down at very large distances, is nowhere valid in hydrogen and media containing a high proportion of hydrogen. In the same section, a brief survey was given of the chief methods used to determine the neutron distribution at moderate distances from the source in hydrogenous media. It was also remarked that the breakdown of age theory for hydrogen is connected with the fact that, even at moderate distances, the neutron distribution has features normally belonging to the asymptotic distribution at very large distances. This suggests that the relation (25.7) should be valid for hydrogen at considerably smaller distances from the source than for heavy nuclei. If this occurs before the neutron population has decreased too greatly, a method of determining the neutron distribution in hydrogenous media would exist which is in some respects more satisfactory than those of Chapter XXIII. We shall now describe this method, leaving to the next chapter the last remaining method of dealing with neutrons at large distances.

It should be remarked at once that the method discussed here, which is due to Spencer and Fano (44), was originally designed for X-rays rather than for neutrons. The problem of the penetration of X-rays is in many respects very similar to that of neutron transport, but the problems differ appreciably in some ways. The application of Spencer and Fano's method has been successful in the case of X-rays, but it is not yet known how useful it will be for neutrons.

The general assumptions given in § 1.1 are as applicable to X-rays as they are to neutrons, and the Boltzmann equation therefore has much the same form (2.4′) as in the case of neutrons; the energy, of course, can no longer be expressed in terms of the velocity, but is expressed in terms of the wavelength of the corresponding photons, whose velocity is that of light. The laws governing the results of individual collisions, and therefore the form of $f(\lambda', \Omega' \to \lambda, \Omega)$ in (2.4′), on the other hand, are markedly different from those found in the case of neutrons. Forward scattering is strongly predominant, and the energy degradation is very appreciable, except in small-angle scattering; the absorption increases rapidly with decreasing energy. Hence, for a given energy and position, the X-rays moving directly away from the source strongly predominate, and the distribution of X-rays at any distance from the source should

have the same character as the neutron distribution at very large distances from the source. We should thus expect an approximate relation similar to (25.7) to become valid for X-rays even at fairly moderate distances from the source.

On the other hand, we have remarked that it is the possibility of applying (25.7) before the number of particles has been too much reduced that is made use of in Spencer and Fano's method. This shows that the success of the method for X-rays does not imply that it will be equally successful for neutrons. The scattering law and the expected form of the solution, however, differ in the X-ray case from those in the neutron case in the same direction, although more so, as do the scattering law and the expected form of the solution for hydrogen from those for heavy nuclei. Thus, even if Spencer and Fano's method in its present form is unsuitable for neutrons in non-hydrogenous media, it may still be suitable for neutrons in hydrogenous media.

25.5.2. *Spatial moments and orthogonal polynomials in spatial coordinates*

We shall start by returning to the spatial moments of the neutron distribution. The equations governing these moments have been derived in § 22.3, and the moments can be determined by numerical integration of these equations, even when no analytical solution is available. Once the spatial moments are known, it is always possible to reconstruct the complete spatial distribution. In practice this would be done as follows. Suppose that we have selected some function $w(x)$, everywhere positive, such that, as $|x| \to \infty$, it tends to zero faster than any power of x, and have introduced two sets of polynomials $p_s(x)$ and $p_s^\dagger(x)$, related by

$$\int_{-\infty}^{\infty} w(x) p_s(x) p_{s'}^\dagger(x)\, dx = \delta_{ss'}. \tag{25.45}$$

It may be assumed without loss of generality that $p_0(x)$ is constant and equal to unity.

Further, let the products $p_s(x)w(x)$ form a complete set of functions which tend to zero as $|x| \to \infty$. We can then expand the neutron density $N_0(x, v)$ in terms of these functions, obtaining

$$N_0(x, v) = \sum_s a_{s,0}(v) p_s(x) w(x); \tag{25.46}$$

multiplying this by $p_s^\dagger(x)$, integrating over x, and using (25.45), we obtain

$$a_{s,0}(v) = \int_{-\infty}^{\infty} N_0(x, v) p_s^\dagger(x)\, dx. \tag{25.47}$$

That is, each $a_{s,0}(v)$ is given by a linear combination of the spatial moments of the neutron distribution. In particular, for an isotropic plane source, the polynomial $p_s^{\dagger}(x)$ being of order not greater than $2s$, the expression for $a_{s,0}(v)$ is

$$a_{s,0}(v) = p_{0,s}m_{2s,0}(v)+p_{1,s}m_{2s-2,0}(v)+\ldots+p_{s,s}m_{0,0}(v), \quad (25.48)$$

where the $p_{r,s}$ are some constants, namely the coefficients of the even powers of x in the $p_s^{\dagger}(x)$, and the $m_{2s',0}(v)$ have the same meaning as in § 22.3.2.

Two facts should be noted regarding the expression (25.48). Firstly, if $w(x)$ happens to coincide with the exact $N_0(x, v)$ for a particular $v = v_1$, say, then, since $p_0(v) = 1$, a comparison of (25.45) and (25.47) shows at once that $a_{s,0}(v_1) = \delta_{s0}$. Similarly, if $w(x)$, though not equal to $N_0(x, v_1)$, is a good approximation to it, the $a_{s,0}(v_1)$ for $s \neq 0$ will be small. Again, if $w(x)$ is a good approximation to $N_0(x, v_1)$ and $w(x)p_1(x)$ is a good approximation to $N_0(x, v_1)p_1(x)$, then $a_{s,0}(v_1)$ for $s \geqslant 2$ will be much smaller than $a_{1,0}(v_1)$, and so on. This suggests that, the more closely $w(x)$ resembles $N_0(x, v_1)$, the more rapidly will the series (25.46) converge for $v = v_1$. In the same way, we can readily see that, if $N_0(x, v)$ is approximately given by

$$N_0(x, v) \cong n_0(x)f(v), \quad (25.49)$$

and $w(x)$ is a good approximation to $n_0(x)$, then all the $a_{s,0}(v)$ for $s \neq 0$ will be small; the closer $w(x)f(v)$ approximates $N_0(x, v)$, the more rapid will be the convergence of the series (25.46) for any v for which (25.49) is valid.

It should be borne in mind, of course, that the convergence of (25.46) is of little use if the $m_{2s',0}(v)$ are first found by numerical integration and then substituted in (25.48). After a certain s', the $m_{2s',0}(v)$ increase with s'. The smallness of the $a_{s,0}(v)$ therefore means simply that $a_{s,0}(v)$ is calculated as the difference between two nearly equal large quantities, and therefore the difficulty of evaluating $a_{s,0}(v)$ balances the gain due to the rapid convergence of (25.46). This shows that the rapid convergence of (25.46) is not of much use unless the $a_{s,0}(v)$ can be calculated directly, without finding the $m_{2s',0}(v)$ first.

Here, however, we come to the second feature of the expression (25.48). We have seen in § 22.3 that the successive moments $m_{2s,0}(v)$ satisfy equations which all have the same kernel and differ only in their free terms. This can be seen by comparing (22.46 a) and (22.10); the equations for the higher spatial moments were not written explicitly in § 22.3,

but it can be seen from the description of their construction that they are of the form stated. Thus, if all the $p_{s',s}$ are constant, $a_{s,0}(v)$ will satisfy an equation having the same kernel, and differing from the equations for the $m_{2s',0}(v)$ only in the free term. Consequently, it can in fact be determined directly without using the $m_{2s',0}(v)$. The free term of the equation for $a_{s,0}(v)$ will, of course, be a linear combination of the free terms of the equations for $m_{2s,0}(v)$, $m_{2s-2,0}(v)$,..., $m_{0,0}(v)$, and will therefore, by § 22.3.4, contain a linear combination of $m_{2s-1,1}(v)$, $m_{2s-3,1}(v)$,..., $m_{1,1}(v)$, with constant coefficients. The various $m_{2s'+1,1}(v)$ themselves satisfy equations with the same kernel and different free terms. This linear combination of $m_{2s-1,1}(v)$, $m_{2s-3,1}(v)$, etc., can thus be again determined directly, without using the $m_{2s'+1,1}(v)$ themselves, and so on.

Thus we see that, if the various $p_{s',s}$ in (25.48) are constant, i.e. if $w(x)$ in (25.45), etc., depends on x only and is independent of v, the difficulty involved in calculating the $a(v)$ from the $m(v)$ can be avoided, and we can utilize the most rapid attainable convergence of (25.46).

This completes our account of the basic ideas underlying Spencer and Fano's method. It also explains the importance of the condition that (25.7) should hold for a significant range of values of x. For, unless this is so, it would not be possible to find a $w(x)$ independent of v that would make (25.46) converge rapidly. If, on the other hand, $w(x)$ depends on v, it would not in general be possible to determine the $a_{s,0}(v)$ directly without using the $m_{2s',0}(v)$.†

25.5.3. *The choice of the weight function*

We shall now discuss the details of Spencer and Fano's method. These depend to some extent on the kind of source involved, whether a plane or a point, and isotropic or collimated. For simplicity, we shall take only the case of a plane isotropic source. The extension to other types of source will be evident.

The first topic is the choice of $w(x)$. For pure hydrogen, according to (25.44), $n_0(x)$ in (25.7) is given by $n_0(x) \sim |x|^{-1-(g_0/\gamma)}e^{-|x|/k(v_0)}$, where g_0 can be calculated as explained in § 25.4. The calculation is extremely lengthy, however, and the factor $|x|^{-1-(g_0/\gamma)}$ varies comparatively slowly,

† The above arguments refer, of course, only to the case where $N_0(x, v)$ is to be determined for all v in a certain range. If we are interested in $N_0(x, v)$ for a certain value of v (v_1, say) only, the restriction that (25.7) must be a fairly good approximation can be removed. For, since the determination of $a_{s,0}(v_1)$ requires a knowledge of $a_{s',n}(v)$ only for $s' < s$, the accuracy with which $a_{s,0}(v_1)$ can be determined is independent of the rate of convergence of (25.46) for $v \neq v_1$. It therefore does not matter whether the behaviour of $w(x)$ is similar to that of $N_0(x, v)$ for $v \neq v_1$. However, there has as yet been no investigation which has made use of this fact.

so that the convergence of (25.46) cannot be much affected by disregarding this factor. We therefore put simply

$$w(x) = e^{-\alpha|x|} \tag{25.50}$$

with

$$\alpha = 1/l(v_0). \tag{25.51}$$

If we have not pure hydrogen but a hydrogenous mixture, the arguments leading to the choice of $w(x)$ are somewhat different. Since a mixture of hydrogen and deuterium is unlikely to be of practical interest, we shall consider only mixtures of hydrogen and heavy nuclei. In this case, we can provisionally neglect the slowing down due to collisions with the heavy nuclei up to moderate distances from the source, and regard collisions with hydrogen as the only cause of slowing down, as in Chapter XX. In this approximation, the distribution of neutrons of the initial energy is given by the solution of a fictitious one-velocity problem in which all hydrogen collisions are regarded as captures and the other nuclei are considered to be infinitely heavy. If $L^*(v_0)$ is the diffusion length (see § 5.2) in this fictitious problem, the solution of that problem at a distance of more than a few times $l(v_0)$ from the source is a constant times $e^{-|x|/L^*(v_0)}$. If $l(v)$ decreases with v, it is easily seen, as in § 25.1, that for any v and the distances concerned $N_0(x, v)$ is approximately given by

$$N_0(x, v) \cong e^{-|x|/L^*(v_0)} f(v), \tag{25.52}$$

which is similar in form to (25.7). The approximation (25.52) breaks down at distances where the effects of slowing down by collisions with the heavy nuclei become important, i.e. at distances comparable with L_s^*, the slowing-down length for a fictitious medium obtained from the actual medium by replacing all hydrogen collisions by captures. If there is an appreciable admixture of hydrogen (and this is the only case where Spencer and Fano's method would be used, since otherwise age theory is applicable), L_s^* is very large, so that the neutron population at these distances will be very small, and the breakdown of (25.52) cannot noticeably affect the convergence of (25.46). We therefore continue to take $w(x)$ in the form (25.50), but instead of (25.51) we use

$$\alpha = 1/L^*(v_0). \tag{25.53}$$

The arguments leading to the choice (25.50) and (25.51) or (25.53) are somewhat crude. This, however, is of little importance, since the final solution for $N_0(x, v)$ cannot depend on the choice of $w(x)$, which affects only the rate of convergence. Moreover, we can be reasonably confident that the deviation of the $w(x)$ chosen above from the ideal $w(x)$ can have only a slight effect on the rate of convergence of (25.46).

25.5.4. *The polynomials $U_s(y)$ and $V_s(y)$*

We now consider the determination of the polynomials $p_s(x)$ and $p_s^\dagger(x)$ introduced in (25.45), for $w(x)$ given by (25.50), and the corresponding polynomials that would appear in expansions similar to (25.46) for the higher spherical harmonic moments $N_n(x, v)$. To start with, we shall discuss the $p_s(x)$ and $p_s^\dagger(x)$ apearing in the expansion of $N_0(x, v)$. Hitherto we have spoken of both these as polynomials, but we have used only the fact that the $p_s^\dagger(x)$ are polynomials; the $p_s(x)$ might be any simple functions, provided that the $w(x)p_s(x)$ form a complete set and (25.45) is satisfied. In particular, the $p_s(x)$ might be polynomials in $|x|$ instead of x; the $p_s^\dagger(x)$, however, must be polynomials in x, since otherwise we should have integrals of the form

$$\int_{-\infty}^{\infty} |x|^{2s+1} N_0(x, v)\, dx,$$

and these do not satisfy the same recurrence relations as were used for the $m_{2s,0}(v)$.

Since both $N_0(x, v)$ and $w(x)$ are even functions of x, $p_s(x)$ should be so also. Thus either $p_s(x)$ should include only even powers of x, i.e. be a polynomial in x^2, or it should be a polynomial in $|x|$, as mentioned in the last paragraph. The polynomials $p_s^\dagger(x)$ should also be even functions of x, though for a somewhat different reason. If $p_s^\dagger(x)$ contained odd powers of x, these would give no contribution to either (25.45) or (25.47). Only terms containing even powers of x are therefore relevant. In other words $p_s^\dagger(x)$ may be regarded as a polynomial in x^2. Finally, in order that (25.45) should determine the polynomials in question completely, the same number of coefficients should be involved in $p_s(x)$ and $p_s^\dagger(x)$ for a given s. We thus have two alternatives:

(*a*) Both $p_s(x)$ and $p_s^\dagger(x)$ are polynomials of order s in x^2.

(*b*) While $p_s^\dagger(x)$ is such a polynomial, $p_s(x)$ is a polynomial of order s in $|x|$.

It is found that the latter alternative has a number of advantages. In particular, polynomials are obtained whose coefficients are constructed according to a much simpler law, and this greatly simplifies the subsequent algebraic manipulation. With alternative (*b*), it is evident that the functions $p_s(x)w(x)$ form a complete set of even functions tending to zero as $|x|$ tends to infinity. We shall therefore take alternative (*b*). The polynomials $p_s(x)$ and $p_s^\dagger(x)$ occurring in this case are denoted by

$U_s(y)$ and $\alpha U_s^\dagger(y)$ respectively (following Spencer and Fano), where we have put $y = \alpha|x|$.

Before giving the explicit forms of these polynomials, we shall consider what kind of polynomials would be used in corresponding expansions of the higher spherical harmonic moments $N_n(x, v)$. According to the symmetry properties of these moments mentioned in § 22.3, we have

$$N_n(-x, v) = (-1)^n N_n(x, v)$$

so that, if the crudest approximation to $N_0(x, v)$ is of the form constant times $e^{-\alpha|x|}$, the corresponding approximation to $N_n(x, v)$ will be of the form constant times (sign $x)^n e^{-\alpha|x|}$. In other words, the same $w(x)$ can be used in determining $N_n(x, v)$, but $p_s(x)$ and $p_s^\dagger(x)$ satisfy the symmetry conditions

$$p_s(-x) = (-1)^n p_s(x), \qquad p_s^\dagger(-x) = (-1)^n p_s^\dagger(x).$$

Thus, for even n, we can use the same polynomials as before, i.e. $U_s(y)$ and $U_s^\dagger(y)$, while for odd n slightly different polynomials must be used, because of the different symmetry properties. It can be shown that, for odd n, $p_s^\dagger(x)$ should involve only odd powers of x, while $p_s(x)$ should be of the form

$$\text{sign } x \times \text{polynomial in } |x|. \tag{25.54}$$

All the $N_n(x, v)$ for odd n vanish at $x = 0$, so that the polynomials appearing in (25.54) may be restricted to those which vanish for $|x| = 0$. Thus (25.54) may be replaced by

$$x \times \text{polynomial in } |x|. \tag{25.55}$$

Following Spencer and Fano, we denote the resulting $p_s(x)$ and $p_s^\dagger(x)$ for odd n by $\alpha x V_s(y)$ and $\alpha V_s^\dagger(y)$ respectively.‡

We shall now determine explicitly the polynomials introduced above. For the polynomials $U_s^\dagger(y)$, the condition (25.45) gives

$$\int_0^\infty y^{s'} U_s^\dagger(y) e^{-y}\, dy = 0 \quad (s' = 0, 1, \ldots, s-1); \tag{25.56}$$

i.e. if

$$g_s(k) = \int_0^\infty U_s^\dagger(y) e^{-ky}\, dy,$$

we have $d^{s'} g_s(k)/dk^{s'} = 0$ for $k = 1$ and $s' = 0, 1, \ldots, s-1$, and therefore $g_s(k)$ has an s-fold zero at $k = 1$. Changing the signs of y and k, we see that $g_s(k)$ also has such a zero at $k = -1$. Since $U_s^\dagger(y)$ is a polynomial of

‡ It is of interest to notice that the Laguerre polynomials, which might have been expected to appear with our choice of $w(x)$, do not do so, because they do not have the necessary symmetry properties. The polynomials we have introduced are, of course, closely related to the Laguerre polynomials and have many similar properties; cf. the recurrence relations (25.59).

order $2s$ in y, $g_s(k)$ is a polynomial of order $2s+1$ in $1/k$, and vanishes for $k = \infty$. This gives

$$g_s(k) = \frac{C_s^\dagger}{k}\left(1-\frac{1}{k^2}\right)^s, \tag{25.57}$$

where $C_s^\dagger$ is some constant. Since $U_s^\dagger(y)$ can be arbitrarily normalized, provided that $U_s(y)$ is suitably normalized, we can take $C_s^\dagger$ in (25.57) to be unity. Applying the formula for the inverse Laplace transformation to $g_s(k)$, we obtain

$$\left.\begin{aligned} U_s^\dagger(y) &= \sum_{s'=0}^{s} \frac{1}{(2s')!}\binom{s}{s'}(-y^2)^{s'}, \\ \text{and similarly we have} \\ V_s^\dagger(y) &= -\sum_{s'=1}^{s+1} \frac{1}{(2s'-1)!}\binom{s+1}{s'}\frac{(-y^2)^{s'}}{y}. \end{aligned}\right\} \tag{25.58}$$

These formulae lead to the following recurrence relations:

$$\left.\begin{aligned} dU_s^\dagger(y)/dy &= -V_{s-1}^\dagger(y) \\ dV_s^\dagger(y)/dy &= \sum_{s'=0}^{s} U_{s'}^\dagger(y) \end{aligned}\right\}, \tag{25.59}$$

which closely resemble the recurrence relation for the Laguerre polynomials.

We now consider the polynomials $U_s(y)$ and $V_s(y)$. For these, the condition (25.45) and its analogue for odd n give

$$\int_0^\infty y^{2s'} U_s(y) e^{-y}\, dy = 0 \quad (s' = 0, 1, \ldots, s-1), \tag{25.60 a}$$

and

$$\int_0^\infty y^{2s'+2} V_s(y) e^{-y}\, dy = 0 \quad (s' = 0, 1, \ldots, s-1), \tag{25.60 b}$$

the subscript s in each case giving the order of the polynomial concerned. To use these relations, we put

$$U_s(y) = C_s(y^s + u_{s,1}\, y^{s-1} + \ldots + u_{s,s}), \tag{25.61}$$

and thence find

$$\begin{aligned} \int_0^\infty y^t U_s(y) e^{-y}\, dy &= C_s[\Gamma(t+s+1) + u_{s,1}\,\Gamma(t+s) + \ldots + u_{s,s}\,\Gamma(t+1)] \\ &= C_s\,\Gamma(t+1)\{(t+s)(t+s-1)\ldots(t+1) + u_{s,1}(t+s-1)\ldots(t+1) + \ldots + u_{s,s}\}. \end{aligned} \tag{25.62}$$

The quantity in the braces in (25.62) is a polynomial of order s in t with

the leading coefficient equal to unity; according to (25.60 a), it should vanish for $t = 0, 2, \ldots, 2(s-1)$. This quantity is therefore equal to $t(t-2)\ldots(t-2s+2)$. Hence

$$u_{s,s} = [t(t-2)\ldots(t-2s+2)]_{t=-1} = (-1)^s 2^s \Gamma(s+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}),$$

$$u_{s,s-1} = \left[\frac{t(t-2)\ldots(t-2s+2)-(-1)^s(2s-1)\ldots 3.1}{t+1}\right]_{t=-2}$$

$$= (-1)^{s-1}2^s[s!-\Gamma(s+\tfrac{1}{2})/\Gamma(\tfrac{1}{2})],$$

and in general

$$u_{s,s-j} = (-1)^{s-j}2^s \sum_{i=0}^{j} \frac{(-1)^{j-i}}{i!(j-i)!} \frac{\Gamma(s+\frac{1}{2}i+\frac{1}{2})}{\Gamma(\frac{1}{2}i+\frac{1}{2})}.$$

Substituting in (25.61), determining C_s from the normalization condition (25.45) with (25.58), and rearranging the resulting expression, we can rewrite the expression for $U_s(y)$ as

$$\left.\begin{aligned} U_s(y) &= \frac{(-1)^s}{2^s s!}\left(\frac{\partial}{\partial y}-1\right)^{2s} \sum_{j=0}^{s} \frac{(s+j)!}{j!(s-j)!2^j} y^{s-j}, \\ \text{and similarly we derive from (25.60 b)} \quad V_s(y) &= \frac{1}{2(s+1)}\left(\frac{\partial}{\partial y}-1\right)^2 U_s(y). \end{aligned}\right\} \tag{25.63}$$

25.5.5. *The equations for the moments*

We can now write explicitly the equations for the various $a_{s,n}(v)$. These could be derived by starting with the equations for the $m_{s,n}(v)$ given in § 22.3, combining them with (25.48) and (25.58). However, it is simpler to derive the equation for $a_{s,n}(v)$ *ab initio*, starting directly from (22.2). It will now be more convenient to change the suffixes according to the order of the corresponding polynomials $p_s^\dagger(v)$. Since $p_s^\dagger(v)$ is of the order $2s$ or $2s+1$ according as n is even or odd, the quantities called $a_{s,n}(v)$ in § 25.5.2 will now be called $a_{2s,n}(v)$ or $a_{2s+1,n}(v)$, according as n is even or odd.

With this changed notation, and using the particular $w(x)$, $p_s(x)$, and $p_s^\dagger(x)$ chosen, the series (25.46) and its analogues become

$$\left.\begin{aligned} N_n(x,v) &= e^{-\alpha|x|} \sum_{s=0}^{\infty} a_{2s,n}(v) U_s(\alpha|x|) \qquad \text{if } n \text{ is even} \\ \text{and} \quad N_n(x,v) &= e^{-\alpha|x|} \sum_{s=0}^{\infty} a_{2s+1,n}(v) \alpha x V_s(\alpha|x|) \quad \text{if } n \text{ is odd} \end{aligned}\right\}, \tag{25.64}$$

whilst we have instead of (25.47)

$$\left.\begin{aligned} a_{2s,n}(v) &= \tfrac{1}{2}\alpha \int_{-\infty}^{\infty} N_n(x,v) U_s^{\dagger}(\alpha x)\, dx \quad \text{if } n \text{ is even} \\ a_{2s+1,n}(v) &= \tfrac{1}{2}\alpha \int_{-\infty}^{\infty} N_n(x,v) V_s^{\dagger}(\alpha x)\, dx \quad \text{if } n \text{ is odd} \end{aligned}\right\}. \tag{25.65}$$

Multiplying equation (22.2) for even n by $\frac{1}{2}\alpha U_s^{\dagger}(\alpha|x|)$, integrating, eliminating $\partial N_{n+1}(x,v)/\partial x$ and $\partial N_{n-1}(x,v)/\partial x$ by integration by parts, and using the recurrence relations (25.59) and the formulae (25.65), we obtain

$$\alpha\big[(n+1)a_{2s-1,n+1}(v)+na_{2s-1,n-1}(v)\big]+(2n+1)\Big[\frac{1}{l_0(v)}+\sum_k \frac{1}{l_{s,k}(v)}\Big]a_{2s,n}(v)$$

$$= (2n+1)\sum_k \frac{(M_k+1)^2}{2M_k} \int_{\frac{M_k+1}{M_k-1}v}^{\infty} \frac{dv'}{v' l_{s,k}(v')} a_{2s,n}(v') P_n\Big[\frac{(M_k+1)v^2-(M_k-1)v'^2}{2vv'}\Big]+$$

$$+\frac{\alpha S}{2v_0}\delta(v-v_0)U_s^{\dagger}(0)\delta_{n0} \quad \text{if } n \text{ is even}, \tag{25.66 a}$$

and similarly

$$-\alpha\sum_{s'=0}^{s}\big[(n+1)a_{2s',n+1}(v)+na_{2s',n-1}(v)\big]+(2n+1)\Big[\frac{1}{l_0(v)}+\sum_k \frac{1}{l_{s,k}(v)}\Big]a_{2s+1,n}(v)$$

$$= \text{the right side of (25.66 a) with } a_{2s+1,n} \text{ instead of } a_{2s,n}, \text{ if } n \text{ is odd.} \tag{25.66 b}$$

Using (22.45), and rewriting (25.48) in the new suffix convention, it can be seen that $a_{2s,n}(v)$ and $a_{2s+1,n}(v)$ will vanish for $2s < n$ and $2s+1 < n$ respectively. This can also be seen directly from the form of the equations (25.66). Thus the various $a(v)$ can be determined one by one in the following order: $a_{0,0}(v)$, $a_{1,1}(v)$, $a_{2,0}(v)$, $a_{2,2}(v)$, $a_{3,1}(v)$, $a_{4,0}(v)$, $a_{3,3}(v)$, etc.

Note added in second impression. An important modification of Spencer and Fano's method is described by J. Certaine, Nuclear Development Associates report NYO 6270, 1956. This modification concerns the manner in which $N_0(x,v)$ is reconstructed from the moments.

XXVI

HOLTE'S METHOD

26.1. Holte's method

THE last method we shall consider is that of Holte (24). This, in its present form, is restricted to the case of a single non-capturing element, the mean free path being assumed to tend to a finite limit as the velocity tends to zero. This method is more complex than those we have previously discussed, and its applications hitherto have therefore been very limited. It deserves attention, however, since it is the only method which covers the entire range of velocities from $v = v_0$ to $v = 0$, and is not restricted to the case where the mean free path decreases monotonically with the velocity.

For a single non-capturing element, it is clearly more convenient to take the variation of the mean free path as given by (25.3), i.e.

$$l(v) = l(v_0) \sum_j B_j (v/v_0)^{\beta_j}, \tag{26.1}$$

and not by (25.1). Whereas Holte considers the case of a point source in an infinite medium, we shall describe the method as applied to the problem of a plane source, which we have studied in the preceding chapters. The two cases are related by the expression (5.44). With $l(v)$ given by (26.1), the solution of the plane-source problem begins from the equations (25.5):

$$ipl(v_0) \sum_j B_j[(n+1)\pi_{n+1}(p, \eta+\beta_j)+n\pi_{n-1}(p, \eta+\beta_j)]+ \\ +(2n+1)h_n(\eta)\pi_n(p, \eta) = S\delta_{n0}/v_0^2, \tag{26.2}$$

where we have put $h_n(\eta)$ for $1-g_n(\eta)$. It may be noticed that $\sum B_j = 1$, and $\beta_0 = 0$.

26.2. The case of constant mean free path

26.2.1. *The determination of the transforms*

We shall first illustrate Holte's method for constant mean free path. The equations (26.2) then become

$$ipl(v_0)[(n+1)\pi_{n+1}(p, \eta)+n\pi_{n-1}(p, \eta)]+(2n+1)h_n(\eta)\pi_n(p, \eta) = S\delta_{n0}/v_0^2, \tag{26.3}$$

since $B_0 = 1$. The solution of these equations is given by the continued fraction (24.4):

$$\pi_0(p,\eta) = \frac{S/v_0^2}{h_0(\eta)+}\,\frac{p^2l^2}{3h_1(\eta)+}\,\frac{4p^2l^2}{5h_2(\eta)+}\cdots. \tag{26.4}$$

We have already remarked (see the discussion following (24.4)) that, if $h_n(\eta)$ tends to 1 as n tends to infinity, the continued fraction converges, except at isolated poles, in the complex p-plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l$, and from $p = i/l$ to $p = i\infty$.

The transform $\pi_0(p,\eta)$ is inverted by the method of §§ 24.2.2 and 24.2.3 to give $N_0(x,v)$ (see (25.4)). It is seen from (26.4) that, when $h_n(\eta) \cong 0$, the continued fraction $\pi_0(p,\eta)$ has poles for which p is small, and these give a large residue because the index of the exponential factor e^{-ipx} is small. The value of p at the principal pole $p_0 = -i/L(\eta)$ is small when $h_0(\eta) = 0$, i.e. when $\eta = 2$. We shall write

$$p_0 = -i\omega_0(\eta)/l. \tag{26.5}$$

Holte considers p_0 as a function of $\zeta = \eta - 2$. When $\zeta \to \infty$ (neutrons of the initial energy), $\omega_0(\zeta) \to 1$. Near $\zeta = 0$, $[\omega_0(\zeta)]^2$ can be expanded as a power series. From (26.4), the leading term in this expansion is ζ, and therefore

$$[\omega_0(\zeta)]^2 = C^2\zeta(1-b_1\zeta-b_2\zeta^2-\ldots)^2. \tag{26.6}$$

The constants C^2, b_1, b_2,... depend on the values of the $h_n(\zeta)$ and their derivatives at the point $\zeta = 0$.

If the solution is not to be too restricted, it is necessary to know $\omega_0(\zeta)$ for much larger values of ζ. This is achieved by analytic continuation† of the function $\omega_0(\zeta)$.

If p lies in the cut p-plane, and ζ in that part of the ζ-plane where $h_n(\zeta)$ tends to 1 as ζ tends to infinity, and if $1/\pi_0(p,\zeta)$ is regular in p and ζ in these domains, the analytic continuation of $p_0(\zeta)$ $(= -i\omega_0(\zeta)/l)$ satisfies

$$1/\pi_0[p_0(\zeta),\zeta] = 0.$$

This analytic continuation, when performed along the real positive ζ-axis and then along the same path of integration as was used in inverting $\phi_0(p,\eta)$ by the saddle-point method, always gives the

† We have seen in § 24.4 (Bethe's method) that, if we start with the integral equation, the convergence of the series involved is much improved; the number of $h_n(\eta)$ needed is therefore many fewer than when (26.4) is used directly. This implies that the exact form of $h_n(\eta)$ for large n cannot be important, and it should therefore be possible to remove the effects of the detailed behaviour of such $h_n(\eta)$, leaving only their general features, without actually using the integral equation. The analytic continuation of $\omega_0(\zeta)$ and of $A(\zeta)$ in (26.7) is the means of doing this employed in Holte's method.

principal pole $p_0(\zeta)$, since no branch-points of $1/\pi_0[p_0(\zeta), \zeta]$ occur on these paths.

The residue of π_0 at the principal pole is found to be

$$-3Sh_1 A(\zeta)/2\omega_0(\zeta)v_0^2 il, \tag{26.7}$$

where $h_1 \equiv h_1(0)$; this expression contains a function $A(\zeta)$ such that $A(0) = 1$. For small ζ, $A(\zeta)$ can be expanded as a power series

$$A(\zeta) = 1+a_1\zeta+a_2\zeta^2+..., \tag{26.8}$$

and, for points outside the radius of convergence of this series, $A(\zeta)$ can be calculated by analytic continuation.

Thus $N_0(x, v)$ is given by

$$N_0(x,v) = \frac{3Sh_1}{4\pi v_0^2 i}\int_{L_\zeta} \frac{A(\zeta)}{\omega_0(\zeta)} e^{u+\frac{1}{2}u\zeta-x\omega_0(\zeta)M}\, d\zeta, \tag{26.9}$$

where $u = 2\log(v_0/v)$, and L_ζ is a straight line parallel to the imaginary axis and lying to the right of all singularities.

The coefficients involved in the above expressions can be determined as follows. The constant C^2 is given by

$$C^2 = 3h_0' h_1, \tag{26.10}$$

where $h_0' \equiv [\partial h_0/\partial\zeta]_{\zeta=0}$. For $M \gg 1$,

$$C \cong \sqrt{\left\{\frac{3}{M}\left(1-\frac{2}{3M}+\frac{1}{6M^2}+...\right)\right\}}.$$

Next, let γ and δ be defined as follows:

$$\gamma = \sqrt{(M^2-1)}, \qquad \delta = \log[(M+1)/(M-1)].$$

It can be shown from formula (24.3) that

$$e^{-\frac{1}{2}\delta\eta}\frac{d^i}{d\eta^i}[e^{\frac{1}{2}\delta\eta}g_n(\eta)] = \frac{(M+1)^2}{2M}\int_{-1}^{1} P_n(\mu)\log^i\left\{\sqrt{\frac{M+1}{M-1}}G(\mu)\right\}\frac{[G(\mu)]^\eta}{\sqrt{(\mu^2+\gamma^2)}}\, d\mu. \tag{26.11}$$

It may be noted here that this formula differs from Holte's formula (57) in that our η corresponds to his 2η. This is because Holte uses the Laplace transform with respect to the lethargy, and not the Mellin transform with respect to the velocity. In (26.11)

$$G(\mu) = \{\mu+\sqrt{(\mu^2+\gamma^2)}\}/(M+1),$$

and therefore

$$\sum_{k=0}^{i}\binom{i}{k}(\tfrac{1}{2}\delta)^k(g_n)_{\eta-2}^{(i-k)} = \frac{\gamma(M+1)}{2M}\int_{-1}^{1} P_n(\mu)\{D(\mu/\gamma)\}^i E(\mu/\gamma)\,d\mu, \tag{26.12}$$

where

$$D(t) = \log\{\sqrt{(1+t^2)}+t\}, \qquad E(t) = 1+t/\sqrt{(1+t^2)}.$$

The integrals in (26.12) are easily calculated by expanding D and E:

$$D(t) = \sum_{\nu=0}^{\infty}\binom{-\frac{1}{2}}{\nu}\frac{t^{2\nu+1}}{2\nu+1}, \qquad E(t) = 1+t\sum_{\nu=0}^{\infty}\binom{-\frac{1}{2}}{\nu}t^{2\nu},$$

from which we see that $tD(t)$ and $\{E(t)-1\}/t$ are functions of $t^2 = \mu^2/\gamma^2$. We have to calculate the terms appearing in the series expansion of

$$D^i(t)E(t) = D^i(t)+[E(t)-1]D^i(t).$$

Even for $M = 2$, only the first few terms of these expansions need be computed. As i increases (n being small), the integral becomes very small. The resulting integrals are of the type

$$I_{nk} = \int_{-1}^{1} P_n(\mu)\mu^k\,d\mu.$$

This vanishes except when $k-n$ is a positive even integer or zero, and then

$$I_{0k} = 2/(k+1), \quad I_{1k} = 2/(k+2), \quad I_{n+2,k} = I_{nk}(k-n)/(k+n+3).$$

It is now useful to divide the function D^iE into an even and an odd function. When $i-n$ is odd, the expansion of the integral in terms of $1/\gamma^2$ begins with $1/\gamma^{i+1}$ if $i+1 \geqslant n$, and with $1/\gamma^n$ otherwise, except for $i = 0$, when the integral vanishes for $n \geqslant 3$. When $i-n$ is even, the leading term in the integral is of the order of $1/\gamma^i$ if $i \geqslant n$ and $1/\gamma^n$ otherwise. Since δ and $1/\gamma$ are of the order of $1/M$, $g_n^{(i)}$ (the ith derivative of $g_n(\eta)$ for $\eta = 2$) is $O(1/M^i)$ for $i \geqslant n$ and $O(1/M^n)$ for $i \leqslant n$. It then follows, by repeated differentiation of the continued fraction, that b_i is of the order of $1/M^i$, and so is the coefficient $\bar{b}_i$ of $-\zeta^i$ in the series obtained by explicitly squaring the series in parentheses in (26.6).

The coefficients a_i in the series (26.8) can be calculated from the formula

$$a_i = -\partial(h_0'\bar{b}_i)/\partial h_0', \tag{26.13}$$

which is proved by Holte.

Thus the coefficients b_i, $\bar{b}_i$, and a_i are all of the order of $1/M^i$. The radius of convergence of these power series is therefore approximately proportional to M. It follows from numerical results that this radius is

nearly $\frac{1}{2}M$ for all M and for both $\omega_0(\zeta)$ and $A(\zeta)$. When ten terms of the series are taken, the results are accurate for $|\zeta| \leqslant 0{\cdot}4M$. For $M > 1$ only the first few $g_n(\eta)$ need be taken into account; those as far as $n = 4$ for $M = 12$ (carbon), and $n = 6$ for $M = 2$ (deuterium) are the only ones required.

The analytic continuation of (26.6) and (26.8) is done by the classical method of Taylor series, taking $1/\zeta$ as the variable instead of ζ, and putting

$$y = 1/\zeta, \qquad H(y) = \omega_0(1/y), \qquad L(y) = A(1/y).$$

Taking a real positive y_0 such that $\zeta_0 = 1/y_0$ is fairly small compared with the radius of convergence, we calculate the values of $H(y)$ and $L(y)$ and their derivatives at $y = y_0$, and form the Taylor expansions at this point. Another point y_1 is then taken on the real axis such that $\zeta_1 = 1/y_1$ is greater than ζ_0, but still a little less than the radius of convergence. The Taylor series at y_1 are calculated from those at y_0. We also put

$$H(y) = H_0 + H_1(y-y_1) + \ldots, \qquad L(y) = L_0 + L_1(y-y_1) + \ldots . \tag{26.14}$$

The intermediate point y_1 is taken for convenience in numerical work. The number of terms in the series depends, of course, on the number of the b_i, $\bar{b}_i$, and a_i that have been computed, and this also determines the accuracy of the values obtained for large ζ. This accuracy can be tested by putting $y = 0$ in (26.14), since $H(0) = 1$ and $L(0) = 0$, from the above.

However, although the series (26.14) seem to converge well even for $y = 0$, more accurate values can be obtained for very small $|y|$ ($\zeta \to \infty$) by using expansions about $y = 0$. The analytic continuation is extended to $y = 0$, calculating $H(0)$, $H'(0)$,..., $L(0)$, $L'(0)$, etc., and completing (26.14) if necessary with extra terms such that $H(0) = 1$ and $L(0) = 0$ exactly. Although this procedure is arbitrary, the correction terms are found in practice to be small:

$$H(y) = 1 - k_1 y - k_2 y^2 - \ldots, \qquad L(y) = m_1 y + m_2 y^2 + \ldots . \tag{26.15}$$

The radius of convergence of these series for $\zeta = 0$, $\zeta = \infty$, and $\zeta = \zeta_1$ can be investigated numerically; it seems that they have a common region of convergence in the ζ-plane near $\zeta = 0$ and $\zeta = \infty$.

Some simple approximate formulae $\omega_s(\zeta)$ for $\omega_0(\zeta)$ are given by Holte:

$$\omega_s(\zeta) = C\sqrt{\zeta}(1 - b_1\zeta) \quad \text{for small } \zeta, \tag{26.16}$$

$$\omega_s(\zeta) = a - b/(c+\zeta) \quad \text{for } \zeta \text{ real and positive,}$$

where the constants a, b, and c are found numerically. Their values are different in various ranges of ζ, but in each range the approximation is

good, not only to $\omega_0(\zeta)$ but also to $\omega_0'(\zeta)$. This method differs from one due to Wick (54) in that the latter gives only a numerical method of finding $A(\zeta)$ for intermediate values of ζ, each value being taken separately.

26.2.2. *The neutron density*

Having obtained expressions for $A(\zeta)$ and $\omega_0(\zeta)$, we can now perform the integration in (26.9). As has been explained in § 24.2.3, the path of integration L_ζ in the ζ-plane is deformed so as to pass through a saddle-point along a path of steepest descent. In the present case there is only one saddle-point, which is always on the positive real axis. Since the integral is desired as an analytical function of u and x, we proceed as follows. The exponential function in (26.9), which is the rapidly varying function, is taken with $\omega_0(\zeta)$ replaced by $\omega_s(\zeta)$, the difference being so small that even the exponential of $(-x/l$ times this difference) varies slowly, as does $A(\zeta)$.

To find the saddle-point, we differentiate the index of the exponential in (26.9), with $\omega_s(\zeta)$ instead of $\omega_0(\zeta)$, and obtain

$$\tfrac{1}{2}u-(x/l)\omega_s'(\zeta) = 0. \tag{26.17}$$

Let the root of this equation be $\zeta = \zeta_s$. Since $\omega_0'(\zeta)$ and $\omega_s'(\zeta)$ are real positive monotonically decreasing functions when ζ varies along the positive real axis, and tend to zero as ζ tends to infinity, ζ_s must be real and positive, increasing monotonically with $2x/ul$, and $\zeta_s = 0$ for $2x/ul = 0$. The division of the real ζ-axis into intervals with various values of a, b, and c corresponds to a similar division of the values of $2x/ul$.

To evaluate the integral, we now expand the integrand about $\zeta = \zeta_s$, and expand the exponential in all but the first and third terms.

A slight modification is necessary since $\omega_0(\zeta)$ has a branch point at $\zeta = 0$, and this means that, as $2x/ul$ tends to zero, i.e. $\zeta_s \to 0$, we must put $\zeta = \phi^2$ and $\phi_s = +\sqrt{\zeta_s}$. Then

$$N_0(x,v) = \frac{3Sh_1}{2\pi i v_0^2}\int \frac{\phi A(\phi^2)}{\omega_0(\phi^2)} e^{u+\frac{1}{2}u\phi^2-x\omega_s(\phi^2)/l}\, d\phi, \tag{26.18}$$

where the integral is taken along the appropriate path, and for small ζ_s, where

$$\omega_s(\zeta) \cong C\sqrt{\zeta}(1-b_1\zeta),$$

we have

$$\sqrt{\zeta_s} = \sqrt{2xC/ul}\{1+\sqrt{(1+12C^2b_1x^2/l^2u^2)}\} \quad \text{for small } x/ul. \tag{26.19}$$

The last relation is applicable if

$$\zeta_s = \phi_s^2 \leqslant 1/12b_1,$$

and this may be taken as an approximate upper limit to the validity of (26.16). Hence the condition of applicability is

$$x/ul \leqslant \sqrt{(2/3C^2b_1)} \quad (\sim \tfrac{1}{2}M \text{ for } M \text{ large}).$$

For larger values of x/ul, the expression $a-b/(c+\zeta)$ must be used, with the appropriate choice of a, b, and c.

When ζ_s has been found, $\frac{1}{2}u\phi^2-(x/l)\omega_0(\phi^2)$ is expanded in a Taylor series about ϕ_s, and the same is done for $\phi A(\phi^2)$, taking $\phi = \phi_s+it$. The following expression is obtained for $N_0(x, v)$:

$$N_0(x, v) = \frac{3Sh_1[\frac{1}{2}u-x\omega_s''(\phi_s^2)/2l\phi_s]^{-\frac{1}{2}}\phi_s}{2\sqrt{\pi}\, v^2\omega_0(\phi_s^2)} e^{\frac{1}{2}iu\zeta_s-x\omega_s(\zeta_s)/l} \times$$

$$\times \text{a slowly-varying function of } x/ul. \quad (26.20)$$

If $\omega_s(\zeta) = C\sqrt{\zeta}$, $\omega_s'(\zeta) = C/2\sqrt{\zeta} = ul/2x$, and so $\sqrt{\zeta_s} = Cx/ul$. Thus

$$\tfrac{1}{2}u\zeta_s-x\omega_s(\zeta_s)/l = -x^2C^2/2ul^2,$$

and $\omega_s(\phi_s^2) = C^2x/2ul$. For small x/ul, the function of x/ul in (26.20) can be approximated by unity, and so

$$N_0(x, v) = \frac{3Sh_1}{\sqrt{(2\pi u)}Cv^2} e^{-C^2x^2/2ul^2}, \quad (26.21)$$

the result given by age theory. The higher approximations give correction terms (§ 23.6); these were first derived by Marshak (38).

For $x/ul > \sqrt{(2/3C^2b_1)}$, we have $\omega_s(\zeta) = a-b/(c+\zeta)$, so that

$$\zeta_s = \sqrt{(2bx/ul)}-c,$$

and

$$N_0(x, v) = \frac{3Sh_1}{2\sqrt{\pi}\, v^2} e^{-\frac{1}{2}iuc-(ax/l)+\sqrt{(2bxu/l)}} \times \text{a s.v. function of } x \text{ and } u. \quad (26.22)$$

As $x/ul \to \infty$, $a \to 1$, $b \to k_1$, $c \to -k_2/k_1$, and we obtain Wick's asymptotic formula (§ 24.5.4).

26.3. The case of variable mean free path

26.3.1. *A survey of the singularities*

We now take the case of a variable mean free path, and as before we shall concentrate on the determination of $\pi_0(p, \eta)$. Let this function be represented in the form

$$\pi_0(p, \eta) = \pi_{0,\lambda(0)}(p, \eta)G_0(p, \eta), \quad (26.23)$$

where $\pi_{0,\lambda(0)}(p, \eta)$ is the value of $\pi_0(p, \eta)$ for a fictitious medium in which the mean free path is constant and equal to its actual value for $v = 0$. That is, $\pi_{0,\lambda(0)}(p, \eta)$ is the solution of (26.2) with $B_j = 0$ for $j \geqslant 1$. Of course, this $\pi_{0,\lambda(0)}(p, \eta)$ is defined only in the complex p-plane cut along

the imaginary axis from $p = -i\infty$ to $p = -i/l(0)$ and from $p = i/l(0)$ to $p = i\infty$. Thus $G_0(p, \eta)$ also is defined only in this cut plane. Let us first survey the singularities of $G_0(p, \eta)$ as a function of p in the cut plane, η being fixed.

It has been shown in § 25.2 that singularities of $\pi_0(p, \eta)$ can arise only in one of two ways: either because p is an eigenvalue of (25.13), or because some $\pi_n(p, \eta+\beta_j)$ for $j \geqslant 1$ is singular. (It is assumed as before that $\beta_0 = 0$ and $\beta_j > 0$ for $j \geqslant 1$.) In the former case $\pi_{0,K(0)}(p, \eta)$ has in general the same kind of singularity as $\pi_0(p, \eta)$, and $G_0(p, \eta)$ is therefore regular.† The singularities of $G_0(p, \eta)$ for some $\eta = \eta_0$ (say) can arise only if some $\pi_n(p, \eta_0+\beta_j)$ for $j \geqslant 1$ is itself singular.‡ This, in turn, can occur only in one of two ways: either p is an eigenvalue of (25.13) for $\eta = \eta_0+\beta_j$ or $\eta = \eta_0+\beta_j+\beta_{j'}$ or etc., or else

$$\pi_n\Big(p, \eta_0+\sum_{k=0}^{\infty}\beta_{j_k}\Big) \quad \text{for } j_k \geqslant 1,$$

i.e. $\pi_n(p, \infty)$, is singular for the value of p concerned.

Since $\pi_n(p, \infty)$ corresponds to neutrons of the initial energy, it should be regular in the complex p-plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l(v_0)$, and from $p = i/l(v_0)$ to $p = i\infty$, while $G_0(p, \eta)$ is (as we have seen) defined only in the complex p-plane cut along the imaginary axis from $p = -i\infty$ to $p = -i/l(0)$, and from $p = i/l(0)$ to $p = i\infty$. We are thus led to consider $G_0(p, \eta)$ only in the complex p-plane cut from $p = -i\infty$ to $p = -i/\max\{l(0), l(v_0)\}$ and from $p = i/\max\{l(0), l(v_0)\}$ to $p = i\infty$. It follows from the above arguments that the only singularities of $G_0(p, \eta_0)$ in this cut plane are poles at the eigenvalues of (25.13) for $\eta = \eta_0+\beta_j$, $\eta = \eta_0+\beta_j+\beta_{j'}$, etc. $(j, j', \ldots \geqslant 1)$. Since these eigenvalues cause the poles of $\pi_{n,K(0)}(p, \eta)$, the result obtained can be expressed as follows. Let $p = \pm i\omega_\nu(\eta)/l(0)$ be the poles of $\pi_{n,K(0)}(p, \eta)$ as a function of p for η fixed. Then the only possible singularities of $G_0(p, \eta)$ in the complex p-plane cut as above are the poles at the points $p = \pm i\omega_\nu(\eta+\beta_j)/l(0)$, $p = \pm i\omega_\nu(\eta+\beta_j+\beta_{j'})/l(0)$, etc. $(j, j', \ldots \geqslant 1)$. If $l(0) < l(v_0)$, then, for any η, $G_0(p, \eta)$ can have only a finite number of poles in the strip $-1/l(v_0) < \operatorname{im} p < 1/l(v_0)$. This

† For general n, situations may be found where the singularity of $\pi_n(p, \eta)$ is due to both causes, i.e. p is an eigenvalue of (25.13) for the given η, and one of the $\pi_n(p, \eta+\beta_j)$ is singular. However, it can be shown that this cannot occur for $n = 0$.

‡ This statement assumes, of course, that $\pi_{0,l(0)}(p, \eta)$ does not vanish; if it did vanish for some p and η, while $\pi_0(p, \eta)$ were not zero, $G_0(p, \eta)$ would again have a singularity. Existing calculations, however, seem to confirm that $\pi_{0,l(0)}(p, \eta)$ has no zeros in the region of (p, η) space concerned. Nevertheless, Holte's method could easily be modified to take account of such extra singularities, should they exist.

follows from the results of § 25.2: we saw in that section that, if p lies in this strip and re η is sufficiently large ($> \sigma_\eta$, say), $\pi_0(p, \eta)$, and therefore $G_0(p, \eta)$, are regular. This means that $\pm i\omega_\nu(\eta+\beta_j)/l(0)$ can lie in this strip only for re $\eta < \sigma_\eta - \beta_j$, $\pm i\omega_\nu(\eta+\beta_j+\beta_{j'})/l(0)$ can lie in this strip only for re $\eta < \sigma_\eta - \beta_j - \beta_{j'}$, and so on. If $l(0) > l(v_0)$, on the other hand, all the poles lie in this strip (see § 24.2).

The following results can be given regarding the position of the poles lying in this strip. The points $p = \pm i\omega_\nu(\eta+\beta_j)/l(0)$ are the singularities of $\pi_{0,\chi(0)}(p, \eta+\beta_j)$, i.e. of the Fourier–Mellin transform for a problem where the mean free path is constant; their position can thus be examined by the methods of § 24.2. In particular, applying the argument preceding (24.5), it can be seen that, if for some β_j and some real η any of the $\pm i\omega_\nu(\eta+\beta_j)/l(0)$ lies in this strip, and none lies on the real axis, then the $\pm i\omega_\nu(\eta+\beta_j)/l(0)$ nearest the real axis should be purely imaginary. On the other hand, we should expect from the arguments of § 24.2.4 that, as re η increases, $|\omega_\nu(\eta)|$ also increases. Thus, if β_1 is the smallest non-zero β_j, and $|\omega_0(\eta)|$ is the smallest $|\omega_\nu(\eta)|$, then $\pm i\omega_0(\eta+\beta_1)/l(0)$ is nearer the real axis than any other $\pm i\omega_\nu(\eta+\beta_j)/l(0)$, and than any $\pm i\omega_\nu(\eta+\beta_j+\beta_{j'})/l(0)$ ($j, j' \geqslant 1$), and so on. That is, if $G_0(p, \eta)$ has, for some real η, poles in the strip

$$-1/\max\{l(0), l(v_0)\} < \operatorname{im} p < 1/\max\{l(0), l(v_0)\}, \tag{26.24}$$

and none of these is on the real axis, the two poles nearest the real axis are at

$$p = \pm i\omega_0(\eta+\beta_1)/l(0).$$

26.3.2. *The determination of the transform of the solution, for small p*

Having ascertained the region where $G_0(p, \eta)$ is defined and its singularities in that region, we shall now determine $G_0(p, \eta)$ explicitly, taking first the case of small p.

For small p, the equations (26.2) give at once

$$\pi_n(p, \eta) = \frac{S}{v_0^2}\left\{\frac{\delta_{n0}}{(2n+1)h_n(\eta)} - ipl(v_0)\sum_{j=0} B_j\left[\frac{(n+1)\delta_{n+1,0}}{(2n+3)h_{n+1}(\eta+\beta_j)} + \right.\right.$$
$$\left.\left. + \frac{n\delta_{n-1,0}}{(2n-1)h_{n-1}(\eta+\beta_j)}\right] + \ldots\right\}.$$

Putting $n = 0$ and dividing through by the corresponding series for $\pi_{0,\chi(0)}(p, \eta)$, we have

$$G_0(p, \eta) = \sum_{\nu=0}^{\infty} (ip)^{2\nu} G_{0,2\nu}(\eta). \tag{26.25}$$

The amount of algebraic work needed to evaluate the successive $G_{0,2\nu}(\eta)$ can be somewhat reduced by using certain symbolic operators, but for simplicity we omit this treatment, which can be found in Holte's paper.

The radius of convergence of (26.25) is determined by the position of the singularity of $G_0(p, \eta)$ nearest the origin. If $\mathrm{re}\,\eta > 2-\beta_1$ and p is real, the integral (25.4) defining $\pi_{0,l(0)}(p, \eta+\beta_1)$ converges, i.e. $\pi_{0,l(0)}(p, \eta+\beta_1)$ is regular for all real p. It therefore follows from § 26.3.1, assuming that $\pi_{0,l(0)}(p, \eta)$ has no zeros, that the singularities of $G_0(p, \eta)$ nearest the origin for real $\eta > 2-\beta_1$ are either

$$\left.\begin{aligned} p &= \pm i/l(v_0) \\ \text{or} \qquad p &= \pm i\omega_0(\eta+\beta_1)/l(0) \end{aligned}\right\}, \tag{26.26}$$

whichever is the smaller.

If, however, it is possible that $\pi_{0,l(0)}(p, \eta)$ may have zeros, the singularities of $G_0(p, \eta)$ mentioned in § 26.3.1 must be augmented by the singularities due to these zeros, and therefore we can draw no immediate conclusions concerning the radius of convergence of (26.25). However, when the coefficients in (26.25) have been calculated numerically, we can find their rate of increase and easily discover from this whether $\pi_{0,l(0)}(p, \eta)$ has any zeros in the circle

$$|p| < \min[1/l(v_0),\ \omega_0(\eta+\beta_1)/l(0)]. \tag{26.27}$$

It has been found that, in the applications of the method made hitherto, there are in fact no such zeros.†

26.3.3. *The case of moderate distances from the source when the mean free path varies only slightly*

The subsequent analysis depends on whether the poles

$$p = \pm i\omega_0(\eta)/l(0)$$

of $\pi_{0,l(0)}(p, \eta)$ nearest the real axis lie

(i) well within the circle of convergence of (26.25);

(ii) inside this circle but near its circumference;

(iii) outside this circle.

† Holte distinguishes the case where all the B_j are positive from that where some of them are negative. In the former case, for η real and greater than 2 ($\zeta > 0$), Holte shows that all the $G_{0,2\nu}(\eta)$ in (26.25) are positive, so that the singularities of $G_0(p, \eta)$ nearest to the origin lie on the imaginary axis. He therefore concludes that in this case the singularities in question are given by (26.26), whereas when some B_j are negative he fails to draw this conclusion. The distinction is difficult to understand, however, since the real criterion is whether $\pi_{0,l(0)}(p, \eta)$ has any zeros in the circle (26.27), and this cannot be affected by the signs of the B_j: the definition of $\pi_{0,l(0)}(p, \eta)$ involves only B_0, which is always positive.

We take first case (i), and start by determining the physical conditions to which it corresponds. It is assumed throughout the following analysis that $\pi_{0,\chi(0)}(p, \eta)$ has no zeros in the circle (26.27), so that the latter circle is the circle of convergence of (26.25).

If $\pm i\omega_0(\eta)/l(0)$ lies well within the circle (26.27), it also lies well within the strip (26.24). Then, applying the inversion formula to the double transform $\pi_0(p, \eta)$ and evaluating the inverse Fourier transform integral by the method of residues, we can easily verify the following facts as for the case of constant mean free path.

Firstly, the main contribution to $N_0(x, v)$ comes from the pole at $p = -i\omega_0(\eta)/l(0)$. The contribution from the next most important pole, that at $p = -i\omega_0(\eta+\beta_1)/l(0)$, is of the order of $\exp[-\beta_1 \log(v_0/v)]$, and so on, so that, if $\beta_1 \log(v_0/v)$ is sufficiently large, the contribution of all poles except the first may be neglected.

Secondly, the evaluation of the inverse Mellin transform integral for the pole at $p = -i\omega_0(\eta)/l(0)$ can be carried out by the method of steepest descent† (see § 24.2.3).

Thirdly, if η_0 is the saddle-point, η_0 tends to infinity with x. If η tends to infinity, $\omega_0(\eta)$ tends to unity, and so does $\omega_0(\eta+\beta_1)$, so that, if x is large, $-i\omega_0(\eta)/l(0)$ cannot lie well within the circle (26.27). Thus case (i) can be realized only for small or moderate distances from the source. If $l(0) < l(v_0)$, it is easily seen that case (i) can occur only if $\omega_0(\eta)$ is considerably less than $l(0)/l(v_0)$. The available range of η, and therefore of x, is thereby further restricted, and for $l(0)/l(v_0)$ sufficiently small there are no values of x which give case (i). This case can therefore arise only for $l(0)/l(v_0)$ greater than or not much less than unity, and for x fairly small. It can be shown that, under these conditions, case (i) does in fact occur.

We shall now determine the solution for this case. It is clear from the above that the leading term in this solution differs from that for constant mean free path only by the appearance, in the residue at the pole concerned, of an extra factor

$$G_0[i\omega_0(\eta)/l(0), \eta] = \bar{G}_0(\eta), \text{ say.} \tag{26.28}$$

Thus, if the residue of $\pi_{0,\chi(0)}(p, \eta)$ at $p = -i\omega_0(\eta)/l(0)$ is

$$-3Sh_1 A_0(\zeta)/2v_0^2 i\omega_0(\eta)l(0),$$

where $A_0(0) = 1$ and h_1 is the value of $h_1(\zeta)$ for $\zeta = 0$, see (26.7), the

† Though this is not quite accurate, the necessary correction (see § 26.3.4) is unimportant for the present argument.

corresponding residue of $\pi_0(p,\eta)$ is $-3Sh_1 A(\zeta)/2v_0^2 i\omega_0(\eta)l(0)$, where

$$A(\zeta) = A_0(\zeta)\bar{G}_0(\eta). \tag{26.29}$$

The quantity $\bar{G}_0(\eta)$ could have been calculated from (26.25). This, however, would involve needless algebra, and Holte suggests that this can be avoided by expanding $\bar{G}_0(\eta)$ in a Taylor series about $\eta = 2$, i.e. $\zeta = 0$. We now write $\bar{G}_0(\zeta)$. It can then be verified that, for $\zeta = 0$, $\omega_0(\zeta) = 0$, so that $\bar{G}_0(0) = G_0(0,0)$. For $p = 0$, $\pi_{0,K(0)}(p,\eta)$ is the same as $\pi_0(p,\eta)$, by the formula preceding (26.25), and therefore $G_0(0,0) = 1$. Thus

$$\bar{G}_0(\zeta) = 1 + \bar{G}_{01}\zeta + \bar{G}_{02}\zeta^2 + \ldots, \tag{26.30}$$

where the coefficients can be determined from those in (26.25). The radius of convergence of (26.30) can be estimated as follows. The singularities of $G_0(p,\zeta)$ are, as we have seen, at $p = \pm i/l(v_0)$ and at the various $p = \pm i\omega_\nu(\eta+\beta_j)/l(0)$ $(j \geqslant 1)$, $p = \pm i\omega_\nu(\eta+\beta_j+\beta_{j'})/l(0)$ $(j,j' \geqslant 1)$, etc. We know, however, that $\omega_0(\eta)$ cannot be equal to any $\omega_\nu(\eta+\beta_j)$ $(j \geqslant 1)$, etc. Thus these latter singularities of $G_0(p,\zeta)$ cannot give rise to singularities of $\bar{G}_0(\zeta)$, and the only singularity of the latter is at $\zeta = \zeta^*$, where ζ^* is the root of

$$\omega_0(\zeta^*) = l(0)/l(v_0). \tag{26.31}$$

The series (26.30) therefore converges for $|\zeta| < |\zeta^*|$.

Hitherto we have discussed only the contributions from the poles of $\pi_0(p,\eta)$. However, since $\pi_0(p,\eta)$ is defined in the cut p-plane, there will also be a contribution from the integral along the two sides of the cut. If $l(0) \geqslant l(v_0)$, this contribution is always negligibly small, as in the case of constant mean free path. If $l(v_0) > l(0)$, on the other hand, this contribution may become important. Holte has investigated the order of magnitude of such a contribution, and has obtained the following simple criterion. If $|\zeta_0| \ll |\zeta^*|$, where ζ_0 is the saddle-point and ζ^* is the solution of (26.31), the contribution from the cut may be neglected. If this inequality does not hold, then the cut contribution is comparable with that from the pole at $p = i\omega_0(\zeta_0)/l(0)$. We shall return to the latter case in § 26.4; meanwhile, we shall examine the case where $|\zeta_0| \ll |\zeta^*|$.

26.3.4. *The application of the method of steepest descent*

There is a certain correction which must be made to the solution so far developed. In applying the method of steepest descent, the integrand is divided into a fairly simple factor which varies rapidly and a slowly varying factor which is assumed to have little or no effect on the position of the saddle-point. Hitherto we have regarded $\bar{G}_0$ in (26.29) as a slowly

varying function, but a closer examination shows that this is a poor approximation. It has been found† that a much better one is obtained by putting

$$\bar{G}_0(\zeta) = G^*(\zeta)e^{a\zeta}, \tag{26.32}$$

where $$a = \frac{1}{2}\sum_{j=0}\sum_{j'=0}\frac{B_j B_{j'}}{B_0^2(\beta_j+\beta_{j'})} \quad (j = j' = 0 \text{ excluded}),$$

and $G^*(\zeta)$ is regarded as the slowly varying function.

This means that the position of the saddle-point ζ_0 is given (approximating ω_0 by ω_s as in § 26.2) by

$$\tfrac{1}{2}u + a = \{x/l(0)\}\omega_s'(\zeta_0), \tag{26.33}$$

and not by (26.17).

It can easily be shown that, on terminating the expansions involved at the leading term, and using (26.33) instead of (26.17), we reach the same expression for $N_0(x, v)$ as is given by age theory, whereas such agreement is not obtained on using (26.17), i.e. $\bar{G}_0(\zeta)$ and not $G^*(\zeta)$. On retaining further terms in the expansions, we find correction terms whose use carries us beyond the range of applicability of age theory. If $l(v_0) < l(0)$, analytic continuation can be used to obtain a formula analogous to (26.22) which is applicable at all distances. If, however, $l(0)/l(v_0) < 1$, then, no matter how close to unity this ratio is, ζ_0 will become very close to ζ^* beyond a certain distance; physically speaking, neutrons which have had a mean free path near $l(v_0)$ for most of their lifetimes will predominate. The procedure required in this case will be described in the following section.

26.4. The case where the mean free path decreases with the velocity

26.4.1. *Preliminary transformations*

Hitherto we have considered only case (i) of those mentioned at the beginning of § 26.3.3. We shall now consider cases (ii) and (iii). It is obvious from our previous remarks that these correspond physically to the situation where $l(0) < l(v_0)$ and x is fairly large. In case (i), the main contribution to the inverse Fourier transform comes from one of the poles of $\pi_0(p, \eta)$, but now it will come from the integral along the cut. For a fixed p, on the other hand, $\pi_0(p, \eta)$ is a meromorphic function of η, i.e. its only singularities are poles. It is therefore more profitable to invert the Mellin transform (i.e. integrate with respect to η) first, using the method of residues, and then to invert the Fourier transform.

† Holte states (without proof) that, as M, the nuclear mass, tends to infinity, $\bar{G}_0(\zeta)$ tends to $e^{a\zeta}$.

The function $\pi_0(p, \eta)$ has, of course, an infinite number of poles, but if we again assume that $\beta_1 \log(v_0/v)$ is fairly large we may neglect all poles except the 'main' one, i.e. that for which $\mathrm{re}\,\eta$ is greatest. This pole obviously corresponds to $p = \pm i\omega_0(\zeta)/l(0)$. If this pole is at $\zeta_0(p)$, we find by solving (26.6) for ζ

$$\zeta_0(p) = -\frac{[pl(0)]^2}{C^2} + \frac{\bar{b}_1[pl(0)]^4}{C^4} + \ldots, \tag{26.34}$$

and for large p (26.14) gives

$$\zeta_0(p) = \left\{y_1 + \frac{ipl(0) - H_0}{H_1} - \frac{H_2[ipl(0) - H_0]^2}{H_1^3} + \ldots\right\}^{-1}. \tag{26.35}$$

Denoting by $\tilde{A}(p)$ the result of substituting (26.34) into (26.29) (for p inside the circle (26.27)), or the analytic continuation of this function (otherwise), and applying the Fourier transform inversion formula, we have similarly to (26.18)

$$\frac{N_0(x, v)}{l(v)} = -\frac{3Sh_1}{4\pi v^2 l^2(0)} \int_{-\infty}^{\infty} \frac{\tilde{A}(p)\zeta_0'(p)e^{-ixp + \zeta_0(p)\log(v_0/v)}}{p}\, dp. \tag{26.36}$$

26.4.2. *The behaviour of the integrand*

Before we can proceed to evaluate (26.36) we have to examine the behaviour of the integrand, and in particular of $\tilde{A}(p)$. We have seen above that the important singularity of $G_0(\zeta)$, and therefore of $A(\zeta)$, is at $\zeta = \zeta^*$. According to the definition of ζ^* (see (26.31)) and of $\tilde{A}(p)$, the important singularities of $\tilde{A}(p)$ are therefore at $p = \pm i/l(v_0)$. If $l(v)$ varies monotonically and $dl(v)/dv \neq 0$ for $v = v_0$, we can use the analysis of § 25.4 (in particular, formula (25.43)) to show that the behaviour of $\tilde{A}(p)$ at these singularities is of the form

$$\tilde{A}(p) = Z(p)/[1 - ipl(v_0)]^\alpha = P(p)/[1 + p^2 l^2(v_0)]^\alpha, \tag{26.37}$$

where α is $|g_0|/\gamma$ in the notation of (25.43), $Z(p)$ is bounded at $p = -i/l(v_0)$, and $P(p)$ is bounded at both $p = \pm i/l(v_0)$.

The derivation of (25.43) shows that the first part of (26.37) can be written more explicitly as

$$\tilde{A}(p) = \sum_{g_m < 0} \frac{B_m(p)}{[1 - ipl(v_0)]^{|g_m|/\gamma}} + \tilde{B} + O[\{1 - ipl(v_0)\}\log\{1 - ipl(v_0)\}], \tag{26.38}$$

say, where the g_m and γ have the same meaning as in § 25.4, $\tilde{B}$ is some constant, and the $B_m(p)$ are regular at $p = -i/l(v_0)$.

The available numerical results concerning the eigenvalues of (25.36) suggest that, whenever (25.36) has more than one eigenvalue, the difference between the two most important eigenvalues is greater than γ. Thus, by comparing (26.38) with (26.37), we see that, near $p = -i/l(v_0)$, $Z(p)$ can be expanded as

$$Z(p) = Z_0+\{-ip+1/l(v_0)\}Z_1+\{ip-1/l(v_0)\}^{\alpha'}\bar{Z}+ \\ +\text{terms tending to zero faster than } (p+i/l(v_0))^{\max(1,\alpha')}, \quad (26.39)$$

where $\alpha' = \alpha$ if (25.36) has only one negative eigenvalue and $\alpha' = (|g_0|-|g_1|)/\gamma$, say, if there are two or more negative eigenvalues and g_1 is the second most important of these, and Z_0, Z_1, and $\bar{Z}$ are some constants.

26.4.3. *The evaluation of the integral*

The above results enable us to rewrite (26.36) in a more convenient form. We substitute the first part of (26.37) in (26.36), and expand all quantities about $p = -i/l(v_0)$, putting

$$ip-1/l(v_0) \equiv \bar{k}, \quad \zeta_{00} \equiv \zeta_0(\bar{k}) \text{ for } \bar{k} = 0, \quad \zeta'_{00} \equiv \zeta'_0(\bar{k}) \text{ for } \bar{k} = 0, \text{ etc.},$$

$$Z(p)\zeta'_0(p) \equiv \bar{K}(\bar{k}) \text{ [in Holte's notation } \bar{k} = k/l(0)-1/l(v_0), \ \zeta_{00} = 2\theta_m, \\ \zeta'_{00} = 2l(0)\theta'_m, \text{ and } \bar{K}(\bar{k}) = il(0)K(k)],$$

and noting that formula (26.39) leads to a similar expression for $\bar{K}(\bar{k})$, say

$$\bar{K}(\bar{k}) = \bar{K}_0+\bar{K}_1\bar{k}+\bar{K}(-\bar{k})^{\alpha'}+\text{smaller terms.} \quad (26.40)$$

The smaller terms not written explicitly in (26.40) are rather complex in nature, but they can be roughly taken as proportional to $\bar{k}^2$, say $\bar{K}_2\bar{k}^2$. For convenience we shall henceforward write the formulae in terms of the lethargy $u = 2\log(v_0/v)$. The integral (26.36) then becomes

$$\frac{N_0(x,v)}{l(v)} = -\frac{3Sh_1[l(v_0)]^{1-\alpha}}{4\pi v^2 l^2(0)} e^{\frac{1}{2}u\zeta_{00}-x/l(v_0)}\times \\ \times \int_{C_k} (-\bar{k})^{-\alpha}[\bar{K}_0+\bar{K}_1\bar{k}+\bar{K}_2\bar{k}^2+\bar{K}(-\bar{k})^{\alpha'}][1+\bar{k}l(v_0)]^{-1}\times \\ \times \exp[-(x-\tfrac{1}{2}u\zeta'_{00})\bar{k}+\tfrac{1}{4}u\zeta''_{00}\bar{k}^2+O(\bar{k}^3)]\,d\bar{k}, \quad (26.41)$$

where C_k is the path of integration in the complex $\bar{k}$-plane cut along the real axis from $\bar{k} = 0$ to $\bar{k} = +\infty$. In choosing this path, we try to make the integrand decrease as rapidly as possible as $|\bar{k}|$ increases, with the

condition that the path must lie to the left of the cut. When $x-\frac{1}{2}u\zeta'_{00}<0$, we can use the simpler method of §§ 26.3.3 and 26.3.4. We shall therefore be concerned with the case where $x-\frac{1}{2}u\zeta'_{00}\geqslant 0$; it can also be shown that $\zeta''_{00}>0$.

The choice of C_k and of the procedure for integrating in (26.41) depends on which factor in the integrand is the dominant one. There are four cases, according as the dominant factor is

$$\left.\begin{array}{ll} \exp[\frac{1}{4}u\zeta''_{00}\bar{k}^2] & (\text{case } (a)) \\ \exp[-(x-\frac{1}{2}u\zeta'_{00})\bar{k}] & (\text{case } (b)) \\ \{1+\bar{k}l(v_0)\}^{-1}(-\bar{k})^{-\alpha} & (\text{case } (c)) \\ \text{or} \quad \bar{K}(\bar{k}) & (\text{case } (d)) \end{array}\right\}. \tag{26.42}$$

Below we shall consider only cases (a) and (b). Holte has discussed case (c), but his procedure seems somewhat uncertain,† while case (d) does not appear to be of any interest in practice.

Case (a)

We first consider the case where $\exp[\frac{1}{4}u\zeta''_{00}\bar{k}^2]$ is the dominant factor. It can easily be shown that this will be so if

$$\alpha l(v_0)\ll x-\tfrac{1}{2}u\zeta'_{00}\leqslant(\tfrac{1}{4}u\zeta''_{00})^{\frac{1}{2}}. \tag{26.43}$$

A further examination of (26.43) shows that this condition requires a fairly large ζ''_{00} ($\geqslant 10l^2(0)$, say) and applies only to moderate changes in the mean free path. Here C_k is taken as a straight line parallel to and to the left of the imaginary axis, but infinitely close to it. All factors except the dominant one can then be expanded in powers of $\bar{k}$, and we obtain a series each term of which involves integrals of the form

$$\int_{C_k}\exp[\tfrac{1}{4}u\zeta''_{00}\bar{k}^2](-\bar{k})^s\,d\bar{k},$$

† In dealing with case (c), Holte returns to (26.36), substitutes for $\tilde{A}(p)$ from the second part of (26.37), and evaluates the resulting integral by the method of steepest descent, regarding $P(p)$ as a slowly varying function and including $\{1+p^2l^2(v_0)\}^{-\alpha}$ in the rapidly varying factor which determines the position of the saddle point. The uncertainty is that we might expect the path of steepest descent in this case to have so great a curvature that it cannot be approximated by a straight line. In other words, on calculating the correction terms as in the derivation of (24.10′), we might find that these terms were not small. The following fact supports this idea. If the procedure is applicable, there seems to be no reason why it should not be applied for $x<\frac{1}{2}u\zeta'_{00}$, whilst, if we use it for small x, the result, as Holte remarks, is in some disagreement with the result of age theory. Only a further investigation can show whether this is because case (c) corresponds to such a rapid variation of the mean free path that age theory is inapplicable, or whether the method of steepest descent is unsuitable in the present case.

which are closely related to the gamma function. The result is

$$\frac{N_0(x,v)}{l(v)} = -\frac{3Sh_1 i[l(v_0)]^{1-\alpha}}{4v^2l^2(0)}\overline{K}_0 e^{\frac{1}{2}u\zeta_{00}-\alpha/\overline{k}v_0)}(\tfrac{1}{4}u\zeta''_{00})^{-\frac{1}{2}(1-\alpha)}\times$$

$$\times\left\{\frac{1}{\Gamma(\frac{1}{2}\alpha+\frac{1}{2})}+\text{terms small for } (\tfrac{1}{4}u\zeta''_{00})^{\frac{1}{2}} \gg \max[x-\tfrac{1}{2}u\zeta'_{00}, 1]\right\}. \tag{26.44}$$

Case (b)

The second case in (26.42) is where $\exp[-(x-\frac{1}{2}u\zeta'_{00})\overline{k}]$ is the dominant factor for small and moderate $\overline{k}$. If this is so,

$$\left.\begin{aligned} x-\tfrac{1}{2}u\zeta'_{00} &\gg \alpha l(v_0), \\ &> (\tfrac{1}{4}u\zeta''_{00})^{\frac{1}{2}}, \\ &> \overline{K}_1/\overline{K}_0. \end{aligned}\right\} \tag{26.45}$$

These three inequalities state that for small $\overline{k}$ the second factor listed in (26.42) is dominant over the third, first, and fourth factors respectively. For sufficiently large $\overline{k}$, however, the first factor is always the dominant (i.e. most rapidly varying) one. We should therefore try to make the product of the first two factors decrease as rapidly as possible, i.e. to make

$$-(x-\tfrac{1}{2}u\zeta'_{00})\overline{k}+\tfrac{1}{4}u\zeta''_{00}\overline{k}^2$$

as nearly real and negative as possible, under the condition that the path lies to the left of the cut. The path will then start from $\overline{k} = \overline{k}_0 - i\infty$, where $\overline{k}_0 = [x-\frac{1}{2}u\zeta'_{00}]/2[\frac{1}{4}u\zeta''_{00}]^2$, pass parallel to the imaginary axis to a point just below the real axis, thence just below the real axis to the origin, round the origin, returning just above the real axis to $\overline{k} = \overline{k}_0$, and thence parallel to the imaginary axis to $\overline{k} = \overline{k}_0 + i\infty$. We denote by C_a the part of the path which follows re $\overline{k} = \overline{k}_0$, and by C_c the path along the cut and round the origin. Thus

$$\int_{C_b} = \int_{C_a} + \int_{C_c}.$$

The integral along C_a is a small correction which may be neglected in the first approximation; if necessary, it can be evaluated by the same means as the integrals in case (*a*). The dominant factor in the integral along C_c is $\exp[-(x-\frac{1}{2}u\zeta'_{00})\overline{k}]$. Expanding all other factors in powers of $\overline{k}$, we obtain a series of which each term involves an integral of the type

$$\frac{1}{i}\int_C \exp(-\gamma\overline{k}).(-\overline{k})^{-s}\,d\overline{k} \quad (\gamma = x-\tfrac{1}{2}u\zeta'_{00}),$$

and these are closely related to the incomplete gamma function,

$$Q(1-s,y) = \int_y^\infty e^{-t} t^{-s}\, dt. \tag{26.46}$$

The result is

$$\frac{N_0(x,v)}{l(v)} = -\frac{3Sh_1\, i\bar{K}}{2v^2 l^2(0)} (x-\tfrac{1}{2}u\zeta'_{00})^{\alpha-1}[l(v_0)]^{1-\alpha} e^{\frac{1}{2}u\zeta_{00}-x/l(v_0)} \times$$

$$\times \left\{\nu(\alpha,y) - \frac{\bar{K}_1}{(x-\tfrac{1}{2}u\zeta'_{00})\bar{K}_0}\nu(\alpha-1,y)+\ldots\right\} + \frac{[N_0(x,v)]_s}{l(v)}, \tag{26.47}$$

where
$$\nu(s,y) = \frac{1}{\Gamma(s)}\left[1 - \frac{Q(1-s,y)}{\Gamma(1-s)}\right]$$

and
$$y = \left[8\frac{x-\frac{1}{2}u\zeta'_{00}}{u^2(\zeta''_{00})^2} - \frac{1}{l(v_0)}\right](x-\tfrac{1}{2}u\zeta'_{00});$$

$Q(1-s,y)$ is defined by (26.46), and $[N_0(x,v)]_s/l(v)$ is the contribution from C_s.

The asymptotic expression for $N_0(x,v)$ as x tends to infinity can easily be obtained by retaining in (26.47) only the terms dominant in this limit. This gives

$$\frac{N_0(x,v)}{l(v)} = -\frac{3Sh_1\, i\bar{K}_0}{v^2 l^2(0)\Gamma(\alpha)} (x-\tfrac{1}{2}u\zeta'_{00})^{\alpha-1}[l(v_0)]^{1-\alpha} e^{\frac{1}{2}u\zeta_{00}-x/l(v_0)}, \tag{26.48}$$

which is a refinement of Wick's result (25.44).

APPENDIX A

Some remarks regarding the completeness of the sets of eigenfunctions $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega})$ and $N_{c,n}(\mathbf{r}, v\boldsymbol{\Omega})$

Two recent publications (**60, 61**) have given rise to some further considerations regarding the question whether the eigenfunctions $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega})$ under variable time-constant form a complete set.

For example, let us consider a cylinder of fixed radius and large but finite height h, and examine solutions of the transport equation such that the neutron distribution is almost independent of the axial coordinate except near the ends of the cylinder. Let $\lambda_n(h)$ and $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; h)$ be the nth eigenvalue and eigenfunction under variable time-constant, and $\gamma_n(h)$ and $N_{c,n}(\mathbf{r}, v\boldsymbol{\Omega}; h)$ those under variable composition; let

$$\lambda_n(\infty) = \lim_{h\to\infty} \lambda_n(h), \qquad \gamma_n(\infty) = \lim_{h\to\infty} \gamma_n(h), \quad \text{etc.}$$

Now, although $\gamma_n(\infty)$ and $N_{c,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$ are in fact the nth eigenvalue and eigenfunction for an infinite cylinder with the neutron distribution independent of the axial coordinate, the same is not necessarily true of $\lambda_n(\infty)$ and $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$. If

$$\mathrm{re}\,\lambda_n(\infty) > -v/l_{\mathrm{tot}}(v)$$

for all neutron speeds v occurring in the system, then $\lambda_n(\infty)$ and $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$ are the nth eigenvalue and eigenfunction under varying time-constant. If, however, this inequality does not hold, the corresponding fictitious inverse total mean free path (see § 3.2) has a negative real part for some v; the integral (2.36) diverges as $R_0 \to \infty$ (that is, $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$ violates the condition at infinity) for some $\boldsymbol{\Omega}$, and $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$ is not a possible solution for the infinite system. On the other hand, for all finite h the range of integration in (2.36) is finite, and no restriction need be placed on $\lambda_n(h)$. That is, unless the above inequality holds for all n, some solutions are lost on passing to the limit $h \to \infty$, and so the $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$ do not form a complete set, since the $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; h)$ are evidently linearly independent for any given h. Lehner and Wing (**60**), however, have shown† that the inequality must be violated for some n. The $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega}; \infty)$, therefore, do not form a complete set.

The above restriction on λ_n has a simple physical meaning. Let us again consider a cylinder as above, and suppose it irradiated in a manner which changes abruptly at time $t = 0$. In a rod of finite length h, after a time $t = h/v_{\min}$, there will be no neutrons which originated before $t = 0$. In an infinite rod, however, such neutrons will always be present. They will, it is true, not noticeably affect the first few terms in (3.6); but the terms for which λ_n does not satisfy the above inequality will be entirely dominated by neutrons originating in the parts of the rod added in extending it to infinity. Thus, in obtaining the complete solution of a time-dependent problem (as opposed to the leading terms only), it is not permissible to replace a finite body by one which extends to infinity even in one direction.

We conclude, therefore, that the $N_{t,n}(\mathbf{r}, v\boldsymbol{\Omega})$ form a complete set only for a system finite in all directions. However, the eigenfunction expansions used in this book

† Their discussion relates to an infinite slab, but the argument is unaffected.

are always in terms of the $N_{c,n}(\mathbf{r}, v\Omega)$. Though the above arguments place no restriction on the completeness of the latter set, there may be other considerations which disprove it.

For example, let the scattering be isotropic in the L system. Then, from (2.31) and (2.37), one and only one $N_{c,n}(\mathbf{r}, v\Omega)$ corresponds to each $n_{c,n}(\mathbf{r}, v)$. By (3.29), the $n_{c,n}(\mathbf{r}, v)$ are linearly independent. Let the function $T(\mathbf{r}, v\Omega)$ be such that $\iint T(\mathbf{r}, v\Omega)\, d\Omega = n_{c,n'}(\mathbf{r}, v)$ for a particular n', but $T(\mathbf{r}, v\Omega)$ and $N_{c,n'}(\mathbf{r}, v\Omega)$ are not the same function of Ω. Then $T(\mathbf{r}, v\Omega)$ cannot be expanded in terms of the $N_{c,n}(\mathbf{r}, v\Omega)$, so that the latter do not form a complete set, although the $n_{c,n}(\mathbf{r}, v)$ may or may not do so. Although this conclusion holds for scattering isotropic in the L system (and similarly for the case where the scattering function is a polynomial in $\Omega.\Omega'$), the results of Chapter III (and similarly of Chapter XVII) are unaffected, since they could have been derived in terms of the $n_{c,n}(\mathbf{r}, v)$, starting from the integral equation.

For the eigenfunctions under variable time-constant, however, there is no orthogonality relation for the $n_{t,n}(\mathbf{r}, v)$ (see § 3.7), and so the $n_{t,n}(\mathbf{r}, v)$ need not be linearly independent. The completeness of the set of $N_{t,n}(\mathbf{r}, v\Omega)$ is not, therefore, excluded on these grounds.

APPENDIX B

Carlson's method

CARLSON (58) has recently proposed a new numerical method for solving problems in plane, spherical, and cylindrical geometries, involving any energy distribution and scattering law, and all types of inhomogeneous medium. The method is so lengthy that it can be used only in conjunction with rapid electronic computing machines. With these, however, it is entirely suitable for problems of up to a certain degree of complexity which depends on the speed and storage capacity, and it has been tested on a variety of problems at different institutions.

The principle of the method is to divide the solid angle into n segments, and to approximate the neutron population in each segment by a linear expression defined by its values at the extreme directions within the segment. The value of n determines the order of the approximation, whence the term 'S_n method' generally applied to Carlson's procedure. It has been found that even the S_2 approximation is an improvement over diffusion theory, while the S_4 approximation is adequate for most practical purposes.

The Boltzmann equation is integrated over each separate segment of solid angle; another equation is derived directly from the Boltzmann equation by taking the neutron direction Ω to be along the negative polar axis. The neutrons are divided into any convenient number of groups as regards energy (cf. Chapter XIX) and position (as in the Monte Carlo method), and the spatial dependence of the neutron population is represented by linear functions as with the angular distribution. The Boltzmann equation, which is of the form (2.4), is thus replaced by sets of coupled difference equations in space, time, and angle. The equations for each energy group contain on their right sides sums involving the neutron populations in that group and in groups of higher energy, together with the source functions S in that group. The right sides of the equations are referred to as the *source terms*. The resulting equations are then solved by iteration (for a time-independent problem), starting with any provisional source terms, solving the difference equations for the neutron population, and effecting the summation to give the second approximation to the source terms. The appropriate boundary conditions and symmetry conditions are used in performing this integration. The direction of integration is taken to be along the direction of motion of the neutron, since this tends to reduce the errors of integration. In a critical-size problem, the result of the iteration process will usually be that the source term is ultimately multiplied by a constant factor at each iteration. The calculation is repeated for various sizes of the system, and the results are interpolated to give the size for which this constant factor is unity. In time-dependent problems, the actual source at some initial time is taken and the equations are integrated forward in time. The reader should consult Carlson's report for further details of the method.

Carlson states that some 25 iterations may be necessary for a time-independent S_4 calculation, with one energy group and 25 radial points, the system being spherically symmetric. Spinney (unpublished) has found that this number of iterations does not give such good convergence in similar calculations for a slab of fissile material. The convergence is judged by taking the ratios of successive source terms at each point in the system; these ratios should ultimately become constant. Some 40 iterations were needed to give a ratio differing by less than

0·02 per cent over the system. It should be noticed that the figure of 0·02 per cent is by no means unnecessarily accurate in dealing with critical-size problems. With 2 energy groups and 32 spatial points, Underhill (unpublished) also required 40 iterations to obtain convergence in an S_8 calculation for a spherically symmetric system of two media. It does not seem that even moderately complex problems (for example, a cylindrical geometry) can be handled by Carlson's method without expending an inordinate amount of time, at least with the machines at present available.

However, although Carlson's method is lengthy and less elegant than (say) the spherical harmonics method, it has the advantage of being much more easily adapted for work on an electronic computer when one is available. This feature becomes particularly important when the properties of the medium vary continuously with position, so that the equations of the spherical harmonics method themselves have to be integrated numerically.

For a fuller account of the method, see the discussion by J. H. Tait in *Progress in Nuclear Energy*, Series I (Physics and Mathematics), Volume 1, pp. 257–9, Pergamon Press, London, 1956.

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INDEX OF SYMBOLS

arg	argument.
av	average value, 340.
C system	centre-of-gravity system.
c	mean number of secondaries per collision, 9, 335.
div	divergence of a vector.
E	energy.
E_n	nth exponential integral, 67, 74, 112, 136.
f	scattering probability function, e.g. $f(v'\Omega' \to v\Omega)$, 17. $f(v' \to v)$, 22. $f_{\text{th}\to\text{th}}$, 362.
grad	gradient of a scalar.
I_s	modified Bessel function of order s, 63, 147, 170.
im	imaginary part.
J_0	Bessel function of order zero, 86, 377, 387.
$\mathbf{j}(\mathbf{r})$	neutron current, 41.
K_s	modified Bessel function of order s (Macdonald's definition), 63, 147.
L system	laboratory system.
L	diffusion length, 52, 54, 94, 239.
L_s	slowing-down length, 342, 363.
l, l_{tot}	total mean free path, 8, 10, 372.
l_c	capture mean free path, 10, 54, 323.
l_f	fission mean free path, 10.
l_s	scattering mean free path, 10, 54, 354.
M	mass of nucleus.
$N(\mathbf{r}, v\Omega, t)$	distribution of neutrons in position, speed, direction, and time, 15, 251, 322, 370.
N	number of nuclei per unit volume, 9, 286.
$n(\mathbf{r}, v)$	distribution of neutrons in position and speed, 23, 346, 371.
$n(\mathbf{r})$	distribution of neutrons in position, 41.
O	order of.
P_n	Legendre polynomial.
$Q(\mathbf{r}, v\Omega)$	neutron emission angular distribution, 27.
$Q(\mathbf{r}, v)$	neutron emission density, 24.
Q_n	Legendre function of the second kind, 118, 318.
q	collision interval, 327.
r	radial coordinate.
$\mathbf{r}$	position vector.
re	real part.
S	source strength, 15, 78, 106, 323–4, 351.
s	source strength, 39, 75, 190, 212, 243.
t	time.
u	lethargy, 326, 419.
dV	volume element.
v	speed.
$\mathbf{v}$	velocity.

Bold roman letters and Ω denote physical vectors; bold italic and Greek letters (except Ω) denote non-physical vectors (see p. 261, footnote).

German (Fraktur) letters denote matrices.

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GREAT BRITAIN
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